Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 2, 217–236

GEGENBAUER FRACTIONAL MAXIMAL FUNCTION AND ITS COMMUTATORS ON GEGENBAUER–ORLICZ SPACES

ELMAN J. IBRAHIMOV[1] AND SAADAT AR. JAFAROVA[2]

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In this paper, we find the necessary and sufficient conditions for the boundedness of Gegenbauer fractional (G-fractional) maximal operator M_G^α in Gegenbauer-Olicz (G-Orlicz) spaces. As an application of these results, we consider the boundedness of G-fractional maximal commutator $M_G^{b,\alpha}$ in G-Orlicz spaces.

1. Introduction

Norm inequalities for several classical operators of harmonic type have been widely studied in the context of Orlicz spaces. It is well known that many of such operators fail to have continuity properties when they act between certain Lebesgue spaces and, in some situations, the Orlicz spaces appear as adequate substitutes. For example, the Hardy–Littlewood maximal operator is bounded on L^p for $1 < p \le \infty$, but not on L^1. However, by using Orlicz spaces, we can investigate the boundedness of the maximal operator near $p = 1$ (for more precise statements, see [3, 11, 12] and for the theory of Orlics spaces, see [14, 15, 17]).

Let $0 \le \alpha < n$. The classical fractional maximal operator M_α is given by

$$M_\alpha f(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int\limits_B |f(y)|dy$$

and the fractional maximal commutators of M_α with a locally integrable function b is defined by

$$M_{b,\alpha} f(x) = \sup_{B \ni x} |B|^{-1+\frac{\alpha}{n}} \int\limits_B |b(x) - b(y)||f(y)|dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing x. If $\alpha = 0$, then $M \equiv M_0$ is the Hardy–Littlewood maximal operator and $M_b \equiv M_{b,0}$ is the maximal commutator of M.

For more details about the operator $M_{b,\alpha}$, where $0 \le \alpha < n$ in the Orlicz space, we refer to [1, 4, 16] and references therein.

2. Definition, Notation and Auxiliary Results

Our investigation is based on the Gegenbauer differential operator [7]

$$G \equiv G_\lambda = \left(x^2 - 1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx} \left(x^2 - 1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}, \quad x \in (1, \infty), \quad \lambda \in \left(0, \frac{1}{2}\right).$$

The shift operator A^λ_{chy} generated by G is given in the form [8]

$$A^\lambda_{chy} f(chx) = \frac{\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int\limits_0^\pi f(chxchy - shxshy\cos\varphi)(\sin\varphi)^{2\lambda-1} d\varphi,$$

where $x, y \in \mathbb{R}_+$.

2020 *Mathematics Subject Classification.* 42B25, 42B35, 47B47, 26A16.

Key words and phrases. G-fractional; Maximal function; Commutator; G-Orlicz spaces.

Let $H_r = (0, r) \subset \mathbb{R}_+$. For any measurable set E $\mu E = |E|_\lambda = \int_E sh^{2\lambda} x dx$. For $1 \le p < \infty$, let $L_p(\mathbb{R}_+, G) = L_{p,\lambda}(\mathbb{R}_+)$ be the space of $\mu_\lambda(x) = sh^{2\lambda} x$ measurable function on $\mathbb{R}_+$ with the finite norm

$$\|f\|_{L_{p,\lambda}} = \left(\int_{\mathbb{R}_+} |f(chx)|^p d\mu_\lambda(x) \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

Note that the space $L_{p,\lambda}$ is the Banach space (see [6, Proposition 5.1]).

Throughout the whole paper, the notation $A \lesssim B$ means that there exists a constant $C > 0$ such that $A \lesssim CB$, where C is independent of the appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent. Further, we need the following lemmas.

Lemma 2.1 ([10]). *The following relation:*

$$|H_r|_\lambda = \int_0^r sh^{2\lambda} y dy \approx \left(sh \frac{r}{2} \right)^\gamma, \quad \textit{where} \quad 0 < \gamma \le 2\lambda + 1. \tag{2.1}$$

is true.

Lemma 2.2 ([7]). *If $f \in L_{p,\lambda}(\mathbb{R}_+)$, $(1 \le p \le \infty)$, then for any $y \in \mathbb{R}_+$, the following inequality:*

$$\left\| A_{chy}^\lambda f \right\|_{L_{p,\lambda}(\mathbb{R}_+)} \le \|f\|_{L_{p,\lambda}(\mathbb{R}_+)}$$

holds.

For $f \in L_{1,\lambda}^{\mathrm{loc}}(\mathbb{R}_+)$, the G-fractional maximal operator M_G^α is defined in [9] as follows:

$$M_G^\alpha f(chx) = \sup_{r>0} |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_r} A_{chy}^\lambda |f(chx)| \, d\mu_\lambda(y). \tag{2.2}$$

Let $b \in L_{1,\lambda}^{\mathrm{loc}}(\mathbb{R}_+)$, then the commutator $M_G^{b,\alpha}$, generated by M_G^α and b, is defined in [9] as follows:

$$M_G^{b,\alpha} f(chx) = \sup_{r>0} |H_r|_\lambda^{\frac{\alpha}{\gamma}-1} \int_{H_r} \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y). \tag{2.3}$$

The definition of a G-BMO space is given below, in Section 5.

3. Preliminaries

Before we proceed to proving our results, we shall introduce some preliminary definitions and properties concerning the G-Orlicz spaces.

Definition 3.1. *A* function $\Phi : [0, \infty) \to [0, \infty]$ is called a Young function if Φ is continuous, convex and strictly increasing, and $\Phi(0) = 0$, $\Phi(r) \to \infty$ as $r \to \infty$.

The set of Young functions such that $0 < \Phi(r) < \infty$ for $0 < r < \infty$ will be denoted by Y, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \le s \le \infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}.$$

If $\Phi \in Y$, then Φ is the usual inverse function of Φ. It is well known that

$$r \le \Phi^{-1}(r) \widetilde{\Phi}^{-1}(r) \le 2r, \qquad r \ge 0,$$

where $\widetilde{\Phi}$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\}, & r \in [0, \infty), \\ \infty, & r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2 -condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le c\Phi(r), \qquad r \ge 0,$$

for some $c \geq 2$. If $\Phi \in \Delta_2$, then $\Phi \in Y$. A Young function Φ is said to satisfy the ∇_2- condition, denoted also as $\Phi \in \nabla_2$, if $\Phi(r) \leq \frac{1}{2c}\Phi(cr)$, $r \geq 0$ for some $c > 1$. We can verify the following examples: the function $\Phi(r) = r$ satisfies the Δ_2 -condition, but does not satisfy the ∇_2-condition. If $1 < p < \infty$, then $\Phi(r) = r^p$ satisfies both conditions. The function $\Phi(r) = e^r - r - 1$ satisfies the ∇_2 -condition, but does not satisfy the Δ_2-condition.

Definition 3.2 ([13] G-Orlicz space)**.** For a Young function Φ, the set

$$L_\Phi(\mathbb{R}_+, G) = \left\{ f \in L^{\text{loc}}_{1,\lambda}(\mathbb{R}_+) : \int\limits_{\mathbb{R}_+} \Phi\left(k\,|f(chx)|\right) d\mu_\lambda(x) < \infty \ \text{ for some } \quad k > 0 \right\}$$

is called the Gegenbauer–Orlicz (G-Orlicz) space.

If $\Phi(t) = t^p$, $1 \leq p < \infty$, then $L_\Phi(\mathbb{R}_+, G) = L_{p,\lambda}(\mathbb{R}_+)$. If $\Phi(r) = 0, (0 \leq r \leq 1)$ and $\Phi(r) = \infty$, $(r > 1)$, then $L_\Phi(\mathbb{R}_+, G) = L_\infty(\mathbb{R}_+)$. The space $L^{\text{loc}}_{1,\lambda}(\mathbb{R}_+, G)$ we define as the set of all functions f such that $f\chi_\Omega \in L_\Phi(\mathbb{R}_+, G)$ for all intervals $\Omega \subset \mathbb{R}_+$, where χ_Ω is a characteristic function of the interval Ω.

Definition 3.3 ([13])**.** Let $\Phi \in Y$ and the norm $\|\cdot\|_{L_\Phi}$ on the space $L_\Phi(\mathbb{R}_+, G)$ be defined by

$$\|f\|_{L_\Phi(\mathbb{R}_+,G)} = \inf\left\{ \nu > 0 : \int\limits_{\mathbb{R}_+} \Phi\left(\frac{1}{\nu}\,|f(chx)|\right) d\mu_\lambda(x) \leq 1 \right\}.$$

For a measurable set $\Omega \subset \mathbb{R}_+$, a $\mu_{\overline{\lambda}}$ measurable function f and $t > 0$, let $m(\Omega, f, t) = |\{x \in \Omega : |f(chx)| > t\}|_\lambda$. In case $\Omega = \mathbb{R}_+$, we shortly denote it by $m(f, t)$.

Definition 3.4. The weak G-Orlicz space

$$WL_{\Phi(R_+,G)} = \left\{ f \in L^{\text{loc}}_{1,\lambda}(R_+) : \|f\|_{WL_\Phi(\mathbb{R}_+,G)} < \infty \right\}$$

we define by the norm

$$\|f\|_{WL_\Phi(\mathbb{R}_+,G)} = \inf\left\{ \nu > 0 : \sup_{t>0} \Phi(t) m\Big(\frac{f}{\nu}, t\Big) \leq 1 \right\}.$$

Note that $\|f\|_{WL_\Phi(\mathbb{R}_+,G)} \leq \|f\|_{L_\Phi(\mathbb{R}_+,G)}$ (see [13]).

We prove that

$$\int\limits_\Omega \Big(\frac{|f(chx)|}{\|f\|_{L_\Phi(\Omega,G)}}\Big) d\mu_\lambda(x) \leq 1, \qquad \sup_{t>0} \Phi(t) m\Big(\frac{f}{\|f\|_{WL_\Phi(\Omega_+,G)}}, t\Big) \leq 1, \tag{3.1}$$

and

$$\sup_{t>0} \Phi(t) m(\Omega, f, t) = \sup_{t>0} t m(\Omega, f, \Phi^{-1}(t)) = \sup_{t>0} t m\left(\Omega, \Phi(|f|), t\right), \tag{3.2}$$

where

$$\|f\|_{L_\Phi(\Omega,G)} = \|f\chi_\Omega\|_{L_\Phi(\mathbb{R}_+,G)}$$

and

$$\|f\|_{WL_\Phi(\Omega,G)} = \|f\chi_\Omega\|_{L_\Phi(\mathbb{R}_+,G)}.$$

Inequalities (3.1) immediately follow from Definitions 3.3 and 3.4.

Now, let's prove (3.2). It is known that $\Phi(\Phi^{-1}(t)) \approx \Phi^{-1}(\Phi(t)) \approx t$. Then by the definition, we have

$$\begin{aligned} m(\Omega, f, t) &= |\{x \in \Omega : |f(chx)| > t\}|_\lambda \\ &= |\{x \in \Omega : \Phi(|f|)(chx) > \Phi(t)\}|_\lambda \\ &= m\left(\Omega : \Phi(|f|), \Phi(t)\right). \end{aligned}$$

This implies that

$$\begin{aligned}\Phi(t)m(\Omega,f,t)&=\Phi(t)m(\Omega,(|f|),\Phi(t))\\&\approx\Phi(\Phi^{-1}(t))m(\Omega,(|f|),\Phi^{-1}(t))\\&\approx tm(\Omega,\Phi(|f|),t).\end{aligned}$$

On the other hand,

$$\Phi(t)m(\Omega,f,t)\approx\Phi(\Phi^{-1}(t))m(\Omega,f,\Phi^{-1}(t))\approx tm(\Omega,f,\Phi^{-1}(t)).$$

Thus we have

$$\sup_{t>0}\Phi(t)m(\Omega,f,t)=\sup_{t>0}tm(\Omega,f,\Phi^{-1}(t))=\sup_{t>0}tm(\Omega,\Phi\left|f\right|,t).$$

By the definition of $\widetilde{\Phi}$, we have

$$uv\le\Phi(u)+\widetilde{\Phi}(v),$$

which is called Young's inequality.

Corollary 3.5. *Let Φ and $\widetilde{\Phi}$ be complementary Young functions and let Ω be a measurable subset of $\mathbb{R}_+$. Suppose that $u\in L_\Phi$ and $v\in L_{\widetilde{\Phi}}$. Then $uv\in L_{1,\lambda}(\Omega,G)$ and*

$$\int\limits_\Omega|u(chx)v(chx)|\,d\mu_\lambda(x)\le\int\limits_\Omega\Phi\left(|u(chx)|\right)d\mu_\lambda(x)+\int\limits_\Omega\widetilde{\Phi}\left(|v(chx)|\right)d\mu_\lambda(x).$$

By the definition (see [13]),

$$\|f\|_{L^\Phi(\Omega,G)}=\sup\left\{\int\limits_\Omega|fg|\,d\mu_\lambda:\int\limits_\Omega\widetilde{\Phi}\left(|g|\right)d\mu_\lambda\le1\right\}.\tag{3.3}$$

Theorem 3.6. *Let Φ be a Young function and Ω be a measurable subset of $\mathbb{R}_+$.*
Then $L_\Phi(\Omega,G)=L^\Phi(\Omega,G)$ and for all $f\in L_\Phi(\Omega,G)$,

$$\|f\|_{L_\Phi(\Omega,G)}\le\|f\|_{L^\Phi(\Omega,G)}\le2\|f\|_{L_\Phi(\Omega,G)}.$$

Proof. Let $\widetilde{\Phi}$ be a Young function, complementary to Φ. Let $L_\Phi(\Omega,G)$, $f\neq0$, and let $k=\frac{1}{\|f\|_{L_\Phi(\Omega,G)}}$, so that

$$\int\limits_\Omega\Phi\left(k|f(chx)|\right)d\mu_\lambda(x)\le1.$$

Let g be such that $\int\limits_\Omega\widetilde{\Phi}\left(|g(chx)|\right)d\mu_\lambda(x)\le1$.

By Corollary 3.5,

$$\int\limits_\Omega|kf(chx)g(chx)|d\mu_\lambda(x)\le2.$$

Hence

$$\int\limits_\Omega|f(chx)g(chx)|d\mu_\lambda(x)\le2\|f\|_{L_\Phi(\Omega,G)}$$

and so,

$$\|f\|_{L^\Phi(\Omega,G)}\le2\|f\|_{L_\Phi(\Omega,G)}.$$

To obtain the remaining inequality (3.3) it suffices to prove it for non-negative simple functions, and as this is done, the use of the Fatou property possessed by the Banach function spaces $L^\Phi(\Omega,G)$ and $f\in L_\Phi(\Omega,G)$ gives the general result. Let f be a non-negative simple function with $\|f\|_{L^\Phi(\Omega,G)}>0$. All we have to do is to show that

$$\int\limits_\Omega\Phi(\nu|f(chx)|d\mu_\lambda(x)\le1,$$

where $\nu = \frac{1}{\|f\|_{L^{\Phi}(\Omega,G)}}$. Since f is simple, $\int\limits_{\Omega} \Phi(\nu|f(chx)|d\mu_\lambda(x) < \infty$. Now, represent Φ in the integral form: $\Phi(u(chx)) = \int\limits_0^{u(chx)} \varphi(s)ds$, say . The function g given by $g(chx) = \varphi(\nu f(chx))$ is simple. By Young's equality,

$$\widetilde{\Phi}(g(chx)) + \Phi(\nu f(chx)) = \nu f(chx) g(chx)$$

for all $x \in \Omega$. Hence

$$\int\limits_{\Omega} \Phi(\nu f(chx)) d\mu_\lambda(x) + \int\limits_{\Omega} \widetilde{\Phi}(g(chx)) d\mu_\lambda(x) = \int\limits_{\Omega} \nu f(chx) g(chx) d\mu_\lambda(x) \tag{3.4}$$

and so,

$$\int\limits_{\Omega} \widetilde{\Phi}(g(chx)) d\mu_\lambda(x) < \infty.$$

Since $L_{\widetilde{\Phi}}(\Omega, G)$ is the associate space of $L^{\Phi}(\Omega, G)$, Hölder's inequality (see Theorem 3.7 below and [13, Lemma 2.5])

$$\|f\|_{L_\Phi(\Omega,G)} \le 1 \Longleftrightarrow \int\limits_{\Omega} \Phi\left(|f(chx)|\right) d\mu_\lambda(x) \le 1$$

yields

$$\begin{aligned}\int\limits_{\Omega} (\nu f(chx) g(chx) d\mu_\lambda(x) &\le \|\nu f\|_{L^{\Phi}(\Omega,G)} \|g\|_{L_{\widetilde{\Phi}}(\Omega,G)} \\ &\le \|g\|_{L_{\widetilde{\Phi}}(\Omega,G)} \le \max\left\{1, \int\limits_{\Omega} \widetilde{\Phi}(g(chx)) d\mu_\lambda(x)\right\}.\end{aligned}$$

Thus from (3.4) we have

$$\int\limits_{\Omega} \Phi(\nu f(chx)) d\mu_\lambda(x) + \int\limits_{\Omega} \widetilde{\Phi}(g(chx) d\mu_\lambda(x) \le 1 + \int\limits_{\Omega} \widetilde{\Phi}(g(chx)) d\mu_\lambda(x),$$

and the result follows, since $\nu = \frac{1}{\|f\|_{L^{\Phi}(\Omega,G)}}$. □

Theorem 3.7 ([13])**.** *Let $\Omega \in \mathbb{R}_+$ be a measurable set and the functions f and g be its complementary $\widetilde{\Phi}$. Then the following inequality*

$$\int\limits_{\Omega} |f(chx) g(chx)| \, d\mu_\lambda(x) \le 2 \|f\|_{L_\Phi(\Omega,G)} \|g\|_{L_{\widetilde{\Phi}}(\Omega,G)}$$

is valid for all $f \in L_\Phi(\Omega, G)$ and $g \in L_{\widetilde{\Phi}}(\Omega, G)$.

Lemma 3.8. *Let Φ be a Young function and $H_r = (0, r)$ be a set in $\mathbb{R}_+$. Then*

$$\|\chi_{H_r}\|_{WL_\Phi(H_r,G)} = \|\chi_{H_r}\|_{L_\Phi(H_r,G)} = \frac{1}{\Phi^{-1}\left(|H_r|_\lambda^{-1}\right)}.$$

Proof. The description will be given in terms of the right-continuous inverse Φ^{-1} of Φ which is defined by

$$\Phi^{-1}(t) = \sup\left\{s \ge 0 : \Phi(s) \le t\right\}, \qquad 0 \le t < \infty.$$

By Definition 3.3 and (3.2), we have

$$\begin{aligned}\|\chi_{H_r}\|_{WL_\Phi(H_r,G)} &= \|\chi_{H_r}\|_{L_\Phi(H_r,G)} \\ &= \inf\left\{\frac{1}{\nu} : \int\limits_{H_r} \Phi\left(\nu \chi_{H_r}(chx)\right) d\mu_\lambda(x) \le 1\right\} \\ &= \inf\left\{\frac{1}{\nu} : |H_r|_\lambda \, \Phi(\nu) \le 1\right\}\end{aligned}$$

$$= \left[\sup \left\{ \nu : \Phi(\nu) \le \frac{1}{|H_r|_\lambda} \right\} \right]^{-1} = \frac{1}{\Phi^{-1}\left(\frac{1}{|H_r|_\lambda}\right)}. \qquad \square$$

Lemma 3.9. *If $0 < |E|_\lambda < \infty$ and Φ is a Young function, then the norm of the characteristic function, χ_E is given as*

$$\|\chi_E\|_{L^\Phi(E,G)} = |E|_\lambda \, \widetilde{\Phi}^{-1}\Big(\frac{1}{|E|_\lambda}\Big), \tag{3.5}$$

$\widetilde{\Phi}$ as usual being the complementary of Φ.

Proof. Indeed, let $g \in L_{\widetilde{\Phi}}(E,G)$ such that $\|g\|_{L_{\widetilde{\Phi}}(E,G)} \le 1$. Then by Jensen's inequality (see [13], Remark 2.7), we get

$$\widetilde{\Phi}\Big(\frac{1}{|E|_\lambda}\int_E |g(chx)|\, d\mu_\lambda(x)\Big)$$
$$\le \frac{\|g\|_{L_{\widetilde{\Phi}}(E,G)}}{|E|_\lambda}\int_E \Big(\frac{|g(chx)|}{\|g\|_{L_{\widetilde{\Phi}}(E,G)}}\Big) d\mu_\lambda(x) \le \frac{1}{|E|_\lambda}\,.$$

Applying $\widetilde{\Phi}^{-1}$ to both sides, we obtain

$$\int_E |g(chx)|\, d\mu_\lambda(x) \le |E|_\lambda \, \widetilde{\Phi}^{-1}\left(|E|_\lambda^{-1}\right).$$

Then by the definition, we have

$$\|\chi_E\|_{L^\Phi(E,G)} = \sup\left\{ \int_\Omega \chi_E(chx) g(chx) d\mu_\lambda(x) : \|g\|_{L_{\widetilde{\Phi}}(E,G)} \le 1 \right\}$$
$$\le \sup\left\{ \int_E |g(chx)|\, d\mu_\lambda(x) : \|g\|_{L_{\widetilde{\Phi}}(E,G)} \le 1 \right\}$$
$$\le |E|_\lambda \, \widetilde{\Phi}^{-1}\left(|E|_\lambda^{-1}\right). \tag{3.6}$$

On the other hand, if $g_0 = \widetilde{\Phi}^{-1}\left(|E|_\lambda^{-1}\right)\chi_E$, then

$$\|g_0\|_{L_{\widetilde{\Phi}}(E,G)} = 1 \;\text{ and so }\; \|\chi_E\|_{L^\Phi(E,G)}$$
$$\ge \int_\Omega \chi_E(chx) g_0(chx) d\mu_\lambda(x) = \widetilde{\Phi}^{-1}\left(|E|_\lambda^{-1}\right)\int_E d\mu_\lambda(x)$$
$$= |E|_\lambda \, \widetilde{\Phi}^{-1}\left(|E|_\lambda^{-1}\right). \tag{3.7}$$

From (3.6) and (3.7), we obtain (3.5). $\square$

Lemma 3.10. *For the complementary pair $\Phi, \widetilde{\Phi}$, the following statement*

$$u \le \Phi^{-1}(u)\widetilde{\Phi}^{-1}(u) \le 2u$$

is valid.

Proof. Let $u = |H_r|_\lambda^{-1}$. Then by Lemma 3.8 and Lemma 3.9, we can write

$$\|\chi_E\|_{L^\Phi(E,G)} = \frac{1}{u}\widetilde{\Phi}^{-1}(u), \qquad \|\chi_E\|_{L_\Phi(E,G)} = \frac{1}{\Phi^{-1}(u)}\,.$$

This implies that

$$\widetilde{\Phi}^{-1}(u) = u\,\|\chi_E\|_{L_\Phi(E,G)} \quad \text{and} \quad \Phi^{-1}(u) = \frac{1}{\|\chi_E\|_{L_\Phi(E,G)}}\,.$$

Then

$$\widetilde{\Phi}^{-1}(u)\Phi^{-1}(u) = u\,\|\chi_E\|_{L_\Phi(E,G)} \cdot \frac{1}{\|\chi_E\|_{L_\Phi(E,G)}}.$$

Using Theorem 3.6, we obtain our statement. □

From Theorem 3.7, Lemma 3.8 and Lemma 3.10, we have the following

Lemma 3.11. *For a Young function Φ and $H_r \subset \mathbb{R}_+$,the following inequality*

$$\int_{H_r} |f(chx)|\, d\mu_\lambda(x) \le 2\,|H_r|_\lambda\, \Phi^{-1}\left(|H_r|_\lambda^{-1}\right) \|f\|_{L_\Phi(H_r,G)}$$

is valid.

The G-fractional integral J_G^α is defined as follows:

$$J_G^\alpha f(chx) = \int_0^\infty \frac{A_{chy}^\lambda f(chx)}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} d\mu_\lambda(y), \qquad 0 < \alpha < \gamma.$$

Lemma 3.12. *Let $0 < \alpha < \gamma$. Then*

$$M_G^\alpha f(chx) \lesssim J_G^\alpha\left(|f|\right)(chx). \tag{3.8}$$

Proof. By the definition of J_G^α and Lemma 2.1, we get

$$\begin{aligned} J_G^\alpha(|f|)(chx) &= \int_0^\infty \frac{A_{chy}^\lambda |f(chx)|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} d\mu_\lambda(y) \\ &\ge \int_0^r \frac{A_{chy}^\lambda |f(chx)|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} d\mu_\lambda(y) \\ &\ge \frac{1}{\left(sh\frac{r}{2}\right)^{\gamma-\alpha}} \int_0^r A_{chy}^\lambda |f(chx)|\, d\mu_\lambda(y) \\ &\approx \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int_0^r A_{chy}^\lambda |f(chx)|\, d\mu_\lambda(y). \end{aligned}$$

Taking the supremum on $r > 0$ of both parts, we obtain our statement. □

4. The Boundedness of G-Fractional Maximal Operator in G-Orlicz Space

In this section, we give the necessary and sufficient condition for the boundedness of G-fractional maximal operator M_G^α on the G-Orlicz space and weak G-Orlicz space. We begin with the boundedness of the G-maximal operator M_G on G-Orlicz spaces, defined by (2.2) for $\alpha = 0$,

$$M_G f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda} \int_{H_r} A_{chy}^\lambda |f(chx)|\, d\mu_\lambda(y).$$

We introduce the following

Definition 4.1. Let $\Phi \in L_\Phi(\mathbb{R}_+, G)$ and $T : L_\Phi \to L_\Phi$ be a sublinear operator.

We say that the operator T is of the weak type (Φ, Φ) if

$$\Phi(\alpha)\,|\{x \in \mathbb{R}_+ : |Tf(chx)| > \alpha\}|_\lambda \le \int_{\mathbb{R}_+} \Phi\left(|f(chx)|\right) d\mu_\lambda(x),$$

for all $f \in L_\Phi(\mathbb{R}_+, G)$ and all $\alpha > 0$.

Definition 4.2. Let $\Phi \in L_\Phi(\mathbb{R}_+, G)$ and $T : L_\Phi \to L_\Phi$ be a sublinear operator. We say that the operator T is of the strong type (Φ, Φ) if there exists a constant $c > 0$ such that

$$\|Tf\|_{L_\Phi(\mathbb{R}_+,G)} \le c\,\|f\|_{L_\Phi(\mathbb{R}_+,G)}\,,$$

for all $f \in L_\Phi(\mathbb{R}_+, G)$.

Lemma 4.3 ([14])**.** *Let Φ be a Young function with the canonical representation*

$$\Phi(r) = \int_0^r \varphi(s)ds, \quad r > 0.$$

(1) *Assume that $\Phi \in \Delta_2$. More precisely, $\Phi(2r) \le c\Phi(r)$ for some $c \ge 2$. Set $\beta = \log_2 c$. If $p > \beta + 1$, then the following inequality*

$$\int_r^\infty \frac{\varphi(s)}{s^p} ds \lesssim \frac{\Phi(r)}{r^p}, \quad r > 0$$

is valid.

(2) *Assume that $\Phi \in \nabla_2$. Then the following inequality*

$$\int_0^r \frac{\varphi(s)}{s} ds \lesssim \frac{\Phi(r)}{r}, \quad r > 0$$

is valid.

Lemma 4.4 ([13])**.** *Let $H_r = (0, r) \subset \mathbb{R}_+$ and let $\Phi : H_r \to \mathbb{R}_+$ be convex and we suppose that $f \in L_{1,\lambda}(\mathbb{R}_+)$ and also $\Phi(A^\lambda_{chy} f) \in L_{1,\lambda}(\mathbb{R}_+)$.*

Then Jensen's type inequality

$$\Phi\left(\frac{1}{|H_r|_\lambda}\int_{H_r} A^\lambda_{chy} f(chx) d\mu_\lambda(y)\right) \le \frac{1}{|H_r|_\lambda}\int_{H_r} \Phi\left(A^\lambda_{chy} f(chx)\right) d\mu_\lambda(y)$$

holds for every $x \in \mathbb{R}_+$.

Theorem 4.5. *Let Φ be a Young function. Then the maximal operator M_G is bounded from $L_\Phi(\mathbb{R}_+, G)$ to $WL_\Phi(\mathbb{R}_+, G)$ and for $\Phi \in \nabla_2$ is bounded in $L_\Phi(\mathbb{R}_+, G)$.*

Proof. First, we prove that the maximal operator M_G is bounded from $L_\Phi(\mathbb{R}_+, G)$ to $WL_\Phi(\mathbb{R}_+, G)$. We take $f \in L_\Phi(\mathbb{R}_+, G)$ satisfying $\|f\|_{L_\Phi(\mathbb{R}_+,G)} = 1$, so that the modular

$$\int_{\mathbb{R}_+} \Phi\left(|f(chx)|\right) d\mu_\lambda(x) \le 1.$$

By Lemma 4.4 and the definition of the maximal operator M_G, we have

$$\Phi(M_G f(chx)) \le M_G \Phi(f(chx)). \tag{4.1}$$

Using (4.1) and weak (1,1) boundedness of the maximal operator (see [14]), we get

$$\begin{aligned} |\{x \in \mathbb{R}_+ : M_G f(chx) > r\}|_\lambda &= |\{x \in \mathbb{R}_+ : \Phi(M_G f(chx)) > \Phi(r)\}|_\lambda \\ &\le |\{x \in \mathbb{R}_+ : M_G \Phi(f(chx)) > \Phi(r)\}|_\lambda \\ &\le \frac{c}{\Phi(r)} \int_{\mathbb{R}_+} \Phi\left(|f(chx)|\right) d\mu_\lambda(x) \\ &\le \frac{c}{\Phi(r)} \le \frac{1}{\Phi\left(\frac{r}{c\|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right)}\,, \end{aligned}$$

since $\|f\|_{L_\Phi(\mathbb{R}_+,G)} = 1$ and $\frac{1}{c}\Phi(r) > \Phi(\frac{r}{c})$, if $c \geq 1$. Since the $\|\cdot\|_{L_\Phi(\mathbb{R}_+,G)}$ norm is homogeneous, the inequality

$$|\{x \in \mathbb{R}_+ : M_G f(chx) > r)\}|_\lambda \leq \frac{1}{\Phi\Big(\frac{r}{c\|f\|_{L_\Phi(\mathbb{R}_+,G)}}\Big)}$$

is true for every $f \in L_\Phi(\mathbb{R}_+, G)$.

Now, we prove that for $\Phi \in \nabla_2$, the maximal operator M_G is bounded in $L_\Phi(\mathbb{R}_+, G)$. Let $\theta > 0$ and $f \in L_\Phi(\mathbb{R}_+, G)$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}_+} \Phi\Big(\frac{M_G f(chx)}{\theta}\Big) d\mu_\lambda(x) &= \int_{\mathbb{R}_+} \int_0^{\frac{M_G f(chx)}{\theta}} \varphi(s) ds d\mu_\lambda(x) \\
&= \int_{\mathbb{R}_+} \int_0^\infty \chi_{\left\{s\in(0,\infty):\frac{M_G f(chx)}{\theta}>s\right\}} \varphi(s) ds d\mu_\lambda(x) \\
&= \int_0^\infty \varphi(s) \Big(\int_{\mathbb{R}_+} \chi_{\{x\in\mathbb{R}_+:M_G f(chx)>\theta s\}} d\mu_\lambda(x) \Big) ds \\
&= \frac{1}{\theta} \int_0^\infty \varphi\Big(\frac{\nu}{\theta}\Big) |\{x \in \mathbb{R}_+ : M_G f(chx) > \nu\}| \, d\nu \\
&= \frac{2}{\theta} \int_0^\infty \varphi\Big(\frac{2\nu}{\theta}\Big) |\{x \in \mathbb{R}_+ : M_G f(chx) > 2\nu\}| \, d\nu.
\end{aligned}$$

By weak (1,1) boundedness of the maximal operator,

$$|\{x \in \mathbb{R}_+ : M_G f(chx) > 2\nu\}|_\lambda \lesssim \frac{1}{\nu} \int_{\{x\in\mathbb{R}_+:|f(chx)|>\nu\}} f(chx) d\mu_\lambda(x)$$

and the change of the order of integration results in

$$\begin{aligned}
\int_{\mathbb{R}_+} \Phi\left(\frac{M_G f(chx)}{\theta}\right) d\mu_\lambda(x) &\lesssim \frac{1}{\nu} \int_{\mathbb{R}_+} \varphi\Big(\frac{2\nu}{\theta}\Big) \Big(\int_{\{x\in\mathbb{R}_+:|f(chx)|>\nu\}} f(chx) d\mu_\lambda(x) \Big) \frac{d\nu}{\nu} \\
&\lesssim \frac{1}{\theta} \int_{\mathbb{R}_+} |f(chx)| \Big(\int_0^{|f(chx)|} \Big(\frac{2\nu}{\theta}\Big) \frac{d\nu}{\nu} \Big) d\mu_\lambda(x).
\end{aligned}$$

Now. we use Theorem 3.7 which yields

$$\int_0^{2\theta^{-1}|f(chx)|} \varphi(\nu) \frac{d\nu}{\nu} \lesssim |f(chx)|^{-1} \theta \Phi\left(\frac{2|f(chx)|}{\theta}\right),$$

if $f(chx) \neq 0$. Recall that $k\Phi(r) \leq \Phi(kr)$ for $k \geq 1$ and $r > 0$, assuming Φ is convex.

Therefore it follows that

$$\begin{aligned}
\int_{\mathbb{R}_+} \Phi\left(\frac{M_G f(chx)}{\theta}\right) d\mu_\lambda(x) &\leq c_0 \int_{\mathbb{R}_+} \Phi\left(\frac{2|f(chx)|}{\theta}\right) d\mu_\lambda(x) \\
&\leq \int_{\mathbb{R}_+} \Phi\left(\frac{c_0|f(chx)|}{\theta}\right) d\mu_\lambda(x).
\end{aligned}$$

Here, c_0 is a constant we would like to ched lihgt on. Chosing $\theta = c_0 \, \|f\|_{L_\Phi(\mathbb{R}_+,G)}$, we get

$$\int\limits_{\mathbb{R}_+} \Phi\left(\frac{M_G f(chx)}{\theta}\right) d\mu_\lambda(x) \le 1.$$

This means that

$$\|M_G f\|_{L_\Phi(\mathbb{R}_+,G)} \le \theta = c_0 \, \|f\|_{L_\Phi(\mathbb{R}_+,G)}$$

from the definition of the norm. □

We recall that for the functions Φ and Ψ from $[0,\infty)$ into $[0,\infty]$, the function Ψ is said to dominate Φ globally if there exists a positive constant c such that $\Phi(s) \le \Psi(cs)$ for all $s \ge 0$.

In the theorem below we also use their notation

$$\widetilde{\Psi}_p(s) = \int\limits_0^s r^{p'-1} \left(B_p^{-1}\left(r^{p'}\right)\right)^{p'} dr, \tag{4.2}$$

where $1 < p \le \infty$ and $\widetilde{\Psi}$ is the Young conjugate function to $\Psi_p(s)$, where $B_p^{-1}(s)$ is inverse to

$$B_p(s) = \int\limits_0^s \frac{\Psi(t)}{t^{1+p'}} dt.$$

In [13], we have found the necessary and sufficient condition for the boundedness of M_G^α on the G-Orlicz spaces.

Theorem 4.6 ([13]). *Let $0 < \alpha < \gamma$ and let Φ and Ψ be Young functions. Then:*

(i) *M_G^α is bounded from $L_\Phi(\mathbb{R}_+,G)$ to $WL_\Psi(\mathbb{R}_+,G)$ if and only if*

$$\Phi \text{ dominates globally the function } \Psi_{\gamma/\alpha}$$

whose inverse is given by

$$\Psi_{\gamma/\alpha}^{-1}(r) = r^{\alpha/\gamma}\Psi^{-1}(r);$$

(ii) *M_G^α is bounded from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$ if and only if*

$$\int\limits_0^1 \frac{\Psi(t)}{t^{1+\gamma/(\gamma-\alpha)}} dt < \infty \text{ and } \Phi \text{ dominates } \Psi_{\gamma/\alpha} \text{ globally.}$$

Here, $\Psi_{\gamma/\alpha}$ is the Young function defined in (4.2).

In order to prove our main theorem, we also need the following

Lemma 4.7. *Let $H_0 = H(0,r_0) \subset H_r(0,r)$. Then*

$$|H_0|_\lambda^{\frac{\alpha}{\gamma}} \le M_G^\alpha \chi_{H_0}(chx) \quad \text{for every} \quad x \in \mathbb{R}_+.$$

Proof. For $x \in H_0$, we get

$$\begin{aligned} M_G^\alpha \chi_{H_0}(chx) &= \sup_{x\in H_0} |H_0|_\lambda^{\frac{\alpha}{\gamma}} |H_r \cap H_0|_\lambda \\ &\ge |H_0|_\lambda^{-1+\frac{\alpha}{\gamma}} |H_0|_\lambda = |H_0|_\lambda^{\frac{\alpha}{\gamma}} . \end{aligned}$$

□

Theorem 4.8. *Let $0 < \alpha < \gamma, \Phi, \Psi$ be Young functions and $\Phi \in Y$. The condition*

$$|H_r|_\lambda^{-\frac{\alpha}{\gamma}} \Phi^{-1}\left(|H_r|_\lambda\right) \le c\Psi^{-1}\left(|H_r|_\lambda\right) \tag{4.3}$$

for all $r > 0$, where $c > 0$ does not depend on r is necessary and sufficient for the boundednes of M_G^α from $L_\Phi(\mathbb{R}_+,G)$ to $WL_\Psi(\mathbb{R}_+,G)$.

Moreover, if $\Phi \in \nabla_2$, condition (4.3) *is necessary and sufficient for the boundedness of M_G^α from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$.*

Proof. For an arbitrary interval H_r we represent f as

$$f = f_1 + f_2, \quad f_1 = f\chi_{H_r}, \quad f_2 = f\chi_{(H_r)^c} = f\chi_{[r,\infty)},$$

and have

$$M_G f(chx) = M_G f_1(chx) + M_G f_2(chx).$$

If $H_s \cap (H_r)^c \neq \varnothing$, then $s > r$, and by Lemmas 2.1, 2.2 and 3.9, we obtain

$$\begin{aligned}
M_G f_2(chx) &= \sup_{s>0} |H_s|_\lambda^{-1+\frac{\alpha}{\gamma}} \int\limits_{H_s\cap(H_r)^c} A_{chy}^\lambda |f_2(chx)|\, d\mu_\lambda(y) \\
&\lesssim \sup_{s>r} |H_s|_\lambda^{-1+\frac{\alpha}{\gamma}} \int\limits_r^s A_{chy}^\lambda |f(chx)|\, d\mu_\lambda(y) \\
&\lesssim \sup_{s>r} |H_s|_\lambda^{-1+\frac{\alpha}{\gamma}} \int\limits_0^s A_{chy}^\lambda |f(chx)|\, d\mu_\lambda(y) \\
&\lesssim \|f\|_{L_\Phi(\mathbb{R}_+,G)} \sup_{s>r} |H_s|_\lambda^{\frac{\alpha}{\gamma}} \, \Phi^{-1}\left(|H_s|_\lambda^{-1}\right) \\
&\lesssim \|f\|_{L_\Phi(\mathbb{R}_+,G)} \sup_{s>r} \left(sh\frac{s}{2}\right)^\alpha \Phi^{-1}\left(|H_s|_\lambda^{-1}\right). \qquad (4.4)
\end{aligned}$$

To estimate $M_G f_1$, we use inequality (3.8) and also $sh(at) \leq asht$ for $0 \leq a \leq 1$.

$$\begin{aligned}
M_G^\alpha f_1(chx) &\lesssim J_G^\alpha\left(|f\chi_{H_r}|\right)(chx) \lesssim \int\limits_0^r \frac{A_{chy}^\lambda |f(chx)|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} d\mu_\lambda(y) \\
&\lesssim \sum_{k=1}^\infty \int\limits_{\frac{r}{2^{k+1}}}^{\frac{r}{2^k}} \frac{A_{chy}^\lambda |f(chx)|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} d\mu_\lambda(y) \\
&\lesssim \sum_{k=0}^\infty \left(sh\frac{r}{2^{k+1}}\right)^\alpha \left(sh\frac{r}{2^{k+1}}\right)^{-\gamma} \int\limits_0^{\frac{r}{2^k}} A_{chy}^\lambda |f(chx)|\, d\mu_\lambda(y) \\
&\lesssim \left(sh\frac{r}{2}\right)^\alpha M_G f(chx) \sum_{k=0}^\infty 2^{-k\alpha} \\
&\lesssim |H_r|_\lambda^{\frac{\alpha}{\gamma}} \, M_G f(chx). \qquad (4.5)
\end{aligned}$$

From (4.4) and (4.5), we obtain

$$M_G f(chx) \lesssim |H_r|_\lambda^{\frac{\alpha}{\gamma}} \, M_G f(chx) + \|f\|_{L_\Phi(\mathbb{R}_+,G)} \sup_{s>r} |H_s|_\lambda^{\frac{\alpha}{\gamma}} \, \Phi^{-1}\left(|H_s|_\lambda^{-1}\right).$$

Thus by (4.3), we have

$$M_G^\alpha f(chx) \lesssim M_G f(chx) \frac{\Psi^{-1}\left(|H_r|_\lambda^{-\gamma}\right)}{\Phi^{-1}\left(|H_r|_\lambda^{-\gamma}\right)} + \Psi^{-1}\left(|H_r|_\lambda^{-\gamma}\right) \|f\|_{L_\Phi(\mathbb{R}_+,G)}.$$

Choose

$$\Phi^{-1}\left(|H_r|_\lambda^{-\gamma}\right) = \frac{M_G f(chx)}{c_0 \|f\|_{L_\Phi(\mathbb{R}_+,G)}}.$$

Then

$$\frac{\Psi^{-1}\left(|H_r|_\lambda^{-\gamma}\right)}{\Phi^{-1}\left(|H_r|_\lambda^{-\gamma}\right)} = \frac{(\Psi_0^{-1}\Phi)\left(\frac{M_G f(chx)}{c_0\|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right)}{\frac{M_G f(chx)}{c_0\|f\|_{L_\Phi(\mathbb{R}_+,G)}}}.$$

Therefore we get

$$M_G^\alpha f(chx) \le c_1 \|f\|_{L_\Phi(\mathbb{R}_+,G)} (\Psi_0^{-1}\Phi) \left(\frac{M_G f(chx)}{c_0 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \right),$$

whence it follows that

$$\frac{M_G^\alpha f(chx)}{c_1 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \le (\Psi_0^{-1}\Phi) \left(\frac{M_G f(chx)}{c_0 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \right)$$

and

$$\Psi \left(\frac{M_G^\alpha f(chx)}{c_1 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \right) \le \Phi \left(\frac{M_G f(chx)}{c_0 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \right).$$

Then by (3.1) and Theorem 4.5, we have

$$\begin{aligned}
\|M_G^\alpha f\|_{WL_\Psi(\mathbb{R}_+,G)} &= \sup_{r>0} \Psi(r) m \left(\frac{M_G^\alpha f(chx)}{c_1 \|f\|_{L_\Phi(\mathbb{R}_+,G)}}, r \right) \\
&= \sup_{r>0} rm \left(\Psi \left(\frac{M_G^\alpha f(chx)}{c_1 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \right), r \right) \\
&\le \sup_{r>0} rm \left(\Phi \left(\frac{M_G f(chx)}{c_0 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \right), r \right) \\
&= \sup_{r>0} \Phi(r) m \left(\frac{M_G f(chx)}{c_0 \|M_G f\|_{WL_\Phi(\mathbb{R}_+,G)}}, r \right) \le 1.
\end{aligned}$$

Thus

$$\|M_G^\alpha f\|_{WL_\Psi(\mathbb{R}_+,G)} \le \|M_G f\|_{WL_\Phi(\mathbb{R}_+,G)} \lesssim \|M_G f\|_{L_\Phi(\mathbb{R}_+,G)}.$$

Since $\Phi \in \nabla_2$, by Theorem 4.5, we get

$$\begin{aligned}
\int_{\mathbb{R}_+} \Psi \left(\frac{M_G f(chx)}{c_1 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \right) d\mu_\lambda(x) &\le \int_{\mathbb{R}_+} \Phi \left(\frac{M_G f(chx)}{c_0 \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \right) d\mu_\lambda(x) \\
&\le \int_{\mathbb{R}_+} \Phi \left(\frac{M_G f(chx)}{c_0 \|M_G f\|_{L_\Phi(\mathbb{R}_+,G)}} \right) d\mu_\lambda(x) \le 1
\end{aligned}$$

i.e.,

$$\|M_G^\alpha f\|_{L_\Psi(\mathbb{R}_+,G)} \lesssim \|f\|_{L_\Phi(\mathbb{R}_+,G)}.$$

We now prove the necessity. Let $H_0 = (0, r_0)$ and $x \in H_0$. By Lemma 4.7 and (2.1), we have $|H_0|_\lambda^{\frac{\alpha}{\gamma}} \approx \left(sh\frac{r_0}{2}\right)^\alpha \lesssim M_G^\alpha \chi_{H_0}(chx)$. Therefore by Lemma 3.8 and (2.1) , we obtain

$$\begin{aligned}
\left(sh\frac{r_0}{2}\right)^\alpha &\lesssim \Psi^{-1}\left(|H_0|_\lambda^{-1}\right) \|M_G^\alpha \chi_{H_0}\|_{WL_\Psi(H_0,G)} \\
&\lesssim \Psi^{-1}\left(|H_0|_\lambda^{-1}\right) \|M_G^\alpha \chi_{H_0}\|_{WL_\Psi(\mathbb{R}_+,G)} \\
&\lesssim \Psi^{-1}\left(|H_0|_\lambda^{-1}\right) \|\chi_{H_0}\|_{L_\Phi(\mathbb{R}_+,G)} \\
&= \frac{\Psi^{-1}\left(|H_0|_\lambda^{-1}\right)}{\Phi^{-1}\left(|H_0|_\lambda^{-1}\right)} \lesssim \frac{\Psi^{-1}\left(\left(sh\frac{r_0}{2}\right)^{-\gamma}\right)}{\Phi^{-1}\left(\left(sh\frac{r_0}{2}\right)^{-\gamma}\right)},
\end{aligned}$$

and

$$\begin{aligned}
\left(sh\frac{r_0}{2}\right)^\alpha &\lesssim \Psi^{-1}\left(|H_0|_\lambda^{-1}\right) \|M_G^\alpha \chi_{H_0}\|_{L_\Psi(H_0,G)} \\
&\lesssim \Psi^{-1}\left(|H_0|_\lambda^{-1}\right) \|M_G^\alpha \chi_{H_0}\|_{L_\Psi(\mathbb{R}_+,G)}
\end{aligned}$$

$$\lesssim \Psi^{-1}\left(|H_0|_\lambda^{-1}\right)\|\chi_{H_0}\|_{L_\Psi(\mathbb{R}_+,G)}$$
$$\lesssim \frac{\Psi^{-1}\left(\left(sh\frac{r_0}{2}\right)^{-\gamma}\right)}{\Phi^{-1}\left(\left(sh\frac{r_0}{2}\right)^{-\gamma}\right)}.$$

Since this is true for every $r_0 > 0$, we are done. □

5. G-Maximal Commutator in G-Orlicz Spaces

In this section, we investigate the boundedness of the G-maximal commutator $M_G^{b,\alpha}$ in G-Orlicz spaces.

We recall the definition of the $G-BMO$ space.

Definition 5.1 ([13])**.** We denote by $BMO_G(\mathbb{R}_+)$ the Gegenbauer-BMO (G-BMO) space as the set of locally integrable functions on $\mathbb{R}_+$ such that

$$\|f\|_G^* = \sup_{x,r\in\mathbb{R}_+}\frac{1}{|H_r|_\lambda}\int\limits_{H_r}|A_{chy}^\lambda f(chx) - f_{H_r}(chx)|d\mu_\lambda(y),$$

where $H_r = (0, r)$,

$$f_{H_r}(chx) = \frac{1}{|H_r|_\lambda}\int\limits_{H_r}A_{chy}^\lambda f(chx)d\mu_\lambda(y)$$

and the set

$$BMO_G(\mathbb{R}_+) = \left\{f \in L_{1,\lambda}^{\text{loc}}(\mathbb{R}_+)\backslash\{\text{const}\} : \|f\|_G^* < \infty\right\}.$$

Before proving our theorems, we need the following lemmas and theorem.

Lemma 5.2. *Let $f \in BMO_G(\mathbb{R}_+)$ and Φ be a Young function. Then*

$$\|f\|_G^* \approx \sup_{x,r\in\mathbb{R}_+}\Phi^{-1}\left(|H_r|_\lambda^{-1}\right)\left\|A_{chy}^\lambda f - f_{H_r}\right\|_{L_\Phi(H_r,G)}.$$

Proof. According to Lemma 3.11, we have

$$\|f\|_G^* = \sup_{x,r\in\mathbb{R}_+}\frac{1}{|H_r|_\lambda}\int\limits_{H_r}|A_{chy}^\lambda f(chx) - f_{H_r}(chx)|d\mu_\lambda(y)$$
$$\leq \sup_{x,r\in\mathbb{R}_+}\frac{1}{|H_r|_\lambda}2|H_r|\Phi^{-1}\left(|H_r|_\lambda^{-1}\right)\left\|A_{chy}^\lambda f - f_{H_r}\right\|_{L_\Phi(H_r,G)}$$
$$\lesssim \sup_{x,r\in\mathbb{R}_+}\Phi^{-1}\left(|H_r|_\lambda^{-1}\right)\left\|A_{chy}^\lambda f - f_{H_r}\right\|_{L_\Phi(H_r,G)}.$$

For the reverse inequality, we use equality (3.3).

Let $f \in BMO_G$ and $g(chy) = \frac{\chi_{H_r}(chy)}{\|\chi_{H_r}\|_{L_{\widetilde{\Phi}}(\mathbb{R}_+,G)}}$, thus $\|g\|_{L_{\widetilde{\Phi}}(\mathbb{R}_+,G)} \leq 1$, and we have

$$\left\|A_{chy}^\lambda f - f_{H_r}\right\|_{L_\Phi(H_r,G)} \lesssim \left|\int\limits_{H_r}\frac{\chi_{H_r}(chy)}{\|\chi_{H_r}\|_{L_{\widetilde{\Phi}}(\mathbb{R}_+,G)}}(A_{chy}^\lambda f(chx) - f_{H_r}(chx))d\mu_\lambda(y)\right|$$
$$\lesssim \frac{1}{\|\chi_{H_r}\|_{L_{\widetilde{\Phi}}(\mathbb{R}_+,G)}}\int\limits_{H_r}|A_{chy}^\lambda f(chx) - f_{H_r}(chx)|d\mu_\lambda(y)$$
$$\lesssim \frac{|H_r|_\lambda\,\|f\|_G^*}{\|\chi_{H_r}\|_{L_{\widetilde{\Phi}}(\mathbb{R}_+,G)}}.$$

Using the equality (see [2, Chapter 2, Theorem 5.2])

$$\|\chi_{H_r}\|_{L_\Phi(\mathbb{R}_+,G)}\|\chi_{H_r}\|_{L_{\widetilde{\Phi}}(\mathbb{R}_+,G)} = |H_r|_\lambda,$$

we get

$$\left\|A^\lambda_{chy} f - f_{H_r}\right\|_{L_\Phi(H_r,G)} \lesssim \frac{\|f\|^*_G}{\Phi^{-1}\left(|H_r|^{-1}_\lambda\right)}. \qquad \square$$

Lemma 5.3. *Let $b \in BMO_G(\mathbb{R}_+)$ and $\Phi \in \nabla_2 \cap Y$. Then the operator M^b_G is bounded on $L_\Phi(\mathbb{R}_+, G)$ and the inequality*

$$\left\|M^b_G f\right\|_{L_\Phi(\mathbb{R}_+,G)} \lesssim \|b\|^*_G \, \|f\|_{L_\Phi(\mathbb{R}_+,G)}$$

holds for all $f \in L_\Phi(\mathbb{R}_+, G)$.

Proof. From (2.3), we get

$$M^b_G f(chx) = \sup_{r>0} \frac{1}{|H_r|_\lambda} \int\limits_{H_r} |A^\lambda_{chy} b(chx) - b_{H_r}(chx)| A^\lambda_{chy} |f(chx)| d\mu_\lambda(y).$$

For $\delta > 0$ and $f \in L^{\rm loc}_\delta(\mathbb{R}_+)$, denote by

$$M_{G,\delta} f(chx) = \sup_{r>0} \left(\frac{1}{|H_r|_\lambda} \int\limits_{H_r} |A^\lambda_{chy} f(chx)|^\delta d\mu_\lambda(y) \right)^{\frac{1}{\delta}}.$$

Let $\delta < \varepsilon < 1$. We use Hölder's inequality with the exponents r and r', where $r = \frac{\varepsilon}{\delta} > 1$, and also the relation

$$\sup_{x,r\in\mathbb{R}_+} \left(\frac{1}{|H_r|_\lambda} \int\limits_{H_r} \left|A^\lambda_{chy} f(chx) - f_{H_r}(chx)\right|^p d\mu_\lambda(y) \right)^{\frac{1}{p}} \approx \|f\|^*_G, \qquad 1 < p < \infty.$$

$$\left(\frac{1}{|H_r|_\lambda} \int\limits_{H_r} \left|A^\lambda_{chy} b(chx) - b_{H_r}(chx)\right| A^\lambda_{chy} |f(chx)| d\mu_\lambda(y) \right)$$

$$\le \left(\frac{1}{|H_r|_\lambda} \int\limits_{H_r} \left|A^\lambda_{chy} b(chx) - b_{H_r}(chx)\right|^{\delta r'} d\mu_\lambda(y) \right)^{\frac{1}{\delta r'}}$$

$$\times \left(\frac{1}{|H_r|_\lambda} \int\limits_{H_r} A^\lambda_{chy} |f(chx)|^{\delta r} d\mu_\lambda(y) \right)^{\frac{1}{\delta r}}$$

$$\lesssim \|b\|^*_G \, M_{G,\varepsilon} f(chx) \lesssim \|b\|^*_G \, M_G f(chx),$$

since by Hölder's inequality, for $0 < \varepsilon < 1$, $M_{G,\varepsilon} f(chx) \le M_G f(chx)$.

Thus we have

$$M^b_G f(chx) \lesssim \|b\|^*_G \, M_G f(chx),$$

whence we obtain

$$\left\|M^b_G f\right\|_{L_\Phi(R_+,G)} \lesssim \|b\|^*_G \, \|M_G f\|_{L_\Phi(R_+,G)} \lesssim \|b\|^*_G \, \|f\|_{L_\Phi(\mathbb{R}_+,G)}. \qquad \square$$

Lemma 5.4. *If $b \in L^{\rm loc}_{1,\lambda}(\mathbb{R}_+)$, $H_{r_0} = (0, r_0)$, then*

$$\left(sh\frac{r_0}{2}\right)^\alpha \left|b_{H_r}(chx) - b_{H_{r_0}}(chx)\right| \lesssim M^{b,\alpha}_G \chi_{H_{r_0}}(chx),$$

for every $x \in \mathbb{R}_+$.

Proof. We choose c_0 so large that the inequality $r \le c_0 r_0$ would hold. When $0 \le t \le c_0$, the inequality $t \le sht \le e^{c_0} t$ holds and according to (2.1), we have

$$|H_r|_\lambda^{1-\frac{\alpha}{\gamma}} = \left(\int\limits_0^r sh^{2\lambda}\, t dt\right)^{1-\frac{\alpha}{\gamma}} \le \left(\int\limits_0^{c_0 r_0} sh^{2\lambda}\, t dt\right)^{1-\frac{\alpha}{\gamma}} \approx \left(sh\frac{c_0 r_0}{2}\right)^{\gamma-\alpha}$$

$$\lesssim \left(e^{c_0}\frac{c_0 r_0}{2}\right)^{\gamma-\alpha} \lesssim (c_0 e^{c_0})^{\gamma-\alpha} \lesssim |H_{r_0}|_\lambda^{1-\frac{\alpha}{\gamma}}.$$

Then $\sup\limits_{r>0} |H_r|_\lambda^{-1+\frac{\alpha}{\gamma}} \gtrsim |H_{r_0}|_\lambda^{-1+\frac{\alpha}{\gamma}}$ and by (2.3) and (2.1), we have

$$\begin{aligned}
M_G^{b,\alpha}\chi_{H_{r_0}}(chx) &= \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_{H_r} \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| A_{chy}^\lambda \chi_{H_{r_0}}(chx) d\mu_\lambda(y) \\
&= \sup_{r>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_{H_r\cap H_{r_0}} \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| d\mu_\lambda(y) \\
&\gtrsim \frac{1}{|H_{r_0}|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_{H_{r_0}\cap H_{r_0}} \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| d\mu_\lambda(y) \\
&\geq \left|\frac{1}{|H_{r_0}|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_{H_{r_0}} \left(A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right) d\mu_\lambda(y)\right| \\
&= \left|\frac{|H_{r_0}|_\lambda^{\frac{\alpha}{\gamma}}}{|H_{r_0}|_\lambda} \int\limits_{H_{r_0}} \left(A_{chy}^\lambda b(chx) d\mu_\lambda(y) - b_{H_r}(chx)\, |H_{r_0}|^{\frac{\alpha}{\gamma}}\right) d\mu_\lambda(y)\right| \\
&= |H_{r_0}|_\lambda^{\frac{\alpha}{\gamma}} \left|b_{H_r}(chx) - b_{H_{r_0}}(chx)\right| \\
&\approx \left(sh\frac{r_0}{2}\right)^\alpha \left|b_{H_r}(chx) - b_{H_{r_0}}(chx)\right|.
\end{aligned}$$

□

Lemma 5.5. *Let* $f \in L_\Phi(\mathbb{R}_+, G)$*, then the inequality*

$$\left\|A_{cht}^\lambda f\right\|_{L_\Phi(\mathbb{R}_+,G)} \leq \|f\|_{L_\Phi(\mathbb{R}_+,G)}$$

is true for every $t \in (0, \infty)$.

Proof. By the definition of A_t^λ, we have

$$\int\limits_1^\infty \Phi\biggl(\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} \int\limits_0^\pi f\left(xt - \sqrt{x^2-1}\sqrt{t^2-1}\cos\varphi\right)(\sin\varphi)^{2\lambda-1} d\varphi\biggr)(x^2-1)^{\lambda-\frac{1}{2}} dx.$$

Taking the substitution (see [7, proof of Lemma 2])

$$z = xt - \sqrt{x^2-1}\sqrt{t^2-1}\cos\varphi,$$

we will have

$$J = \int\limits_1^\infty \Phi\biggl(\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} (t^{2-1})^{\frac{1}{2}-\lambda} \int\limits_{xt-\sqrt{x^2-1}\sqrt{t^2-1}}^{xt+\sqrt{x^2-1}\sqrt{t^2-1}} \left(1 - x^2 - t^2 - z^2 + 2xtz\right)^{\lambda-1}\biggr) dx.$$

Since

$$\begin{aligned}
xt - \sqrt{x^2-1}\sqrt{t^2-1} \leq z \leq xt + \sqrt{x^2-1}\sqrt{t^2-1} \\
\Longleftrightarrow tz - \sqrt{t^2-1}\sqrt{z^2-1} \leq x \leq tz + \sqrt{t^2-1}\sqrt{z^2-1},
\end{aligned}$$

we obtain

$$J = \int\limits_1^\infty \Phi\biggl(\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)\Gamma\left(\frac{1}{2}\right)} (t^2-1)^{\frac{1}{2}-\lambda} f(z) \int\limits_{tz-\sqrt{x^2-1}\sqrt{t^2-1}}^{tz+\sqrt{x^2-1}\sqrt{t^2-1}} \left(1 - x^2 - t^2 - z^2 + 2xtz\right)^{\lambda-1} dx\biggr) dz.$$

Taking into account

$$1 - x^2 - t^2 - z^2 + 2xtz = \left(tz - \sqrt{t^2-1}\sqrt{z^2-1} - x\right)\left(x - tz + \sqrt{t^2-1}\sqrt{z^2-1}\right)$$

and in view of $x = tz - \sqrt{t^2-1}\sqrt{z^2-1}\cos\varphi$, we get

$$\int\limits_{tz-\sqrt{x^2-1}\sqrt{t^2-1}}^{tz+\sqrt{x^2-1}\sqrt{t^2-1}} \left(1-x^2-t^2-z^2+2xtz\right)^{\lambda-1} dx$$

$$= \int\limits_{tz-\sqrt{x^2-1}\sqrt{t^2-1}}^{tz+\sqrt{x^2-1}\sqrt{t^2-1}} \left(tz+\sqrt{t^2-1}\sqrt{z^2-1}-x\right)^{\lambda-1} \left(x-tz+\sqrt{t^2-1}\sqrt{z^2-1}\right) dx.$$

Using the equality (see [5, p. 299])

$$\int\limits_a^b (x-a)^{\mu-1}(b-x)^{\nu-1}dx = (b-a)^{\mu+\nu-1}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

by $\mu=\nu=\lambda$, we get

$$J = \int\limits_1^\infty \Phi(f(z))(z^2-1)^{\lambda-\frac{1}{2}}dz.$$

Thus we obtain

$$\int\limits_1^\infty \Phi\left(A_t^\lambda|f(x)|\right)(x^2-1)^{\lambda-\frac{1}{2}}dx \le \int\limits_1^\infty \Phi(|f(z)|)(z^2-1)^{\lambda-\frac{1}{2}}dz,$$

or

$$\int\limits_0^\infty \Phi\left(A_{cht}^\lambda|f(chx)|\right)d\mu_\lambda(x) \le \int\limits_0^\infty \Phi(|f(chu)|)d\mu_\lambda(u).$$

From this it follows that

$$\int\limits_0^\infty \left(\frac{A_{cht}^\lambda|f(chx)|}{\|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right)d\mu_\lambda(x) \le \int\limits_0^\infty \left(\frac{|f(chu)|}{\|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right)d\mu_\lambda(u) \le 1.$$

Thus our statement is proved. □

Supposing $\Phi(t)=t^p$, we have the following results (see [7, Lemma 2]).

Corollary 5.6 ([7]). *If $f\in L_{p,\lambda}(\mathbb{R}_+)$, then for any $t\in[0,\infty)$ the following inequality:*

$$\left\|A_{cht}^\lambda f\right\|_{L_{p,\lambda}} \le \|f\|_{L_{p,\lambda}}, \qquad 1\le p\le\infty,$$

holds.

Theorem 5.7. *Let $0<\alpha<\gamma$, $b\in BMO_G(\mathbb{R}_+)$ and Φ, Ψ be Young functions and $\Phi,\Psi\in Y$*

(i) *If $\Phi\in\nabla_2$ and $\Psi\in\Delta_2$, then the condition*

$$\left(sh\frac{r}{2}\right)^\alpha \Phi^{-1}\left(|H_r|_\lambda^{-1}\right) + \sup_{r<t<\infty}\Phi^{-1}\left(|H_t|_\lambda^{-1}\right)\left(sh\frac{t}{2}\right)^\alpha \lesssim \Psi^{-1}\left(|H_r|_\lambda^{-1}\right)$$

for all $r>0$ is sufficient for the boundedness of $M_G^{b,\alpha}$ from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$.

(ii) *If $\Psi\in\Delta_2$, then the condition*

$$\left(sh\frac{r}{2}\right)^\alpha \Phi^{-1}\left(|H_r|_\lambda^{-1}\right) \lesssim \Psi^{-1}\left(|H_r|_\lambda^{-1}\right) \tag{5.1}$$

is necessary for the boundedness of $M_G^{b,\alpha}$ from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$.

(iii) *Let $\Phi\in\nabla_2$ and $\Psi\in\Delta_2$. If the condition*

$$\sup_{r<t<\infty}\Phi^{-1}\left(|H_t|_\lambda^{-1}\right)\left(sh\frac{t}{2}\right)^\alpha \lesssim \left(sh\frac{r}{2}\right)^\alpha \Phi\left(|H_r|_\lambda^{-1}\right) \tag{5.2}$$

is true for all $r>0$, then condition (5.1) is necessary and sufficient for the boundedness of $M_G^{b,\alpha}$ from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$.

Proof. (i). Let $H_r = (0, r)$ and $(H_r)^c = [r, \infty)$.

We write $f = f_1 + f_2$, where $f_1 = f\chi_{H_r}$ and $f_2 = f\chi_{(H_r)^c}$. If $H_t \cap (H_r)^c \neq \emptyset$ and $x \in \mathbb{R}_+$, we have

$$\begin{aligned} M_G^{b,\alpha} f_2(chx) &= \sup_{t>0} \frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_{H_t \cap (H_r)^c} \left| A_{chy}^\lambda b(chx) - b_{H_t}(chx) \right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\ &\leq \sup_{t>r} \frac{1}{|H_t|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_r^t \left| A_{chy}^\lambda b(chx) - b_{H_t}(chx) \right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\ &\leq \sup_{t>r} \frac{1}{|H_t|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_0^t \left| A_{chy}^\lambda b(chx) - b_{H_t}(chx) \right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\ &= \sup_{t>r} \frac{1}{|H_t|_\lambda^{1-\frac{\alpha}{\gamma}}} \int\limits_{H_t} \left| A_{chy}^\lambda b(chx) - b_{H_t}(chx) \right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y). \end{aligned}$$

Therefore, for every $x \in \mathbb{R}_+$, we obtain

$$M_G^{b,\alpha} f_2(chx) \lesssim \sup_{t>r} \left(sh\frac{t}{2} \right)^{\alpha-\gamma} \int\limits_{H_t} \left| A_{chy}^\lambda b(chx) - b_{H_t}(chx) \right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y).$$

Using Hölder's inequality, Lemma 5.2, Lemma 3.10 and Lemma 5.5, we get

$$\begin{aligned} M_G^{b,\alpha} f_2(chx) &\lesssim \sup_{t>r} \left(sh\frac{t}{2} \right)^{\alpha-\gamma} \left\| A_{chy}^\lambda b - b_{H_t} \right\|_{L_{\widetilde{\Phi}}(H_t,G)} \left\| A_{chy}^\lambda f \right\|_{L_\Phi(H_t,G)} \\ &= \sup_{t>r} \left(sh\frac{t}{2} \right)^{\alpha-\gamma} \frac{\widetilde{\Phi}^{-1}\left(|H_t|_\lambda^{-1} \right)}{\widetilde{\Phi}^{-1}\left(|H_t|_\lambda^{-1} \right)} \left\| A_{chy}^\lambda b - b_{H_t} \right\|_{L_{\widetilde{\Phi}}(H_t,G)} \left\| A_{chy}^\lambda f \right\|_{L_\Phi(H_t,G)} \\ &\approx \sup_{t>r} \left(sh\frac{t}{2} \right)^{\alpha-\gamma} \frac{\|b\|_G^*}{\widetilde{\Phi}^{-1}\left(|H_t|_\lambda^{-1} \right)} \left\| A_{chy}^\lambda f \right\|_{L_\Phi(H_t,G)} \\ &\lesssim \|b\|_G^* \sup_{t>r} \left(sh\frac{t}{2} \right)^{\alpha} \Phi^{-1}\left(|H_t^{-1}|_\lambda \right) \|f\|_{L_\Phi(H_t,G)}. \end{aligned} \tag{5.3}$$

To estimate $M_G^{b,\alpha} f_1$, we prove that

$$M_G^{b,\alpha} f(chx) \lesssim J_G^{b,\alpha}\left(|f| \right)(chx), \tag{5.4}$$

where (see [9])

$$J_G^{b,\alpha} f(chx) = \int\limits_0^\infty \frac{\left[A_{chy}^\lambda b(chx) - b_{H_t}(chx) \right]}{\left(sh\frac{t}{2} \right)^{\alpha-\gamma}} A_{chy}^\lambda f(chx) d\mu_\lambda(y), \qquad 0 < \alpha < \gamma$$

for all $x \in \mathbb{R}_+$.

Indeed, by the definition and (2.1), we have

$$\begin{aligned} J_G^{b,\alpha}(|f|)(chx) &= \int\limits_0^\infty \frac{\left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right|}{\left(sh\frac{y}{2} \right)^{\alpha-\gamma}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\ &\geq \int\limits_0^r \frac{\left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right|}{\left(sh\frac{y}{2} \right)^{\alpha-\gamma}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\ &\geq \left(sh\frac{r}{2} \right)^{\alpha-\gamma} \int\limits_0^r \left| A_{chy}^\lambda b(chx) - b_{H_r}(chx) \right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \end{aligned}$$

$$\approx |H_r|_\lambda^{1-\frac{\alpha}{\gamma}} \int\limits_{H_r} \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y).$$

Taking the supremum on $r > 0$ in both parts, we get (5.4).

Using (5.4), we obtain

$$\begin{aligned}
M_G^{b,\alpha} f_1(chx) &\lesssim \int\limits_0^\infty \frac{\left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} A_{chy}^\lambda |f\chi_{H_r}(chx)| d\mu_\lambda(y) \\
&= \int\limits_0^r \frac{\left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\
&\lesssim \sum_{j=0}^\infty \int\limits_{\frac{r}{2^{j+1}}}^{\frac{r}{2^j}} \frac{\left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right|}{\left(sh\frac{y}{2}\right)^{\gamma-\alpha}} A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\
&\lesssim \sum_{j=0}^\infty \left(sh\frac{r}{2^{j+1}}\right)^\alpha \left(sh\frac{r}{2^{j+1}}\right)^{-\gamma} \int\limits_0^{\frac{r}{2^j}} \left|A_{chy}^\lambda b(chx) - b_{H_r}(chx)\right| A_{chy}^\lambda |f(chx)| d\mu_\lambda(y) \\
&\lesssim \left(sh\frac{r}{2}\right)^\alpha M_G^b f(chx) \sum_{j=0}^\infty 2^{-\alpha j} \lesssim \left(sh\frac{r}{2}\right)^\alpha M_G^b f(chx). \qquad (5.5)
\end{aligned}$$

Combining (5.3) and (5.5), we have

$$M_G^{b,\alpha} f(chx) \lesssim \left(sh\frac{r}{2}\right)^\alpha M_G^b f(chx) + \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)} \sup_{t>r} \left(sh\frac{t}{2}\right)^\alpha \Phi^{-1}\left(|H_t|_\lambda\right).$$

According to (5.1), we get

$$M_G^{b,\alpha} f(chx) \lesssim M_G^b f(chx) \frac{\Psi^{-1}\left(|H_r|_\lambda^{-1}\right)}{\Phi^{-1}\left(|H_r|_\lambda^{-1}\right)} + \Psi^{-1}\left(|H_r|_\lambda^{-1}\right) \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}.$$

Choose $r > 0$ so that $\Phi^{-1}\left(|H_r|_\lambda^{-1}\right) = \frac{M_G^b f(chx)}{c_0 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}}$.

Then

$$\frac{\Psi^{-1}\left(|H_r|_\lambda^{-1}\right)}{\Phi^{-1}\left(|H_r|_\lambda^{-1}\right)} = \frac{\left(\Psi^{-1} \circ \Phi\right)\left(\frac{M_G^b f(chx)}{c_0 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right)}{\frac{M_G^b f(chx)}{c_0 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}}}.$$

Therefore

$$M_G^{b,\alpha} f(chx) \lesssim c_1 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)} \left(\Psi^{-1} \circ \Phi\right)\left(\frac{M_G^b f(chx)}{c_0 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right),$$

whence it follows that

$$\frac{M_G^{b,\alpha} f(chx)}{c_1 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}} \leq \left(\Psi^{-1} \circ \Phi\right)\left(\frac{M_G^b f(chx)}{c_0 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right).$$

Consequently,

$$\Psi\left(\frac{M_G^{b,\alpha} f(chx)}{c_1 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right) \leq \Phi\left(\frac{M_G^b f(chx)}{c_0 \|b\|_G^* \|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right).$$

Then by Lemma 5.3, we get

$$\int\limits_{\mathbb{R}_+} \Psi\left(\frac{M_G^{b,\alpha}f(chx)}{c_1\|b\|_G^*\|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right)d\mu_\lambda(x) \le \int\limits_{\mathbb{R}_+} \Phi\left(\frac{M_G^{b}f(chx)}{c_0\|b\|_G^*\|f\|_{L_\Phi(\mathbb{R}_+,G)}}\right)d\mu_\lambda(x)$$
$$\lesssim \int\limits_{\mathbb{R}_+} \Phi\left(\frac{M_G^{b}f(chx)}{\left\|M_G^b f\right\|_{L_\Phi(\mathbb{R}_+,G)}}\right)d\mu_\lambda(x) \le 1,$$

i.e.,

$$\left\|M_G^{b,\alpha}f\right\|_{L_\Psi(\mathbb{R}_+,G)} \lesssim \left\|M_G^b f\right\|_{L_\Phi(\mathbb{R}_+,G)}.$$

(ii) Now, let's prove the second part of the theorem. Let $H_{r_0}=(0,r_0)$ and $x\in H_{r_0}$.
By Lemma 5.4, we get

$$\left(sh\frac{r}{2}\right)^\alpha \lesssim \frac{\left\|M_G^{b,\alpha}f\right\|_{L_\Phi(H_{r_0},G)}}{\left\|b_{H_r}-b_{H_{r_0}}\right\|_{L_\Psi(H_{r_0},G)}} \lesssim \Psi^{-1}\left(|H_{r_0}^{-1}|_\lambda\right)\left\|M_G^{b,\alpha}\chi_{H_{r_0}}\right\|_{L_\Psi(H_{r_0},G)}$$
$$\lesssim \Psi^{-1}\left(|H_{r_0}|_\lambda^{-1}\right)\left\|M_G^{b,\alpha}\chi_{H_{r_0}}\right\|_{L_\Psi(\mathbb{R}_+,G)} \lesssim \Psi^{-1}\left(|H_{r_0}|_\lambda^{-1}\right)\left\|\chi_{H_{r_0}}\right\|_{L_\Psi(\mathbb{R}_+,G)}$$
$$\lesssim \frac{\Psi^{-1}\left(|H_{r_0}|_\lambda^{-1}\right)}{\Phi^{-1}\left(|H_{r_0}|_\lambda^{-1}\right)} \approx \frac{\Psi^{-1}\left(\left(sh\frac{r_0}{2}\right)^{-\gamma}\right)}{\Phi^{-1}\left(\left(sh\frac{r_0}{2}\right)^{-\gamma}\right)}.$$

Since it is valid for every $r_0>0$, the second part of the theorem is proved.

(iii) The third part of the theorem follows from the first and second parts of the theorem. □

Theorem 5.8. *Let $0<\alpha<\gamma$, $b\in L_{1,\lambda}^{\mathrm{loc}}(\mathbb{R}_+)\backslash\{\mathrm{const}\}$ and Φ, Ψ be Young functions and $\Phi,\Psi\in Y$.*

(i) *If $\Phi\in\nabla_2$, $\Psi\in\Delta_2$ and conditions (5.1) and (5.2) hold, then the condition $b\in BMO_G(\mathbb{R}_+)$ is sufficient for the boundedness of $M_G^{b,\alpha}$ from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$.*

(ii) *If $\Psi^{-1}\left(|H_r|_\lambda^{-1}\right)\lesssim\Phi^{-1}\left(|H_r|_\lambda^{-1}\right)|H_r|_\lambda^{\frac{\alpha}{\gamma}}$.*

Then the condition $b\in BMO_G(\mathbb{R}_+)$ is necessary for the boundedness of $M_G^{b,\alpha}$ from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$.

(iii) *If $\Phi\in\nabla_2$, $\Psi\in\Delta_2$, $Psi^{-1}\left(|H_r|_\lambda^{-1}\right)\approx\Phi^{-1}\left(|H_r|_\lambda^{-1}\right)|H_r|_\lambda^{\frac{\alpha}{\gamma}}$ and condition (5.2) holds, then $b\in BMO_G(\mathbb{R}_+)$ is necessary and sufficient for the boundedness of $M_G^{b,\alpha}$ from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$.*

Proof. The first statement follows from the first part of Theorem 5.7.

(ii) We prove the second part.

Let $M_G^{b,\alpha}$ be bounded from $L_\Phi(\mathbb{R}_+,G)$ to $L_\Psi(\mathbb{R}_+,G)$.Choosing every interval $H_r\in\mathbb{R}_+$, by Lemma 3.8 and Lemma 3.11, we have

$$\frac{1}{|H_r|_\lambda}\int\limits_{H_r}\left|A_{chy}^\lambda b(chx)-b_{H_r}(chx)\right|d\mu_\lambda(y)$$
$$=\frac{1}{|H_r|_\lambda^2}\int\limits_{H_r}\left|A_{chy}^\lambda b(chx)-b_{H_r}(chx)\right|d\mu_\lambda(y)\int\limits_{H_r}A_{chx}^\lambda\chi_{H_r}(chy)d\mu_\lambda(x)$$
$$=\frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}}\int\limits_{H_r}\frac{1}{|H_r|_\lambda^{1-\frac{\alpha}{\gamma}}}\int\limits_{H_r}\left|A_{chy}^\lambda b(chx)-b_{H_r}(chx)\right|A_{chy}^\lambda\chi_{H_r}(chx)d\mu_\lambda(y)d\mu_\lambda(x)$$
$$\le\frac{1}{|H_r|_\lambda^{1+\frac{\alpha}{\gamma}}}\int\limits_{H_r}M_G^{b,\alpha}\left(\chi_{H_r}(chx)\right)d\mu_\lambda(x)$$

$$\leq \frac{2}{|H_r|_\lambda^{\frac{\alpha}{\gamma}}} \left\| M_G^{b,\alpha} \chi_{H_r} \right\|_{L_\Psi(\mathbb{R}_+,G)} \Psi^{-1}\left(|H_r|_\lambda^{-1}\right)$$

$$\leq \frac{\Psi^{-1}\left(|H_r|_\lambda^{-1}\right)}{|H_r|_\lambda^{\frac{\alpha}{\gamma}}\, \Phi^{-1}\left(|H_r|_\lambda^{-1}\right)} \lesssim 1.$$

Thus the third part of the theorem follows from the first and second parts of the theorem. □

References

1. M. Agcayazi, A. Gogatishvili, K. Koca, R. Mustafayev, Mustafayev, Rza. A note on maximal commutators and commutators of maximal functions. *J. Math. Soc. Japan* **67** (2015), no. 2, 581–593.
2. C. Bennett, R. Sharpley, *Interpolation of Operators.* Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.
3. S. Chamillo, A note on commutators, *Indiana Univ. Math. J.* **31** (1982) no.1, 7–16.
4. V. S. Guliev, F. Deringoz, S. G. Gasanov, Commutators of a fractional maximal operator on Orlicz spaces. (Russian) *translated from Mat. Zametki* **104** (2018), no. 4, 516–526 *Math. Notes* **104** (2018), no. 3-4, 498–507.
5. I. S. Gradshteyn, I. M. Ryzhik, *Table of Integrals.* series, and products. Translated from the Russian. Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger. With one CD-ROM (Windows, Macintosh and UNIX). Seventh edition. Elsevier/Academic Press, Amsterdam, 2007.
6. V. S. Guliyev, E. J. Ibrahimov, S. Ar. Jafarova, Gegenbauer Transformation Nikolski-Besov spaces Generalized by Gegenbauer Operator and Their Approxmations Characteristics. *Advances in Analysis* **2** (2017), no. 3, 167–195.
7. E. I. Ibrahimov, On Gegenbauer transformation on the half-line. *Georgian Math. J.* **18** (2011), no. 3, 497–515.
8. E. J. Ibrahimov, A. Akbulut, The Hardy-Littlewood-Sobolev theorem for Riesz potential generated by Gegenbauer operator. *Trans. A. Razmadze Math. Inst.* **170** (2016), no. 2, 166–199.
9. E. J. Ibrahimov, G. A. Dadashova, S. Ar. Jafarova, Boundedness of higher order commutators of G-fractional integral and G-fractional maximal operators with G-BMO functions. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 3, 325–341.
10. E. J. Ibrahimov, V. S. Guliyev, S. Ar. Jafarova, Weighted boundedness of the fractional maximal operator and Riesz potential generated by Gegenbauer differential operator. *Trans. A. Razmadze Math. Inst.* **173** (2019), no. 3, 45–78.
11. H. Kita, On maximal functions in Orlicz spaces. *Proc. Amer. Math. Soc.* **124** (1996), no. 10, 3019–3025.
12. H. Kita, On Hardy-Littlewood maximal functions in Orlicz spaces. *Math. Nachr.* **183** (1997), 135–155.
13. V. Kokilashvili, E. J. Ibrahimov, Weak and strong type inequalities criteria for fractional maximal functions and fractional integrals associated with Gegenbauer differential operator. *Georgian Math. J.* 2023. https://doi.org/10.1515/gmj-2023-2031.
14. V. Kokilashvili, M. Krbec, *Weighted Inequalities in Lorentz and Orlicz Spaces.* World Scientific Publishing Co., Inc., River Edge, NJ, 1991.
15. L. Maligranda, *Orlicz Spaces and Interpolation.* Seminars in Mathematics, 5. Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.
16. L. PeiDe, W. MaoFa, Weak Orlicz spaces: Some basic properties and their applications to harmonic analysis. *Science China Mathematics* **56** (2013), no 4, 789–802.
17. M. M. Rao, Z. D. Ren, *Theory of Orlicz Spaces.* Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991.

(Received 25.11.2022)

[1]Institute of Mathematics and Mechanics, ANAS, Baku, Azerbaijan

[2]Azerbaijan State University of Economics (UNEC), Baku, Azerbaijan
Email address: elman@yahoo.com
Email address: saadat.cafarova73@gmail.com