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SECOND-ORDER HARDY-TYPE INEQUALITY AND ITS APPLICATIONS

RYSKUL OINAROV[1*] AND AIGERIM KALYBAY[2]

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In this paper, we characterize a new second-order Hardy-type inequality and find a two-sided estimate for its least constant. As applications of this new inequality, we study oscillatory properties of a fourth-order differential equation and spectral properties of a corresponding differential operator.

1. Introduction

Let $I=(0,\infty)$ and $1<p,q<\infty$. Let r, v and u be positive functions. Moreover, we assume that r is continuously differentiable and v, u, r^{-1} and $v^{1-p'}$ are locally summable on the interval I, where $p'=\frac{p}{p-1}$. Let $C_0^\infty(I)$ be the set of finitely supported functions infinitely differentiable on the interval I. Suppose that $\|g\|_{p,v}=\Big(\int\limits_0^\infty v(t)|g(t)|^p dt\Big)^{\frac{1}{p}}$ is the norm of the Lebesgue space $L_{p,v}(I)$.

The second-order Hardy inequality has the following form:

$$\bigg(\int\limits_0^\infty u(t)|f(t)|^q dt\bigg)^{\frac{1}{q}} \le C\bigg(\int\limits_0^\infty v(t)|f''(t)|^p dt\bigg)^{\frac{1}{p}}, \qquad f\in C_0^\infty(I). \tag{1.1}$$

For functions f such that the right-hand side of (1.1) is finite, i.e., $\|f''\|_{p,v}<\infty$, we assume that $\lim\limits_{t\to0^+} f(t)=f(0)$, $\lim\limits_{t\to0^+} f'(t)=f'(0)$, $\lim\limits_{t\to\infty} f(t)=f(\infty)$ and $\lim\limits_{t\to\infty} f'(t)=f'(\infty)$ if these limits are finite. Depending on the way of posing boundary conditions $f(0)=0$, $f'(0)=0$, $f(\infty)=0$ and $f'(\infty)=0$ at the endpoints of the interval I, the inequality (1.1) was investigated in many papers (see, e.g., [3,7,8,10,12–14,16,17] and the references given therein). Concerning the current knowledge of higher-order Hardy-type inequalities we refer to [9, Chapter 4].

We introduce the differential operations $D_r^2 f(t)=\frac{d}{dt}r(t)\frac{df(t)}{dt}$ and $D_r^1 f(t)=r(t)\frac{df(t)}{dt}$ and consider the following generalization of the second-order Hardy inequality

$$\bigg(\int\limits_0^\infty u(t)|f(t)|^q dt\bigg)^{\frac{1}{q}} \le C\bigg(\int\limits_0^\infty v(t)|D_r^2 f(t)|^p dt\bigg)^{\frac{1}{p}}, \qquad f\in C_0^\infty(I), \tag{1.2}$$

with different combinations of boundary conditions $f(0)=0$, $D_r^1 f(0)=0$, $f(\infty)=0$ and $D_r^1 f(\infty)=0$, where $\lim\limits_{t\to0^+} f(t)=f(0)$, $\lim\limits_{t\to0^+} D_r^1 f(t)=D_r^1 f(0)$, $\lim\limits_{t\to\infty} f(t)=f(\infty)$ and $\lim\limits_{t\to\infty} D_r^1 f(t)=D_r^1 f(\infty)$ if these limits are finite. It is obvious that in the case $r=1$ the inequality (1.2) turns to the inequality (1.1).

The introduction of an additional weight function can help to handle some singularities that can not be handled by the existing weight functions. Thus, from the works [11] and [15] it follows that, in general, for any v providing $\|f''\|_{p,v}<\infty$, there does not exist exactly one limit at infinity $\lim\limits_{t\to\infty} f(t)$. At the same time, according to [5], we can find v and r providing $\|D_r^2 f\|_{p,v}<\infty$, which guarantee the existence of exactly one condition at infinity $\lim\limits_{t\to\infty} f(t)=f(\infty)$. Similarly, for any v, which gives

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*Ryskul Oinarov.

that $\|f''\|_{p,v} < \infty$, there does not exist exactly one finite limit at zero $\lim\limits_{t\to 0^+} f'(t)$. However, we can find v and r giving $\|D_r^2 f\|_{p,v} < \infty$, which guarantee the existence of exactly one finite limit at zero $\lim\limits_{t\to 0^+} D_r^1 f(t) = D_r^1 f(0)$.

The inequalities (1.1) and (1.2) can be investigated in the case when number of boundary conditions is exactly two. This case is called the "standard" case because the inequalities (1.1) and (1.2) are of the second order. In this paper, we consider a more interesting for applications case with three boundary conditions, which is called the "overdetermined" case.

Let $W_{p,v}^2(r,I)$ be a set of functions $f : I \to \mathbb{R}$, having generalized derivatives together with functions $D_r^1 f(t)$ on the interval I, with the finite norm

$$\|f\|_{W_{p,v}^2(r,I)} = \|D_r^2 f\|_{p,v} + |D_r^1 f(1)| + |f(1)|. \tag{1.3}$$

By the conditions on v and r, we have that $C_0^\infty(I) \subset W_{p,v}^2(r)$. Denote by $\mathring{W}_{p,v}^2(r,I)$ the closure of the set $C_0^\infty(I)$ with respect to the norm (1.3). Then the inequality (1.2) is equivalent to the inequality

$$\left(\int_0^\infty u(t)|f(t)|^q dt\right)^{\frac{1}{q}} \le C\left(\int_0^\infty v(t)|D_r^2 f(t)|^p dt\right)^{\frac{1}{p}}, \qquad f \in \mathring{W}_{p,v}^2(r,I), \tag{1.4}$$

in addition, the least constants in (1.2) and (1.4) coincide.

Summing up, in this paper we study the inequality (1.4) in the following four cases:

$$\mathring{W}_{p,v}^2(r,I) = \{f \in W_{p,v}^2(r,I) : D_r^1 f(0) = f(\infty) = D_r^1 f(\infty) = 0\}, \tag{1.5}$$

$$\mathring{W}_{p,v}^2(r,I) = \{f \in W_{p,v}^2(r,I) : f(0) = D_r^1 f(0) = D_r^1 f(\infty) = 0\}, \tag{1.6}$$

$$\mathring{W}_{p,v}^2(r,I) = \{f \in W_{p,v}^2(r,I) : f(0) = f(\infty) = D_r^1 f(\infty) = 0\}, \tag{1.7}$$

$$\mathring{W}_{p,v}^2(r,I) = \{f \in W_{p,v}^2(r,I) : f(0) = D_r^1 f(0) = f(\infty) = 0\}. \tag{1.8}$$

Note that, in view of the results from [11] and [15], in the case $r = 1$ it is impossible to study the inequality (1.4) for the cases (1.5) and (1.8) since there do not exist exactly one finite limit at zero $\lim\limits_{t\to 0^+} D_r^1 f(t) = \lim\limits_{t\to 0^+} f'(t) = f'(0)$ and exactly one finite limit at infinity $\lim\limits_{t\to\infty} f(t) = f(\infty)$, respectively.

In the paper, we prove that for each of the "overdetermined" cases (1.5)–(1.8), there are five characterizations for the fulfillment of the inequality (1.4). These main results are stated in Theorems 2.1, 3.1, 4.1 and 5.1. The crucial part of our main results is the fact that for all four cases the least constant of the inequality (1.4) is estimated by only two of the obtained characterizations. As an application, the estimates of the least constant of (1.4) are used for establishing the strong oscillation and strong non-oscillation of the differential equation

$$D_r^2\big(v(t)D_r^2 y(t)\big) - \lambda u(t)y(t) = 0, \quad t \in I, \;\; \lambda > 0, \tag{1.9}$$

which, when expanded, has the form of the differential equation with intermediate derivatives

$$r(vry'')'' + r(vr'y')'' + r'(vry'')' + r'(vr'y')' - \lambda uy = 0, \quad t \in I.$$

Let us remind that two distinct points t_1 and t_2 of the interval I are called conjugate with respect to the equation (1.9), if there exists its solution y such that $y(t_1) = y_2(t_2) = 0$ and $D_r^1 y(t_1) = D_r^1 y(t_2) = 0$. The equation (1.9) is called oscillatory at infinity (resp. at zero), if for any $T \in I$ there exist conjugate points with respect to the equation (1.9) to the right (resp. left) of T. Otherwise, the equation (1.9) is called non-oscillatory at infinity (resp. at zero). The equation (1.9) is said to be strong oscillatory (resp. non-oscillatory) at zero or at infinity if it is oscillatory (resp. non-oscillatory) for every $\lambda > 0$ at zero or at infinity, respectively. In order to establish the oscillatory properties of the equation (1.9), we consider the following inequalities:

$$\lambda \int_0^T u(t)|f(t)|^2 dt \le \lambda C_T \int_0^T v(t)|D_r^2 f(t)|^2 dt, \;\; f \in \mathring{W}^2_{2,v}(r,(0,T)), \tag{1.10}$$

$$\lambda \int_T^\infty u(t)|f(t)|^2 dt \le \lambda C_T \int_T^\infty v(t)|D_r^2 f(t)|^2 dt, \;\; f \in \mathring{W}^2_{2,v}(r,(T,\infty)). \tag{1.11}$$

The main results concerning the strong oscillation and strong non-oscillation of the equation (1.9) (see Theorems 2.2, 3.2, 4.2 and 5.2) follow as corollaries from the estimates of the least constant of (1.4) and the following statement:

Lemma 1.1. *Let $0 < T < \infty$. Let C_T be the least constant in* (1.10) (*resp.* (1.11)).

(i) *The equation* (1.9) *is strong non-oscillatory at zero* (*resp. at infinity*) *if and only if* $\lim\limits_{T\to 0^+} C_T = 0$ (*resp.* $\lim\limits_{T\to\infty} C_T = 0$).

(ii) *The equation* (1.9) *is strong oscillatory at zero* (*resp. at infinity*) *if and only if* $C_T = \infty$ (*resp.* $C_T = \infty$) *for any* $T > 0$.

Lemma 1.1, based on the inequalities (1.10) and (1.11), clearly shows why it is interesting to consider the "overdetermined" cases for the inequality (1.4). Obviously, for any v and r, we already have two boundary conditions at one endpoint T of each intervals $(0,T)$ and (T,∞). Thus, the inequality (1.10) is a partial case of the inequality (1.4) in the cases (1.5) and (1.7) when there is one extra condition at zero for the interval $(0,T)$, and the inequality (1.11) is a partial case of the inequality (1.4) in the cases (1.6) and (1.8) when there is one extra condition at infinity for the interval (T,∞).

We obtain one more application by considering the following operator

$$Ly(t) = \frac{1}{u(t)} D_r^2\big(v(t) D_r^2 y(t)\big) \tag{1.12}$$

in the space $L_{2,u}(I)$ with inner product $(f,g)_{2,u} = \int_0^\infty f(t)g(t)u(t)dt$. Let the minimal differential operator $L_{\min}$ be generated by the differential expression (1.12), i.e., $L_{\min}$ is an operator with the domain $D(L_{\min}) = C_0^\infty(I)$. It is known that all self-adjoint extensions of the minimal operator have the same spectrum (see [2]).

In this paper, we focus our attention to the boundedness below and discreteness of the operator L. The problem of finding the conditions under which any self-adjoint extension L of the operator $L_{\min}$ has a spectrum, which is discrete and bounded below is one of the most important problems in the theory of singular differential operators since these properties guarantee that the singular operator behaves like a regular one (see [4]).

The relationship between the oscillatory properties of the equation (1.9) and the spectral properties of the operator L is explained in the following statement:

Lemma 1.2 (see [2])**.** *The operator L is bounded below and has a discrete spectrum if and only if the equation* (1.9) *is strong non-oscillatory.*

The main results concerning the boundedness below and discreteness of the operator L are stated in Theorems 2.3, 3.3, 4.3 and 5.3 for the cases (1.5), (1.6), (1.7) and (1.8), respectively.

Remark 1.1. The proofs of the main results for the cases (1.5) and (1.6) are given in [6] and the proofs of the main results for the cases (1.7) and (1.8) are presented in [1].

2. The Case $\mathring{W}^2_{p,v}(r,I) = \{f \in W^2_{p,v}(r,I) : D_r^1 f(0) = f(\infty) = D_r^1 f(\infty) = 0\}$

Let $\tau \in I$ and

$$A_1^+(\tau) = \sup_{z>\tau} \left(\int_\tau^z \left(\int_t^z r^{-1}(x)dx \right)^q u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty v^{1-p'}(s)ds \right)^{\frac{1}{p'}},$$

$$
\begin{aligned}
A_2^+(\tau) &= \sup_{z>\tau}\left(\int_\tau^z u(t)dt\right)^{\frac{1}{q}}\left(\int_z^\infty\left(\int_z^s r^{-1}(x)dx\right)^{p'} v^{1-p'}(s)ds\right)^{\frac{1}{p'}},\\
A_3^+(\tau) &= \left(\int_0^\tau u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty\left(\int_\tau^s r^{-1}(x)dx\right)^{p'} v^{1-p'}(s)ds\right)^{\frac{1}{p'}},\\
E_1^-(\tau) &= \sup_{0<z<\tau}\left(\int_0^z u(t)dt\right)^{\frac{1}{q}}\left(\int_z^\tau\left(\int_s^\tau r^{-1}(x)dx\right)^{p'} v^{1-p'}(s)ds\right)^{\frac{1}{p'}},\\
E_2^-(\tau) &= \sup_{0<z<\tau}\left(\int_z^\tau\left(\int_t^\tau r^{-1}(x)dx\right)^{q} u(t)dt\right)^{\frac{1}{q}}\left(\int_0^z v^{1-p'}(s)ds\right)^{\frac{1}{p'}},\\
A^+(\tau) &= \max\{A_1^+(\tau), A_2^+(\tau), A_3^+(\tau)\}, \quad E^-(\tau) = \max\{E_1^-(\tau), E_2^-(\tau)\},\\
A^+E^- &= \inf_{\tau\in I}\max\{A^+(\tau), E^-(\tau)\}.
\end{aligned}
$$

According to [5], the following conditions on the weights v and r provide (1.5):

$$
v^{1-p'} \in L_1(I), \;\; r^{-1} \in L_1(1,\infty), \;\; r^{-1} \notin L_1(0,1), \qquad \int_0^1\left(\int_t^1 r^{-1}(x)dx\right)^{p'} v^{1-p'}(t)dt = \infty. \tag{2.1}
$$

Since $v^{1-p'} \in L_1(I)$, for any $\tau \in I$ there exists k_τ such that $\int_0^\tau v^{1-p'}(t)dt = k_\tau \int_\tau^\infty v^{1-p'}(t)dt.$

Our main result concerning the inequality (1.4) reads:

Theorem 2.1. *Let $1 < p \le q < \infty$ and the conditions* (2.1) *hold. Then for the least constant C in* (1.4) *the estimates*

$$
\begin{gathered}
4^{-\frac{1}{p}} A^+E^- \le C \le 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A^+E^-,\\
\sup_{\tau\in I}(1+k_\tau^{p-1})^{-\frac{1}{p}} E^-(\tau) \le C \le 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} E^-(\tau_1)
\end{gathered} \tag{2.2}
$$

hold, where $\tau_1 = \inf\{\tau > 0 : A^+(\tau) \le E^-(\tau)\}$.

For the case $p = 2$ the conditions (2.1) can be rewritten in the form:

$$
v^{-1} \in L_1(I), \;\; r^{-1} \in L_1(1,\infty), \;\; r^{-1} \notin L_1(0,1), \qquad \int_0^1\left(\int_t^1 r^{-1}(x)dx\right)^{2} v^{-1}(t)dt = \infty. \tag{2.3}
$$

From the estimate (2.2) for $p = q = 2$ and Lemma 1.1 we have the following result for the strong oscillation and non-oscillation of the equation (1.9):

Theorem 2.2. *Let* (2.3) *hold.*

(i) *The equation* (1.9) *is strong non-oscillatory at zero if and only if*

$$
\lim_{z\to 0^+}\int_0^z u(t)dt\int_z^\infty\left(\int_s^\infty r^{-1}(x)dx\right)^2 v^{-1}(s)ds = 0, \tag{2.4}
$$

$$
\lim_{z\to 0^+}\int_z^\infty\left(\int_t^\infty r^{-1}(x)dx\right)^2 u(t)dt\int_0^z v^{-1}(s)ds = 0. \tag{2.5}
$$

(ii) *The equation* (1.9) *is strong oscillatory at zero if and only if*

$$
\lim_{z\to 0^+}\int_0^z u(t)dt\int_z^\infty\left(\int_s^\infty r^{-1}(x)dx\right)^2 v^{-1}(s)ds = \infty
$$

or

$$\lim_{z\to 0^+}\int\limits_z^\infty\Big(\int\limits_t^\infty r^{-1}(x)dx\Big)^2 u(t)dt\int\limits_0^z v^{-1}(s)ds=\infty.$$

From Theorem 2.2 and Lemma 1.2 we have the following result for the spectral properties of the operator L:

Theorem 2.3. *Let* (2.3) *hold. Then the operator* L *is bounded below and has a discrete spectrum if and only if* (2.4) *and* (2.5) *hold.*

3. The Case $\overset{\circ}{W}{}^2_{p,v}(r,I)=\{f\in W^2_{p,v}(r,I): f(0)=D^1_r f(0)=D^1_r f(\infty)=0\}$

Let $\tau\in I$ and

$$\begin{aligned}
A_1^-(\tau)&=\sup_{0<z<\tau}\Big(\int\limits_z^\tau u(t)\Big(\int\limits_z^t r^{-1}(x)dx\Big)^q dt\Big)^{\frac{1}{q}}\Big(\int\limits_0^z v^{1-p'}(s)ds\Big)^{\frac{1}{p'}},\\
A_2^-(\tau)&=\sup_{0<z<\tau}\Big(\int\limits_z^\tau u(t)dt\Big)^{\frac{1}{q}}\Big(\int\limits_0^z\Big(\int\limits_s^z r^{-1}(x)dx\Big)^{p'} v^{1-p'}(s)ds\Big)^{\frac{1}{p'}},\\
A_3^-(\tau)&=\Big(\int\limits_\tau^\infty u(t)dt\Big)^{\frac{1}{q}}\Big(\int\limits_0^\tau\Big(\int\limits_s^\tau r^{-1}(x)dx\Big)^{p'} v^{1-p'}(s)ds\Big)^{\frac{1}{p'}},\\
E_1^+(\tau)&=\sup_{z>\tau}\Big(\int\limits_z^\infty u(t)dt\Big)^{\frac{1}{q}}\Big(\int\limits_\tau^z\Big(\int\limits_\tau^s r^{-1}(x)dx\Big)^{p'} v^{1-p'}(s)ds\Big)^{\frac{1}{p'}},\\
E_2^+(\tau)&=\sup_{z>\tau}\Big(\int\limits_\tau^z\Big(\int\limits_\tau^t r^{-1}(x)dx\Big)^q u(t)dt\Big)^{\frac{1}{q}}\Big(\int\limits_z^\infty v^{1-p'}(s)ds\Big)^{\frac{1}{p'}},\\
A^-(\tau)&=\max\{A_1^-(\tau),A_2^-(\tau),A_3^-(\tau)\},\quad E^+(\tau)=\max\{E_1^+(\tau),E_2^+(\tau)\},\\
A^-E^+&=\inf_{\tau\in I}\max\{A^-(\tau),E^+(\tau)\}.
\end{aligned}$$

For the case (1.6) to be hold, we need the following conditions:

$$v^{1-p'}\in L_1(I),\;\; r^{-1}\in L_1(0,1),\;\; r^{-1}\notin L_1(1,\infty),\qquad \int\limits_1^\infty\Big(\int\limits_1^t r^{-1}(x)dx\Big)^{p'} v^{1-p'}(t)dt=\infty. \tag{3.1}$$

Theorem 3.1. *Let* $1<p\le q<\infty$ *and the conditions* (3.1) *hold. Then for the least constant* C *in* (1.4) *the estimates*

$$4^{-\frac{1}{p}}A^-E^+\le C\le 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}}A^-E^+,$$

$$\sup_{\tau\in I}(1+k_\tau^{1-p})^{-\frac{1}{p}}E^+(\tau)\le C\le 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}}E^+(\tau_2)$$

hold, where $\tau_2=\sup\{\tau>0: A^-(\tau)\le E^+(\tau)\}$.

Theorem 3.2. *Let* $v^{-1}\in L_1(I)$, $r^{-1}\in L_1(0,1)$, $r^{-1}\notin L_1(1,\infty)$, $\int\limits_1^\infty\Big(\int\limits_1^t r^{-1}(x)dx\Big)^2 v^{-1}(t)dt=\infty$.

(i) *The equation* (1.9) *is strong non-oscillatory at infinity if and only if*

$$\lim_{z\to\infty}\int\limits_z^\infty u(t)dt\int\limits_0^z\Big(\int\limits_0^s r^{-1}(x)dx\Big)^2 v^{-1}(s)ds=0, \tag{3.2}$$

$$\lim_{z\to\infty}\int_0^z\Big(\int_0^t r^{-1}(x)dx\Big)^2 u(t)dt\int_z^\infty v^{-1}(s)ds=0. \tag{3.3}$$

(ii) *The equation* (1.9) *is strong oscillatory at infinity if and only if*

$$\lim_{z\to\infty}\int_z^\infty u(t)dt\int_0^z\Big(\int_0^s r^{-1}(x)dx\Big)^2 v^{-1}(s)ds=\infty$$

or

$$\lim_{z\to\infty}\int_0^z\Big(\int_0^t r^{-1}(x)dx\Big)^2 u(t)dt\int_z^\infty v^{-1}(s)ds=\infty.$$

Theorem 3.3. *Let the conditions of Theorem* 3.2 *hold. Then the operator L is bounded below and has a discrete spectrum if and only if* (3.2) *and* (3.3) *hold.*

4. The Case $\overset{\circ}{W}{}^2_{p,v}(r,I)=\{f\in W^2_{p,v}(r,I): f(0)=f(\infty)=D^1_r f(\infty)=0\}$

Assume that $\rho(t)=\int_0^t r^{-1}(x)dx$, $t\in I$. Let $\tau\in I$ and

$$\begin{aligned}
B_3^+(\tau)=&\frac{1}{\rho(\tau)}\Big(\int_0^\tau \rho^q(t)u(t)dt\Big)^{\frac1q}\Big(\int_\tau^\infty\Big(\int_\tau^s r^{-1}(x)dx\Big)^{p'}v^{1-p'}(s)ds\Big)^{\frac{1}{p'}},\\
F_1^-(\tau)=&\sup_{0<z<\tau}\frac{1}{\rho(\tau)}\Big(\int_0^z \rho^q(t)u(t)dt\Big)^{\frac1q}\Big(\int_z^\tau\Big(\int_s^\tau r^{-1}(x)dx\Big)^{p'}v^{1-p'}(s)ds\Big)^{\frac{1}{p'}},\\
F_2^-(\tau)=&\sup_{0<z<\tau}\frac{1}{\rho(\tau)}\Big(\int_z^\tau\Big(\int_t^\tau r^{-1}(x)dx\Big)^{q}u(t)dt\Big)^{\frac1q}\Big(\int_0^z \rho^{p'}(s)v^{1-p'}(s)ds\Big)^{\frac{1}{p'}},\\
B^+(\tau)=&\max\{A_1^+(\tau),A_2^+(\tau),B_3^+(\tau)\},\quad F^-(\tau)=\max\{F_1^-(\tau),F_2^-(\tau)\},\\
B^+F^-=&\inf_{\tau\in I}\max\{B^+(\tau),F^-(\tau)\}.
\end{aligned}$$

The case (1.7) holds if

$$r^{-1}\in L_1(I),\ \ v^{1-p'}\in L_1(1,\infty),\ \ v^{1-p'}\notin L_1(0,1),\qquad \int_0^1\Big(\int_0^t r^{-1}(x)dx\Big)^{p'}v^{1-p'}(t)dt<\infty. \tag{4.1}$$

Let $\bar v(t)=\rho^{-p}(t)v(t)$, $t\in I$. Since from (4.1) we get that $\bar v^{1-p'}\in L_1(I)$, for any $\tau\in I$ there exists l_τ such that $\int_0^\tau \bar v^{1-p'}(t)dt=l_\tau\int_\tau^\infty \bar v^{1-p'}(t)dt$.

Theorem 4.1. *Let $1<p\le q<\infty$ and the conditions* (4.1) *hold. Then for the least constant C in* (1.4) *the estimates*

$$4^{-\frac1p}B^+F^-\le C\le 11p^{\frac1q}(p')^{\frac{1}{p'}}B^+F^-,$$
$$\sup_{\tau\in I}(1+l_\tau^{p-1})^{-\frac1p}F^-(\tau)\le C\le 11p^{\frac1q}(p')^{\frac{1}{p'}}F^-(\tau_3)$$

hold, where $\tau_3=\inf\{\tau>0: B^+(\tau)\le F^-(\tau)\}$.

Theorem 4.2. *Let $r^{-1}\in L_1(I)$, $v^{-1}\in L_1(1,\infty)$, $v^{-1}\notin L_1(0,1)$, $\int_0^1\Big(\int_0^t r^{-1}(x)dx\Big)^2 v^{-1}(t)dt<\infty$.*

(i) *The equation* (1.9) *is strong non-oscillatory at zero if and only if*

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\frac{1}{\rho^2(\tau)}\int_0^z \rho^2(t)u(t)dt\int_z^\tau\left(\int_s^\tau r^{-1}(x)dx\right)^2 v^{-1}(s)ds=0, \tag{4.2}$$

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\frac{1}{\rho^2(\tau)}\int_z^\tau\left(\int_t^\tau r^{-1}(x)dx\right)^2 u(t)dt\int_0^z \rho^2(s)v^{-1}(s)ds=0. \tag{4.3}$$

(ii) *The equation* (1.9) *is strong oscillatory at zero if and only if*

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\frac{1}{\rho^2(\tau)}\int_0^z \rho^2(t)u(t)dt\int_z^\tau\left(\int_s^\tau r^{-1}(x)dx\right)^2 v^{-1}(s)ds=\infty$$

or

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\frac{1}{\rho^2(\tau)}\int_z^\tau\left(\int_t^\tau r^{-1}(x)dx\right)^2 u(t)dt\int_0^z \rho^2(s)v^{-1}(s)ds=\infty.$$

Theorem 4.3. *Let the conditions of Theorem* 4.2 *hold. Then the operator L is bounded below and has a discrete spectrum if and only if* (4.2) *and* (4.3) *hold.*

5. The Case $\overset{\circ}{W}{}^2_{p,v}(r,I)=\{f\in W^2_{p,v}(r,I): f(0)=D^1_r f(0)=f(\infty)=0\}$

Assume that $\bar\rho(t)=\int_t^\infty r^{-1}(x)dx$, $t\in I$. Let $\tau\in I$ and

$$\begin{aligned}
B_3^-(\tau)=&\frac{1}{\bar\rho(\tau)}\left(\int_\tau^\infty \bar\rho^q(t)u(t)dt\right)^{\frac1q}\left(\int_0^\tau\left(\int_s^\tau r^{-1}(x)dx\right)^{p'} v^{1-p'}(s)ds\right)^{\frac{1}{p'}},\\
F_1^+(\tau)=&\sup_{z>\tau}\frac{1}{\bar\rho(\tau)}\left(\int_\tau^z\left(\int_\tau^t r^{-1}(x)dx\right)^q u(t)dt\right)^{\frac1q}\left(\int_z^\infty \bar\rho^{p'}(s)v^{1-p'}(s)ds\right)^{\frac{1}{p'}},\\
F_2^+(\tau)=&\sup_{z>\tau}\frac{1}{\bar\rho(\tau)}\left(\int_z^\infty \bar\rho^q(t)u(t)dt\right)^{\frac1q}\left(\int_\tau^z\left(\int_\tau^s r^{-1}(x)dx\right)^{p'} v^{1-p'}(s)ds\right)^{\frac{1}{p'}},\\
B^-(\tau)=&\max\{A_1^-(\tau),A_2^-(\tau)\},B_3^-(\tau)\},\quad F^+(\tau)=\max\{F_1^+(\tau),F_2^+(\tau)\},\\
B^-F^+=&\inf_{\tau\in I}\max\{B^-(\tau),F^+(\tau)\}.
\end{aligned}$$

For the last case (1.8) we need the following conditions:

$$r^{-1}\in L_1(I),\ \ v^{1-p'}\notin L_1(1,\infty),\ \ v^{1-p'}\in L_1(0,1),\qquad \int_1^\infty\left(\int_t^\infty r^{-1}(x)dx\right)^{p'}v^{1-p'}(t)dt<\infty. \tag{5.1}$$

Theorem 5.1. *Let $1<p\le q<\infty$ and the conditions* (5.1) *hold. Then for the least constant C in* (1.4) *the estimates*

$$4^{-\frac1p}B^-F^+\le C\le 11p^{\frac1q}(p')^{\frac{1}{p'}}B^-F^+,$$
$$\sup_{\tau\in I}(1+l_\tau^{1-p})^{-\frac1p}F^+(\tau)\le C\le 11p^{\frac1q}(p')^{\frac{1}{p'}}F^+(\tau_4)$$

hold, where $\tau_4=\sup\{\tau>0: B^-(\tau)\le F^+(\tau)\}$.

Theorem 5.2. *Let $r^{-1}\in L_1(I)$, $v^{-1}\notin L_1(1,\infty)$, $v^{-1}\in L_1(0,1)$, $\int_1^\infty\left(\int_t^\infty r^{-1}(x)dx\right)^2 v^{-1}(t)dt<\infty$.*

(i) *The equation* (1.9) *is strong non-oscillatory at infinity if and only if*

$$\lim_{\tau\to\infty}\sup_{z>\tau}\frac{1}{\bar{\rho}^2(\tau)}\int\limits_{\tau}^{z}\left(\int\limits_{\tau}^{t} r^{-1}(x)dx\right)^2 u(t)dt\int\limits_{z}^{\infty}\bar{\rho}^2(s)v^{-1}(s)ds=0, \tag{5.2}$$

$$\lim_{\tau\to\infty}\sup_{z>\tau}\frac{1}{\bar{\rho}^2(\tau)}\int\limits_{z}^{\infty}\bar{\rho}^2(t)u(t)dt\int\limits_{\tau}^{z}\left(\int\limits_{\tau}^{s} r^{-1}(x)dx\right)^2 v^{-1}(s)ds=0. \tag{5.3}$$

(ii) *The equation* (1.9) *is strong oscillatory at infinity if and only if*

$$\lim_{\tau\to\infty}\sup_{z>\tau}\frac{1}{\bar{\rho}^2(\tau)}\int\limits_{\tau}^{z}\left(\int\limits_{\tau}^{t} r^{-1}(x)dx\right)^2 u(t)dt\int\limits_{z}^{\infty}\bar{\rho}^2(s)v^{-1}(s)ds=\infty$$

or

$$\lim_{\tau\to\infty}\sup_{z>\tau}\frac{1}{\bar{\rho}^2(\tau)}\int\limits_{z}^{\infty}\bar{\rho}^2(t)u(t)dt\int\limits_{\tau}^{z}\left(\int\limits_{\tau}^{s} r^{-1}(x)dx\right)^2 v^{-1}(s)ds=\infty.$$

Theorem 5.3. *Let the conditions of Theorem* 5.2 *hold. Then the operator L is bounded below and has a discrete spectrum if and only if* (5.2) *and* (5.3) *hold.*

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[1]L.N. Gumilyov Eurasian National University, 2 Satpayev Str., 010008 Astana, Kazakhstan

[2]KIMEP University, 4 Abay Ave., 050010 Almaty, Kazakhstan
Email address: o_ryskul@mail.ru
Email address: kalybay@kimep.kz