Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 2, 189–204

A LIPSCHITZ VERSION OF DE RHAM THEOREM FOR L_p-COHOMOLOGY

VLADIMIR GOL'DSHTEIN AND ROMAN PANENKO

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. We focus our attention on the de Rham operators' underlying properties which are specified by intrinsic effects of differential geometry structures. And then we apply the procedure of regularization in the context of Lipschitz version of de Rham calculus on metric simplicial complexes with bounded geometry.

1. Introduction

The reasons which lie at the roots of the present text could be abstracted as follows. Despite the fact that the notion of de Rham's regularization operators has a long history and some useful applications, primarily, it was a tool that allowed to reduce cohomology of Banach chain complexes to the case which is more familiar and convenient since it is presented by a subcomplex of smooth objects. It does not leave the impression of clarity. Indeed, de Rham's initial exposition on the subject and further applications, focusing on the analytic aspect of the matter, seem to tend cloud intrinsic elegance and simplicity of that construction. Such situation inherently encourages us to reopen the discourse on the subject in order to embellish the prevailing approach and see how far that construction could be generalized.

Beginning with the first decades of the 20th century when the basic notions of the exterior calculus were formulated thanks to Élie Cartan's works and further through Georges de Rham's contribution, one got the perfectly clear language to talk about global properties of manifolds. In particular, his explorations led up to the emergence of the concept of the so-called de Rham's complex, namely, that work elucidated the analogies between differential forms and chains. One can notice that the concept of chain complex was quite known within the frames of algebraic topology and homological algebra that was being formed at that time. Also, we should attribute to that period (around the thirties) the de Rham's theorem establishing an isomorphism between the cohomogy of differential forms and the singular cohomology. Eventually, those reasons led up to the notion of current generalizing essential characteristics shared by both chains and forms. Later, de Rham extensively developed the theory of currents involving as the foundation of Lauren Schwartz's work on distributions. That yielded subsequently the results on the approximation of currents by smooth forms and required to introduce the regularization operators defined in the weak sense. It was regarded as relying on a duality between currents and compactly supported smooth forms. Let us take a closer look at the subject.

There is the well-known idea to approximate a value of locally integrable function f at every point with its mean value over a bounded neighbourhood of a such point. More generally, using a convolution with a smooth kernel φ such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$, we can define a regularization operator $\Phi_\varepsilon(f) = f * \varphi_\varepsilon$.

Let $\mathscr{D}'(\mathbb{R}^n)$ be a space of continuous functionals on the space $C_0^\infty(\mathbb{R}^n)$ endowed with the usual topology. The convolution $T * \varphi$ of a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ and a function $\varphi(y) \in C_0^\infty(\mathbb{R}^n)$ is defined by $\{T * \varphi\}(x) = T(\varphi(x-y))$, that is, the operator $T \mapsto \langle T, \tau_x\tilde{\varphi}\rangle$ where $\tilde{\varphi}(y) = \varphi(-y)$, $\tau_x\varphi(y) = \varphi(y+x)$.

We can sum up that approach in the following way (see, e.g., [7]).

2020 *Mathematics Subject Classification.* 58A12, 58A15, 53C65, 57Q05, 57R12, 51F30, 46E30.

Key words and phrases. Differential forms; Lipschitz analysis; De Rham complex; Mollifier; Metric simplicial complex; Bounded geometry; De Rham theorem; Whitney form.

Theorem 1. *Let $\varphi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ be a sequence of positive functions such that $\int_{\mathbb{R}^n} \varphi_\varepsilon(x)dx = 1$ and $\operatorname{supp}(\varphi_\epsilon)$ is a ball with radius ε. If $T \in \mathscr{D}'(\mathbb{R}^n)$, it follows that $T * \varphi_\varepsilon \in C^\infty(\mathbb{R}^n)$ and $T * \varphi_\varepsilon \to T$ in $\mathscr{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$.*

In the light of the above-mentioned, it is reasonable to talk about the regularization of distributions in the following sense: $\Phi_\varepsilon(T) = T * \varphi_\varepsilon$. If φ_ε is a symmetric kernel, that is, $\varphi_\varepsilon(x) = \varphi_\varepsilon(-x)$ and $g \in C_0^\infty(\mathbb{R}^n)$, we have $\langle T * \varphi_\varepsilon, g\rangle = \langle T, g * \varphi_\varepsilon\rangle$. Then we can define a regularization operator with a symmetric kernel on the space of distributions as follows: $\{\Phi_\varepsilon(T)\}(g) = T(\Phi_\varepsilon(g))$.

Let M be a differentiable manifold and let $(\Omega^*(M), d)$ be the de Rham DG-algebra (a differential graded algebra) on M, that is, the algebra of smooth differential forms. In particular, there is defined a chain complex

$$0 \longrightarrow C^\infty(M) \longrightarrow \Omega^1(M) \longrightarrow \cdots \longrightarrow \Omega^n(M) \longrightarrow 0\ .$$

Following the de Rham approach, we turn to the subcomplex of compactly supported forms and its dual complex of currents. According to what has been said above, we intend to define a regularization operator on currents analogously to the case of distributions $RT[\omega] = T[R^*\omega]$. In line with the above, there emerges a reasonable question what we should think of a procedure of computing the mean value of a differential form $R^*\omega$. It is quite clear that we need to define the operator under consideration in such a manner that preserves cohomology classes.

First of all, we should clarify the notion of homotopy. Let A be an additive category. Consider the category of chain complexes $\mathsf{Ch}(\mathsf{A})$. We can introduce the homotopy category of chain complexes $\mathsf{K}(\mathsf{A})$ by taking into account a concept of 'equivalent deformation' η of morphisms $f, g \in \operatorname{Hom}_{\mathsf{Ch}(\mathsf{A})}(V, W)$,

$$V \underset{g}{\overset{f}{\rightrightarrows}} W \quad (\Downarrow \eta)\ .$$

Let us take a look at this construction in more detail. The homotopy η we assume to be a set of morphisms $\{\eta^i \in \operatorname{Hom}_{\mathsf{A}}(V^i, W^{i-1})\}$ which satisfy $f^i - g^i = d_W^{i-1}\eta^i + \eta^{i+1}d_V^i$. It should be noted that we do not involve the requirement that η is a chain morphism. The condition of the existence of such a homotopy between morphisms equips the set $\operatorname{Hom}_{\mathsf{Ch}(\mathsf{A})}(V, W)$ with an equivalence relation. So, $\mathsf{K}(\mathsf{A})$ can be introduced as a category of chain complexes with morphisms defined modulo homotopy. We can reveal the point by turning to the well-studied case of Abelian categories which are the classical setting for the treatment of homological algebra. It is not hard to see that homotopic morphisms induce the same morphism between the corresponding cohomology groups, and every homotopy equivalence $f : V \to W$ defines the isomorphism of cohomologies. Thus a two-sided invertible morphism in the category $\mathsf{K}(\mathsf{A})$ corresponds to an isomorphism of cohomologies. In particular, a homotopy equivalence between topological spaces induces isomorphism between the singular chain complexes in $\mathsf{K}(\mathsf{A})$.

Another example of chain homotopy will serve as the central part of our interpretation of the regularization operators. Assume that X is a vector field, then the Lie derivative $\mathcal{L}_X$ is the 0-derivation of the DG-algebra such that there exists a -1-derivation ι_X being the homotopy between $\mathcal{L}_X$ and the zero map

$$(\Omega^*(M), d) \underset{\mathcal{L}_X}{\overset{0}{\rightrightarrows}} (\Omega^*(M), d) \quad (\Downarrow \iota_X)\ .$$

That is to say, the Lie derivative satisfies Cartan's formula $\mathcal{L}_X = d\iota_X + \iota_X d$. At every point x, the map $\omega \mapsto \mathcal{L}_X\omega$ induces a function with values in the exterior power of the cotangent space at x,

$$\mathcal{L}_X\omega_x : \mathbb{R} \to \bigwedge^n T_x^* M,$$

$\mathcal{L}_X\omega$ is precisely nothing else than the instantaneous velocity in the exterior power of the cotangent space, that is, a magnitude of change of the form ω under the infinitesimal translation along an integral curve $\phi_X(t)$. That corresponds to the zero endomorphism of $(\Omega^*(M),\, d)$ in the homotopy category of chain complexes. In other words, for every closed form ω, the form $\mathcal{L}_X\omega$ is an exact form.

We can compute the integral of the function $\mathcal{L}_X\omega_x$. Owing to the linearity of integration and the fact that it commutes with the exterior differential, we can see that the following $\int_0^1 \mathcal{L}_X\omega dt = \phi^*_{t=1}\omega - \omega$ implies the homotopy

$$(\Omega^*(M),\, d) \; \underset{\mathrm{Id}}{\overset{\phi^*_{t=1}}{\rightrightarrows}} \; (\Omega^*(M),\, d) \qquad \Downarrow \; .$$

That makes sense to talk about the procedure of regularization on differential forms. Namely, the pullback of translation along vector fields preserves cohomology classes. And as a result, combining the intrinsic attribute of smooth manifolds expressed in the Cartan formula and the classic idea of mollifier, we can define a form representing the mean value of the given differential form at every point of smooth manifold.

Turning back to de Rham's construction, it is not hard to see that the operator $RT[\omega] = T[R^*\omega]$ defined on currents inherits the property to preserve cohomology classes

$$\begin{aligned} RT[\omega] = T[R^*\omega] &= T[A^*d\omega + dA^*\omega] = T[A^*d\omega] + T[dA^*\omega] \\ &= AT[d\omega] + \partial T[A^*\omega] = \{\partial AT + A\partial T\}[\omega]. \end{aligned}$$

The next step in that direction was made in [4], where the authors focused on a special kind of currents which can be presented as the elements of Sobolev space of differential forms $\Omega^*_{p,\,p}(M)$. It is clear that being a special case of currents, such forms hold all basic properties of the regularization. The crucial result consists in the proof that we have the same diagram

$$(\Omega^*_{p,\,p}(M),\, d) \; \underset{\mathrm{Id}}{\overset{R}{\rightrightarrows}} \; (\Omega^*_{p,\,p}(M),\, d) \qquad \Downarrow$$

in the category of chain complexes of Banach spaces. That allows us to generalize de Rham's theorem to the case of L_p-cohomology of triangulable noncompact manifolds.

We call a simplicial complex K having bounded geometry if every vertex of the 1-skeleton of K has a uniformly bounded degree as a vertex of graph, and the length of every edge is in the interval $[L^{-1},\, L]$ for some $L \geq 1$.

We introduce a class of differential forms $S\mathscr{L}^*_p(K)$ on a simplicial complex K which are locally pullbacks of smooth forms defined on a subsets of $\mathbb{R}^n$ under bi-Lipschitz homomorphisms and having a finite graph norm on the domain of $S\mathscr{L}^k(K) \xrightarrow{d} S\mathscr{L}^{k+1}(K)$ in the sense of L_p-spaces. Let $\Omega^*_{p,\,p}(K)$ denote the closure of that class under a topology induced from the graph norm. The main result of the present work can be summed up in the following two assertions:

- *Let K be a complex of bounded geometry. Then there exists the diagram*

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{d} & \Omega^{k-1}_{p,\,p}(K) & \xrightarrow{d} & \Omega^{k}_{p,\,p}(K) & \xrightarrow{d} & \Omega^{k+1}_{p,\,p}(K) & \xrightarrow{d} \cdots \\
 & \downarrow \mathscr{R} & & \downarrow \mathscr{R} & & \downarrow \mathscr{R} & \\
\cdots & S\mathscr{L}^{k-1}_{p}(K) & \overset{\mathscr{A}}{\swarrow} & S\mathscr{L}^{k}_{p}(K) & \overset{\mathscr{A}}{\swarrow} & S\mathscr{L}^{k+1}_{p}(K) & \cdots \\
 & \hookdownarrow & & \hookdownarrow & & \hookdownarrow & \\
\cdots \xrightarrow[d]{} & \Omega^{k-1}_{p,\,p}(K) & \xrightarrow[d]{} & \Omega^{k}_{p,\,p}(K) & \xrightarrow[d]{} & \Omega^{k+1}_{p,\,p}(K) & \xrightarrow[d]{} \cdots
\end{array}$$

with the commutative squares in the category of Banach spaces Ban_∞. Moreover, the map $\mathscr{R}$ is homotopic to the identity

$$(\Omega^*_{p,\,p}(M),\, d) \underset{\mathrm{Id}_{\Omega^*_{p,\,p}}}{\overset{\mathscr{R}}{\rightrightarrows}} (\Omega^*_{p,\,p}(M),\, d)\,, \qquad \Downarrow \mathscr{A}$$

- *Under the same conditions, the following commutative triangle of isomorphisms takes place in the category $\mathsf{Vec}_\mathbb{R}$,*

$$\begin{array}{ccc}
 & H^k(S\mathscr{L}^*_p(K)) & \\
\overset{\mathscr{R}}{\nearrow} & & \overset{\mathscr{I}}{\searrow} \\
H^k(\Omega^*_{p,p}(K)) & \dashrightarrow & H^k(C^*_p(K))
\end{array}$$

It should be noted that such complexes could be useful as a bi-Lipschitz triangulation of Riemannian manifolds with bounded geometry. It is not hard to see that the obtained results hold for an arbitrary $L \geq 1$. Moreover, a specified type of complexes emerges naturally (see, for example, [1,2]).

Theorem 2. *Let M be an n-dimenthional Riemannian manifold of bounded geometry with geometric bounds a, b, ϵ. Then M admits a triangulation K of bounded geometry (whose geometric bounds depend on n, a, b, ϵ) and an L-bi-Lipcshitz homeomorphism $f\colon K \to M$, where $L = L(n,\, a,\, b,\, \epsilon)$.*

2. Homotopy and Lie Derivative

Most of the content included in this section can be found in [9].

Let $\mathscr{C}^\infty(\mathbb{R}^n)$ be a ring of smooth functions on $\mathbb{R}^n$. The differentiation of smooth functions defines a derivation in $\mathscr{C}^\infty(\mathbb{R}^n)$ with values in $\mathscr{C}^\infty$-module consisting of 1-forms on $\mathbb{R}^n$ $d\colon \mathscr{C}^\infty(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)$, that is, a homomorphism of the respective additive groups which satisfies the condition $d(fg) = fdg + gdf$. As usual, we define all operations in $\mathscr{C}^\infty(\mathbb{R}^n)$ pointwisely. There are the well-known algebraic reasons which imply a number of facts about local structure of $\mathscr{C}^\infty(\mathbb{R}^n)$. Let $\mathscr{C}^\infty_x(\mathbb{R}^n)$ be the space of germs of smooth functions at a point x. Then $\mathscr{C}^\infty_x(\mathbb{R}^n)$ is a commutative local ring, that means that non-invertible elements, namely germs of functions which vanish at x, form a maximal ideal $\mathfrak{m}_x$. As a consequence, the quotient ring $\mathscr{C}^\infty_x(\mathbb{R}^n)/\mathfrak{m}_x$ is a field. As a result, we can conclude that $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a vector space. Assume that $f \in \mathscr{C}^\infty_x(\mathbb{R}^n)$, then due to Taylor's theorem, we can write down the following: $f - f(x) = \langle \nabla_x f, \sum_{i=1}^n (\xi_i - \xi_i(x))\vec{e_i}\rangle + h$, $h \in m_x^2$ and as a result, we obtain the representation of f at the point x as an element of the vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$: $f \mapsto f - f(x) \in \mathfrak{m}_x/\mathfrak{m}_x^2$. To summarize, the fibers $x \mapsto \mathfrak{m}_x/\mathfrak{m}_x^2$ specify a vector bundle. Considering that the components of the vector $f - f(x)$ change smoothly on $\mathbb{R}^n$, we obtain a vector field, corresponding to the element of $\mathscr{C}^\infty(\mathbb{R}^n)$. So, we can sum up that the procedure outlined above allows us to define a derivation in the commutative ring $\mathscr{C}^\infty(\mathbb{R}^n)$ with values in $\mathscr{C}^\infty$-module consisting of vector fields. As a consequence, there exists a related derivation in $\mathscr{C}^\infty(\mathbb{R}^n)$ with values in the dual $\mathscr{C}^\infty$-module consisting of 1-forms $d\colon \mathscr{C}^\infty(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)$. Now, we can define the de Rham DG-algebra on $\mathbb{R}^n$ as a graded algebra $\Omega^*(\mathbb{R}^n) = \bigoplus_{k=0}^n \bigwedge^k \Omega(\mathbb{R}^n)$, $\bigwedge^0 \Omega(\mathbb{R}^n) = \mathscr{C}^\infty(\mathbb{R}^n)$ endowed with an antiderivation, i.e., an endomorphism satisfying the graded Leibnitz rule with a commutator factor -1, which is specified as the exterior derivative d in the usual sense:

- define the derivative in accordance with the derivation in a ring $d\colon \mathscr{C}^\infty(\mathbb{R}^n) \to \Omega(\mathbb{R}^n)$ for 0-forms;
- $d^2 = 0$;
- $d(\omega \wedge \theta) = (d\omega) \wedge \theta + (-1)^{\deg \omega} \omega \wedge (d\theta)$.

Let X be a vector field on $\mathbb{R}^n$ and $\phi_X\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a corresponding flow.

Consider a -1-antiderivation on the graded algebra $\Omega^*(\mathbb{R}^n)$ defined in the following

Definition 1. Let X be a vector field on $\mathbb{R}^n$. The interior product is a map $\iota_X\colon \Omega^n(\mathbb{R}^n) \to \Omega^{n-1}(\mathbb{R}^n)$ satisfying a number of conditions:

- If $\omega \in \Omega^1(\mathbb{R}^n)$, we put $\iota_X \omega = \langle \omega, X \rangle$, i.e., ι is the canonical pairing;
- $\iota(\omega \wedge \theta) = (\iota_X \omega) \wedge \theta + (-1)^{\deg \omega} \omega \wedge (\iota_X \theta)$.

Definition 2. Let X be a vector field on $\mathbb{R}^n$. The Lie derivative is a map $\mathcal{L}_X\colon \Omega^n(\mathbb{R}^n) \to \Omega^n(\mathbb{R}^n)$ defined in the following way: $\mathcal{L}_X\colon \omega \mapsto \frac{d}{dt}\Big|_{t=0} \phi_X^* \omega$.

Theorem 3. *Under the above assumptions, Cartan's magic formula $\mathcal{L}_X \omega = \iota_X \circ d\omega + d \circ \iota_X \omega$ holds.*

Remark 1. The Poincaré lemma, that is, $H^i(\Omega^*(U)) = 0$ for $i < n$, where U is an open ball in $\mathbb{R}^n$, can be derived from Cartan's formula.

Let us consider the change of the form ω along a segment $\phi_X(x, t)\colon [0, 1] \to \mathbb{R}^n$ of the integral curve which starts at the point x. A parametrized differential form $\phi_X^* \omega(t) \in \Omega^k(\mathbb{R}^n)$ defines a family of multilinear skew-symmetric maps on the tangent space $T_x\mathbb{R}^n$ for an integral curve that starts at the point x $f(t) A_x^0(t) \wedge \dots \wedge A_x^{k-1}(t)\colon \bigwedge^k \mathbb{R}^n \to \mathbb{R}$, $A_x^i(t) \in (\mathbb{R}^n)^*$, that is, there is specified function $F(t) = f(t) \det(A_x^i(t) \xi_j)$ at every $\xi_0 \wedge \dots \wedge \xi_{k-1}$, and so, we can define $\frac{d}{dt} F(t)$ and $\int_0^1 F'(t) dt$.

This induces a couple of maps: $\mathcal{L}_X \omega(t)\colon \bigwedge^k \mathbb{R}^n \to \mathbb{R}$, at every point t, where $\{\mathcal{L}_X \omega(t)\}(x) = \mathcal{L}_{X(\phi_t(x))} \{\phi_X^*(t)\} \omega$ and $\int_0^1 \mathcal{L}_X \omega(t) dt\colon \bigwedge^k \mathbb{R}^n \to \mathbb{R}$.

Then we have

$$\int_0^1 \mathcal{L}_X \omega dt = \phi_X^*\big|_{t=1} \omega - \omega$$

$$= \int_0^1 \iota_{X(\phi_t(x))} \circ \{\phi_X^*(t)\}(d\omega) dt + d\bigg(\int_0^1 \iota_{X(\phi_t(x))} \circ \{\phi_X^*(t)\}(\omega) dt \bigg).$$

Definition 3. Let $v \in \mathbb{R}^n$. Define an associated flow $s_v\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ as the translation along v: $s_{tv}(x) = x + tv$.

Then we can use Cartan's formula $s_v^* \omega - \omega = Q_v d\omega + dQ_v \omega$, where $Q_v = \left\{ \int_0^1 dt \right\} \circ \iota_v \circ \phi_X^*(t)$.

3. De Rham Operators on $\Omega^*(\mathbb{R}^n)$

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a compactly supported smooth function such that $\operatorname{supp}(f) \subset B_1$, $\int_{\mathbf{R}^n} f(v) dv^0 \dots dv^{n-1} = 1$, $f(v) \geq 0$ and $f(v) = f(-v)$. Let us put $\tau(v) = f(v) dv^0 \dots dv^{n-1}$. Using the previous argumentation, we can integrate the equation $s_v^* \omega \cdot \tau(v) - \omega \cdot \tau(v) = dQ_v(\omega) \cdot \tau(v) + Q_v(d\omega) \cdot \tau(v)$ at every point x, namely, this integration procedure is induced by integration of forms of the type $g(v) \det(\langle A^i(v), * \rangle) \tau$, where $\omega(x) = g(v) A^0(v) \wedge \dots \wedge A^k(v)(* \wedge \dots \wedge *)$:

$$\int_{\mathbb{R}^n} (s_v^* \omega(x) \cdot \tau(v) - \omega(x) \cdot \tau(v)) = \int_{\mathbb{R}^n} (dQ_v(\omega(x)) \cdot \tau(v) + Q_v(d\omega(x)) \cdot \tau(v)),$$

that is which allows us to specify a chain homotopy $\mathcal{R}(\omega) - \omega = d\mathcal{A}(\omega) + \mathcal{A}(d\omega)$.

There is a diffeomorphism h of $\mathbb{R}^n$ onto the open ball $\mathbf{B}_1$ with centre 0 and radius 1. Let $U \subset \mathbb{R}^n$ and $\mathbf{B}_1 \subset U$. We can define $\mathfrak{s}_v \colon U \to U$ as

$$\mathfrak{s}_v x = \begin{cases} h s_v h^{-1}(x), & \text{if } x \in \mathbf{B}_1; \\ x, & \text{if } x \notin \mathbf{B}_1. \end{cases}$$

It was shown in de Rham's book [6] that $\mathfrak{s}^*_{tv}$ produces a group action of the additive group of real numbers on U, that is, $\mathfrak{s}^*_{(t_0+t_1)v} = \mathfrak{s}^*_{t_1 v} \circ \mathfrak{s}^*_{t_0 v}$. Also, we can say that if $\mathfrak{X}_v = d_{h^{-1}(x)} h(v)$ is a vector field consisting of tangent vectors to $\mathfrak{s}_{tv}(x)$, then we have $\frac{d}{dt}\Big|_{t=0} \mathfrak{s}^*_{tv}\omega = d \circ \iota_{\mathfrak{X}_v}(\omega) + \iota_{\mathfrak{X}_v} \circ d(\omega)$; and $\mathcal{R}_\varepsilon(\omega) - \omega = d\mathcal{A}_\varepsilon(\omega) + \mathcal{A}_\varepsilon(d\omega)$, where $\mathcal{R}_\varepsilon \omega = \int_{\mathbb{R}^n} \mathfrak{s}^*_{\varepsilon v}\omega(x) \cdot \tau(v)$ and $\mathcal{A}_\varepsilon(\omega) = \int_{\mathbb{R}^n} \big(\int_0^1 \iota_{\mathfrak{X}_{\varepsilon v}(\mathfrak{s}_{\varepsilon v t}(x))} (\mathfrak{s}^*_{\varepsilon v t}\omega) dt \big) \cdot \tau(v)$. It was shown [4] that the following lemma holds.

Lemma 1. *For every $\varepsilon > 0$, the maps $\mathcal{R}_\varepsilon$ and $\mathcal{A}_\varepsilon$ are bounded on $\Omega^k_p(\mathbf{B_1})$ with respect to the L_p-norm and, moreover, the following estimations $\|\mathcal{R}_\varepsilon\|_p \le C(\varepsilon)$, and $\|\mathcal{A}_\varepsilon\|_p \le M(\varepsilon)$ hold, where $C(\varepsilon) \to 1$, $M(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

From now on, we will use $\operatorname{res}_{V,U}$ for the restriction map $\Omega^k(U) \to \Omega^k(V)$ where $V \subset U$ if it is well-defined in the context of our consideration.

Lemma 2. *Let $\mathbf{B}_1$ be a closed ball in $\mathbb{R}^n$ with centre 0 and radius 1, $\mathbf{B}_1 \subset U \subset \mathbb{R}^n$. Then for every $\varepsilon > 0$ and any compact $F \subset \operatorname{Int} \mathbf{B}_1$, the map $\operatorname{res}_{F,U} \circ \mathcal{R}_\varepsilon$ is a bounded operator $\Omega^k_p(U) \to \Omega^k_\infty(F)$.*

Proof. Let $\omega \in \Omega^k_p(U)$, then $|\mathcal{R}_\varepsilon\omega|$ is a smooth function which implies that it is bounded and there is a point ξ such that $\sup_{x\in F} |\mathcal{R}_\varepsilon\omega|(x) = |\mathcal{R}_\varepsilon\omega|(\xi)$.

$$|\mathcal{R}_\varepsilon\omega|^p(\xi) \le \Bigg(\int\limits_{\operatorname{supp}(f)} |\mathfrak{s}^*_{\varepsilon v}\omega|(\xi) \cdot \tau(v) \Bigg)^p$$

$$\le C \int\limits_{\operatorname{supp}(f)} |\mathfrak{s}^*_{\varepsilon v}\omega|^p(\xi) dv^0 \dots dv^{n-1} \le C \|\omega\|^p_{\Omega^k_p(U)},$$

where $C = \operatorname{mes}(\operatorname{supp}(f))^{p-1} (\sup_{x \in \operatorname{supp}(f)} f)^p$. □

It follows that taking the closure of $\Omega^k(\mathbf{B_1})$ with respect to the L_p-norm induces bounded maps on the Banach spaces $\Omega^k_p(\mathbf{B_1})$ and there exists the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & \Omega^{k-1}_p(\mathbf{B}_1) & \xrightarrow{d} & \Omega^k_p(\mathbf{B}_1) & \xrightarrow{d} & \Omega^{k+1}_p(\mathbf{B}_1) & \xrightarrow{d} & \cdots \\
 & & \downarrow \mathcal{R}_\varepsilon & \swarrow \mathcal{A}_\varepsilon & \downarrow \mathcal{R}_\varepsilon & \swarrow \mathcal{A}_\varepsilon & \downarrow \mathcal{R}_\varepsilon & & \\
\cdots & \xrightarrow{d} & \Omega^{k-1}_p(\mathbf{B}_1) & \xrightarrow{d} & \Omega^k_p(\mathbf{B}_1) & \xrightarrow{d} & \Omega^{k+1}_p(\mathbf{B}_1) & \xrightarrow{d} & \cdots
\end{array}$$

with commutative squares and morphisms such that the equation $\mathcal{R}(\omega) - \omega = d\mathcal{A}(\omega) + \mathcal{A}(d\omega)$ holds.

Consider the bi-Lipschitz homomorphism $\varphi \colon \mathbf{B}_1 \to B \subset \mathbf{R}^n$. Then we can define the operators $\tilde{\mathcal{R}}_\varepsilon$ and $\tilde{\mathcal{A}}_\varepsilon$:

$$\begin{array}{ccc}
\Omega^k_p(B) & \xrightarrow{\tilde{\mathcal{R}}_\varepsilon} & \bigwedge^k \Omega_{\mathscr{L}}(B) \\
\varphi^* \downarrow & & \uparrow (\varphi^{-1})^* \\
\Omega^k_p(\mathbf{B}_1) & \xrightarrow{\mathcal{R}_\varepsilon} & \Omega^k_{\text{smooth}}(\mathbf{B}_1)
\end{array}
\qquad
\begin{array}{ccc}
\Omega^k_p(B) & \xrightarrow{\tilde{\mathcal{A}}_\varepsilon} & \Omega^{k-1}_p(B) \\
\varphi^* \downarrow & & \uparrow (\varphi^{-1})^* \\
\Omega^k_p(\mathbf{B}_1) & \xrightarrow{\mathcal{A}_\varepsilon} & \Omega^{k-1}_p(\mathbf{B}_1)
\end{array}$$

It is not difficult to see that the commutative squares are the squares in the category of normed Banach spaces because all the arrows are bounded maps and, moreover, we have $\|\tilde{\mathcal{R}}_\varepsilon\|_p \le \tilde{C}(\varepsilon)$ and $\|\tilde{\mathcal{A}}_\varepsilon\|_p \le \tilde{M}(\varepsilon)$.

Just as we did above, take the closure of $\Omega^k(B)$ with respect to the L_p-norm; this induces bounded maps on the Banach spaces $\Omega_p^k(B)$ and there exists the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & \Omega_p^{k-1}(B) & \xrightarrow{d} & \Omega_p^{k}(B) & \xrightarrow{d} & \Omega_p^{k+1}(B) & \xrightarrow{d} & \cdots \\
 & & \downarrow \tilde{\mathcal{R}}_\varepsilon & \tilde{\mathcal{A}}_\varepsilon \swarrow & \downarrow \tilde{\mathcal{R}}_\varepsilon & \tilde{\mathcal{A}}_\varepsilon \swarrow & \downarrow \tilde{\mathcal{R}}_\varepsilon & & \\
\cdots & & \bigwedge^{k-1}\Omega_{\mathscr{L}}(B) & & \bigwedge^{k}\Omega_{\mathscr{L}}(B) & & \bigwedge^{k+1}\Omega_{\mathscr{L}}(B) & & \cdots \\
 & & \hookdownarrow & & \hookdownarrow & & \hookdownarrow & & \\
\cdots & \xrightarrow[d]{} & \Omega_p^{k-1}(B) & \xrightarrow[d]{} & \Omega_p^{k}(B) & \xrightarrow[d]{} & \Omega_p^{k+1}(B) & \xrightarrow[d]{} & \cdots
\end{array}$$

with commutative squares and morphisms such that the equation $\tilde{\mathcal{R}}_\varepsilon(\omega) - \omega = d\tilde{\mathcal{A}}_\varepsilon(\omega) + \tilde{\mathcal{A}}_\varepsilon(d\omega)$ holds.

4. Classes of Differential Forms on a Metric Simplicial Complex

Denote by Fin_+ the category of finite nonempty sets and partial maps and by Set the usual category of sets. A simplicial complex K can be defined as a functor $\mathsf{K}\colon \mathsf{Fin}_+^{\mathsf{op}} \to \mathsf{Set}$, where $\mathsf{Fin}_+^{\mathsf{op}}$ is the opposite category of Fin_+. Fix some set V and put $K([n]) = \{\rho\colon [n] \to V \mid \rho \text{ is a partial injective function}\}$. In other words, the elements of $\mathsf{K}([n]) = K[n]$ serve as indices to n-simplices and $f\colon [m] \to [n]$ induces the embedding of faces of K, $K(f)\colon K[n] \to K[m]$, as follows: $K(f)\langle\rho\rangle = \rho \circ f$.

The condition below was introduced in [5] in the context of studying triangulated Riemannian manifolds.

Definition 4 (The star-boudness condition)**.** We call K star-bounded if there exists $C > 0$ such that for every $v \in K[0]$ the cardinality of a set $\Psi_v = \{\iota \in \operatorname{Hom}(K[0], K[1]) \mid v \in \operatorname{Dom}\iota\}$ satisfies the following $|\Psi_v| \le C$.

Define a geometric realization of the simplicial complex K as a topological space $|K| = \coprod_{i=0}^{n}(\Delta_i \times K[i])/\sim$, where

$$\Delta_n = \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n-1} t_i = 1,\ t_i \ge 0\right\}$$

and $\sim$ is an equivalence relation defined by gluing of simplices. We also can endow $|K|$ with the simplicial metric, that is, Euclidian on each simplex. From now on, we will follow the terminology of [2]. Let every simplex of K be isometric to the standard simplex in the Euclidian space. Thus each morphism $[k] \to [m]$ induces for each m-simplex σ^m an isometric embedding of its face σ^k, that is, $\sigma^k \to \sigma^m$. Also, as was mentioned in [2], we can introduce a length-metric on K in such a manner that each simplex is isometrically embedded in K. In more detail, a piecewise-linear path $\gamma\colon [a, b] \to K$ is a path such that its domain can be broken into finitely many intervals $[a_i, a_{i+1}]$ so that the image $\gamma([a_i, a_{i+1}])$ is a piecewise-linear path contained in a single simplex of K. The length of γ is defined by using Euclidian metric on simplices of K: $|\gamma| = \sum_i |\gamma([a_i, a_{i+1}])|$ and so, we can define the distance as follows: $d(x, y) = \inf_\gamma |\gamma|$, where the infimum is taken over all paths connecting x and y in the class of piecewise-linear maps.

Remark 2. The path-metric d is complete and turns K to a geodesic metric space.

Definition 5. A metric simplicial complex K has bounded geometry if it is connected, star-bounded and there exists $L \ge 1$ such that the length of every edge is in the interval $[L^{-1}, L]$.

Below, we will assume that all complexes have bounded geometry with $L = 1$.

Let $\mathscr{L}_{\mathrm{loc}}(|K|)$ be a space of locally Lipschitz functions on $|K|$. We require that for every morphism $f\colon [k] \to [m]$ there is a restriction $\operatorname{res}_{K[k], K[m]}\colon \mathscr{L}_{\mathrm{loc}}(|K[m]|) \to \mathscr{L}_{\mathrm{loc}}(|K[k]|)$ induced by an isometric embedding of its face which can be implemented by the consecutive vanishing of $m - k$ -sets of barycentric coordinates t_j with indices $j \notin \{j_0, \dots, j_k\}$ on every m-simplex Δ.

Define $\mathscr{C}^\infty\mathscr{L}(|K|) \subset \mathscr{L}_{\mathrm{loc}}(|K|)$ as a space of locally Lipschitz functions from the class $\mathscr{C}^\infty$ on every topological space (Δ_n, α), $\alpha \in K[n]$.

Theorem 4 (Rademacher's theorem)**.** *Let $U \subset \mathbb{R}^n$, and let $f\colon U \to \mathbb{R}$ be locally Lipschitz. Then f is differentiable at almost every point in U.*

Let K be a simplicial complex define barycentric coordinates $t_i\colon |K| \to \mathbb{R}$, where $\sum_i t_i = 1,\ t_i \geq 0$.

The restrictions of coordinate functions to every simplex of K are smooth and the function germs, which are locally Lipschitz on $|K|$ and smooth inside simplices, generate a correctly defined tangent space for every interior point x of each simplex of K. If $x \in |K[n-1]|$, then every coordinate function t_i is not differentiable at x as a function on $|K|$. This implies that every Lipschitz function $f(t_0, \dots, t_n)$ is not differentiable at x, as well. In spite of that fact, Rademacher's theorem allows us to define the $n-1$-dimensional tangent space almost everywhere on the $n-1$-dimensional skeleton of our complex by using the restriction of coordinate functions to the $|K[n-1]|$. It follows that we can define the tangent space almost everywhere on a skeleton of each dimension. Let $\mathscr{C}^\infty\mathscr{L}_c(|K|)$ be a subspace of compactly supported functions in $\mathscr{C}^\infty\mathscr{L}(|K|)$. Suppose $f \in \mathscr{L}_{\mathrm{loc}}(|K|)$ and define df in the sense of distributions. Due to Rademacher's theorem, at almost every point x of $|K|$, we can consider a continuous germ $f - f(x) = \langle \nabla_x f,\, \sum_{i=1}^n (\xi_i - \xi_i(x))\vec{e_i}\rangle + o\big(\big|\sum_{i=1}^n (\xi_i - \xi_i(x))\vec{e_i}\big|\big)$ implying that it is reasonable to assign to f a vector $f - f(x) = \langle \nabla_x f,\, \sum_{i=1}^n (\xi_i - \xi_i(x))\vec{e_i}\rangle (\mod \mathfrak{o}_x)$. So, the function f induces a cotangent vector df at almost every point x, it implies that $\int_U df \wedge h$ is defined for every $U \subset |K|$, where h is of an $n-1$-form defined on every simplex as an exterior product of differentials of functions from $\mathscr{C}^\infty\mathscr{L}_c(|K|)$. Summarizing, we can define df over U as a functional $df(h) = -\int_U f dh$ which holds for every h $\int_U df \wedge h = -\int_U f dh$. As a result, locally almost everywhere we have a finitely generated L_∞-module and an epimorphism $L_\infty^n \to \Omega_{\mathscr{L}}$ induced by the map $df \mapsto \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n$, where $\Omega_{\mathscr{L}}$ is the L_∞-module of Lipschitz 1-form.

Definition 6. We use the following notation for L_p-norms on $\mathscr{L}_{\mathrm{loc}}(|K|)$:

- $f \in \mathscr{L}_{\mathrm{loc}}(|K|)$, $\|f\|_{L_p} = (\sum_{T \in K([n])} \int_T |f(x)|^p dx)^{\frac{1}{p}}$.
  In the case $p = \infty$, we put $\|f\|_{L_\infty} = \operatorname{ess\,sup} |f(x)|$.

Definition 7. We use the following notation for L_p-norms on spaces of differential forms on $|K|$:

- $\omega \in \bigwedge^k \Omega_{\mathscr{L}}(K)$, $\|\omega\|_{\Omega_{p,\,p}} = (\||\omega|\|^p + \||d\omega|\|^p)^{\frac{1}{p}}$;
- $\omega \in \bigwedge^k \Omega_{\mathscr{L}}(K)$, $\|\omega\|_{\Omega_{\infty,\,\infty}} = \max\{\||\omega|\|_{\Omega_\infty}, \||d\omega|\|_{\Omega_\infty}\}$;
- $\omega \in \bigwedge^k \Omega_{\mathscr{L}}(K)$, $\|\omega\|_{S\mathscr{L}_p} = \big(\sum_{T \in K([n])} \|\omega\|^p_{\Omega_{\infty,\,\infty}(T)}\big)^{\frac{1}{p}}$.

Definition 8. We define Sobolev spaces of differential forms on $|K|$ as follows: $\Omega^k_{p,\,p}(K) = \left(\overline{\bigwedge^k \Omega_{\mathscr{L}}(K)}\right)_{\Omega_{p,\,p}}$, i.e., $\Omega^k_{p,\,p}(K)$ is the closure of the graded module of Lipschitz forms with respect to the norm of Sobolev spaces.

Lemma 3. *Let Δ be a simplex and $\partial\Delta$ be its boundary. Then any $\omega \in \bigwedge^k \Omega_{\mathscr{L}}(\partial\Delta)$ can be extended to the whole Δ in such a way that $\tilde{\omega} \in \bigwedge^k \Omega_{\mathscr{L}}(\Delta)$ and $\|\tilde{\omega}\|_{\Omega^*_{p,\,p}(\Delta)} \leq \|\omega\|_{\Omega^*_{p,\,p}(\partial\Delta)}$.*

Proof. Let $I = [0, 1]$. Consider a Lipschitz form $\omega \in \bigwedge^k \Omega_{\mathscr{L}}(\partial I^n)$. Our aim is to define $\tilde{\omega} \in \bigwedge^k \Omega_{\mathscr{L}}(\partial I^n \times I)$ in such a manner that $\tilde{\omega}\big|_{\partial I^n} = \omega$ and $\|\tilde{\omega}\|_{\Omega^*_{p,\,p}(\partial I^n \times I)} \leq \|\omega\|_{\Omega^*_{p,\,p}(\partial I^n)}$.

Define a functions $f\colon I \to \mathbb{R}$ as $t \mapsto 1-t$. Then we can define $\tilde{\omega}$ as the following $\tilde{\omega}(x,\, t) = f(t)\omega(x)$

$$\|\tilde{\omega}\|^p_{\Omega^*_p(\partial I^n \times I)} = \int\limits_{\partial I^n \times I} |\tilde{\omega}|^p dx dt = \int\limits_I dt \int\limits_{\partial I^n} |f(t)|^p |\omega(x)|^p dx$$

$$= \int\limits_0^1 |1-t|^p dt \int\limits_{\partial I^n} |\omega(x)|^p dx \leq \frac{1}{(1+p)^p} \|\omega\|^p_{\Omega^*_p(\partial I^n)}$$

$$d\tilde{\omega} = df \wedge \omega + f d\omega = (1-t) d\omega(x) - dt \wedge \omega(x)$$

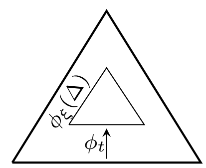
$$|d\tilde{\omega}| \leq |1-t||d\omega(x)| + |\omega(x)|$$

$$\|d\tilde{\omega}\|_{\Omega^*_p(\partial I^n \times I)} \leq \|\omega\|_{\Omega^*_p(\partial I^n \times I)} + \|(1-t) d\omega\|_{\Omega^*_p(\partial I^n \times I)}$$

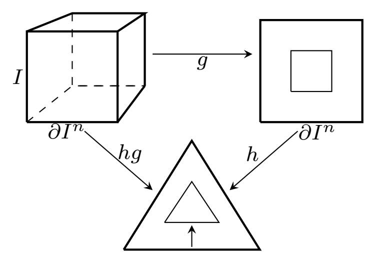
$$\|(1-t) d\omega\|^p_{\Omega^*_p(\partial I^n \times I)} = \int\limits_I dt \int\limits_{\partial I^n} |1-t|^p |d\omega(x)|^p dx = \frac{1}{(1+p)^p} \|d\omega\|^p_{\Omega^*_p(\partial I^n)}$$

$$\|d\tilde{\omega}\|_{\Omega^*_p(\partial I^n\times I)} \le \|\omega\|_{\Omega^*_p(\partial I^n)} + \|d\omega\|_{\Omega^*_p(\partial I^n)}$$
$$\|d\tilde{\omega}\|^p_{\Omega^*_p(\partial I^n\times I)} \le 2^{p-1}\|\omega\|^p_{\Omega^*_p(\partial I^n)} + 2^{p-1}\|d\omega\|^p_{\Omega^*_p(\partial I^n)}$$
$$\|\tilde{\omega}\|^p_{\Omega^*_p(\partial I^n\times I)} + \|d\tilde{\omega}\|^p_{\Omega^*_p(\partial I^n\times I)} \le 2^{p-1}\|\omega\|^p_{\Omega^*_p(\partial I^n)} + 2^{p-1}\|d\omega\|^p_{\Omega^*_p(\partial I^n)} + \|\omega\|^p_{\Omega^*_p(\partial I^n)}$$
$$\le (2^{p-1}+1)\|\omega\|^p_{\Omega^*_{p,\,p}(\partial I^n)}.$$

Assume that Δ is an n-dimensional simplex and $\omega \in \bigwedge^k \Omega_{\mathscr{L}}(\partial\Delta)$ is a Lipschitz form.



There exist Lipschitz map $\gamma\colon \partial I^n \to \partial\Delta$ and a couple of Lipschitz maps g, h as illustrated below



We can extend $\gamma^*\omega$ as is shown above. Then we put $\tilde{\omega} = (g^{-1}h^{-1})^*\tilde{\gamma^*\omega}$. Hence we have $\tilde{\omega}\big|_{\phi_\xi(\partial\Delta)} = 0$ and consider $\tilde{\omega}$ to be zero over $\phi_\xi(\Delta)$. □

Lemma 4. *Let K be an n-dimensional simplicial complex and $K[m]$ be its m-dimensional skeleton. Then any $\omega \in \Omega^*_{p,\,p}(K[m])$ can be extended to the whole K in such a way that $\tilde{\omega} \in \Omega^*_{p,\,p}(K)$ and $\|\tilde{\omega}\|_{\Omega^*_{p,\,p}(K)} \le \|\omega\|_{\Omega^*_{p,\,p}(K[m])}$.*

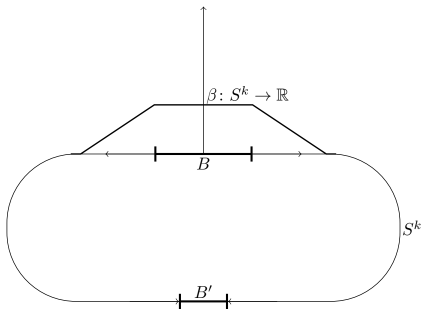
Proof. Suppose that $\omega \in \Omega^*_{p,\,p}(K[m])$ and there is $\{\omega_i\} \subset \bigwedge^k \Omega_{\mathscr{L}}(K[m])$ such that $\|\omega_i - \omega\|_{\Omega^*_{p,\,p}(K[m])} \to 0$ as $i \to \infty$.

In the light of previous lemma, there exists $\tilde{\omega_i} \in \Omega^*_{p,\,p}(K[m+1])$ which satisfies the following estimation: $\|\tilde{\omega_i}\|_{\Omega^*_{p,\,p}(K[m+1])} \le \|\omega_i\|_{\Omega^*_{p,\,p}(K[m])}$ for each i. It is not hard to see that $\{\tilde{\omega_i}\}$ is a Cauchy sequence since the procedure of extension is linear: $\|\tilde{\omega_i} - \tilde{\omega_j}\|_{\Omega^*_{p,\,p}(K[m+1])} \to 0$, $i, j \to \infty$. So, $\lim \|\tilde{\omega_i}\|_{\Omega^*_{p,\,p}(K[m+1])} \le \lim \|\omega_i\|_{\Omega^*_{p,\,p}(K[m])} = \|\omega\|_{\Omega^*_{p,\,p}(K[m])}$. Denote a limit of the sequence as the following $\lim \tilde{\omega_i} = \tilde{\omega}$. Repeating this construction for every dimension as a result we obtain an extension to the whole complex. □

Remark 3. It is not hard to see that the same argument holds for $S\mathscr{L}_p(K)$.

Lemma 5. *Let S^k be a k-sphere and $B \subset S^k$ be a k-ball. Any Lipschitz k-form $\omega \in \Omega^k(B)$ can be extended by zero to the Lipschitz k-form on S^k.*

Proof. There is a homotopy $\varphi\colon S^k \times [0,\,1] \to S^k$ such that $\varphi(x,\,0) = \mathrm{Id}$ and $\varphi(B,\,1) = B'$, where $B \cap B' = \varnothing$.



So, we can extend any k-form $\omega \in \Omega^k(B)$ by a zero k-form to the whole S^k. Define first ω over B' as follows: $\tilde{\omega} = (\varphi_{t=1}^{-1})^*\omega$. Let $\omega = f^0 df^1 \wedge \cdots \wedge df^k$. Then for every $a \in \partial B$, we can put $f^i(\varphi(a,t)) = f^i(a)$. Thus as a result, we have the following. Let v_1 be a tangent vector to the curve $\varphi(a, t)\colon [0, 1] \to S^k$. Consider a basis $v_1, \dots, v_k$. Then ∇f^i has the zero component corresponding to the direction v_1. So, we obtain $df^i = f^i_{v_2} v_2^* + \cdots + f^i_{v_k} v_k^*$ and as a result, $df^1 \wedge \cdots \wedge df^k = 0$. Now, let $\beta\colon S^k \to \mathbb{R}$ be a 'bump' function such that $\beta(B) = \{1\}$ and $\operatorname{supp}(\beta) \subset S^k \setminus B'$. Hence $\operatorname{supp}(\beta\tilde{\omega}) \subseteq B$ and $d(\beta\tilde{\omega}) = 0$. □

5. De Rham Operators on Simplicial Complexes

Theorem 5. *Let K be a complex of bounded geometry with $L = 1$. Then there is the diagram*

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{d} & \Omega^{k-1}_{p,p}(K) & \xrightarrow{d} & \Omega^{k}_{p,p}(K) & \xrightarrow{d} & \Omega^{k+1}_{p,p}(K) & \xrightarrow{d} \cdots \\
& \downarrow \mathscr{R} & \swarrow \mathscr{A} & \downarrow \mathscr{R} & \swarrow \mathscr{A} & \downarrow \mathscr{R} & \\
\cdots & S\mathscr{L}^{k-1}_p(K) & & S\mathscr{L}^{k}_p(K) & & S\mathscr{L}^{k+1}_p(K) & \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \xrightarrow{d} & \Omega^{k-1}_{p,p}(K) & \xrightarrow{d} & \Omega^{k}_{p,p}(K) & \xrightarrow{d} & \Omega^{k+1}_{p,p}(K) & \xrightarrow{d} \cdots
\end{array}$$

with commutative squares in the category of Banach spaces Ban_∞. Moreover, the equality $\mathscr{R} - \mathrm{Id}_{\Omega^*_{p,p}} = d\mathscr{A} + \mathscr{A}d$ holds.

In order to prove the above theorem, we state a number of lemmas about the arrows of the diagram. Let K be a star-bounded complex. Assume that K' is the first barycentric subdivision of K. Let Σ'_i be the star of vertex e_i in K'. Let φ_i be a bi-Lipschitz homeomorphism $\varphi_i\colon \operatorname{Int}\Sigma_i \to U$ such that $\mathbf{B_1} \subset U$ and $\Sigma'_i \subset \operatorname{Int}\varphi^{-1}(\mathbf{B}_1)$.

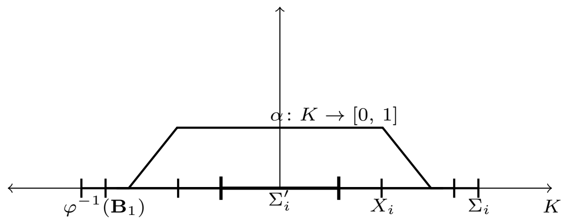
Given $\varepsilon > 0$, define operators $\mathcal{R}_i$ and $\mathcal{A}_i$,

$$\mathcal{R}_i\omega = \begin{cases} (\varphi_i^{-1})^*\mathcal{R}_\varepsilon\varphi_i^*\omega & \text{on } \Sigma_i \\ \omega, & \text{otherwise} \end{cases}; \qquad \mathcal{A}_i\omega = \begin{cases} (\varphi_i^{-1})^*\mathcal{A}_\varepsilon\varphi_i^*\omega & \text{on } \Sigma_i \\ 0, & \text{otherwise} \end{cases}$$

Consider the operators $\mathscr{R}\omega = \lim\limits_{i\to\infty} \mathcal{R}_1\mathcal{R}_2\dots\mathcal{R}_i\omega$ and $\mathscr{A}\omega = \sum\limits_{i=1}^{\infty} \mathcal{R}_1\mathcal{R}_2\dots\mathcal{R}_{i-1}\mathcal{A}_i\omega$.

Lemma 6. *The arrow $\Omega^*_{p,p}(K) \xrightarrow{\mathscr{R}} S\mathscr{L}^k_p(K)$ is a morphism in the category Ban_∞, namely, $\mathscr{R}$ is a bounded operator.*

Proof. Consider a star Σ_i of K. Assume that there is a set X_i such that $\Sigma'_i \subset \operatorname{Int} X_i \subset \operatorname{Int}\varphi^{-1}(\mathbf{B}_1)$. We can represent ω as a sum $\omega = \omega_1 + \omega_2$,



where $\omega_1 = \alpha\omega$ and $\omega_2 = (1-\alpha)\omega$, i.e., $\operatorname{supp}(\omega_2) \subset K \setminus X_i$.

For any $\mathcal{R}_j$ and $\eta \in \Omega^k(K)$ such that $\operatorname{supp}(\eta) \subset K \setminus X_i$, choosing $\varepsilon > 0$ sufficiently small, we can achieve $\operatorname{supp}(\mathcal{R}_j\eta) \subset K \setminus \Sigma_i'$ that implies $R_j\eta = 0$ on Σ_i'. Due to this fact, for each j, we can choose ε_j in the definition of the operator $\mathcal{R}_j$ in such a way that $\operatorname{supp}(\mathcal{R}_1 \dots \mathcal{R}_j\omega_2) \subset K \setminus \Sigma_i'$ and, correspondingly, $\mathscr{R}\omega = \mathscr{R}\omega_1$ on Σ_i'.

Let Σ_i be spanned by points $\{e_{j_1}, \dots, e_{j_n}\}$. For every form θ such that $\operatorname{supp}(\theta) \subset \varphi^{-1}(\mathbf{B}_1) \subset \Sigma_\mathrm{i}$, we can see that only for $k \in \{j_1, \dots, j_n\}$ the operator $\mathcal{R}_k$ is distinct from the identity. Choosing sufficiently small ε each time we face such an operator $R_k \in \{R_{j_1} \dots, R_{j_n}\}$ in the composition $\mathcal{R}_1\mathcal{R}_2 \dots \mathcal{R}_j$ we obtain a map preserving the support of a form derived at this step inside $\operatorname{Int}\Sigma_i$. Then we have $\mathcal{R}_1 \dots \mathcal{R}_j\omega_1 = \mathcal{R}_{j_1} \dots \mathcal{R}_{j_n}\omega_1$ for each j. So, $\mathscr{R}\omega_1 = \mathcal{R}_{j_1} \dots \mathcal{R}_{j_n}\omega_1$.

We know that $\mathcal{R}_{j_k} \colon \Omega_p^*(\Sigma_i) \to \Omega_p^*(\Sigma_i)$ and, moreover, the operator $\|\mathcal{R}_{j_k,(\varepsilon)}\|_p \leq 1 + \varepsilon$, $\varepsilon \to 0$. Then there exists $\varepsilon > 0$ such that $\|\mathscr{R}\|_p = \|\mathcal{R}_{j_1} \dots \mathcal{R}_{j_n}\|_p \leq 1 + O(\varepsilon)$, $\varepsilon \to 0$. It should be noted that $d\mathscr{R} = \mathscr{R}d$ and the above argument holds for $d\omega$. As a result, for each i, we have $\|\operatorname{res}_{\Sigma_i', K} \circ \mathscr{R}\omega\|_{\Omega^k_{p,p}(\Sigma_i')} \leq (1 + O(\varepsilon_i))\|\operatorname{res}_{\Sigma_i, K}\omega\|_{\Omega^k_{p,p}(\Sigma_i)}$, $\varepsilon_i \to 0$. Due to the star-boundedness of the complex, we can choose $\varepsilon = \min_i \varepsilon_i$. Assume that $\omega \in \Omega^k_{p,p}(K)$,

$$\begin{aligned}\|\mathscr{R}\omega\|_{\Omega^k_{p,p}(K)} = \sum_i \|\operatorname{res}_{\Sigma_i', K} \circ \mathscr{R}\omega\|_{\Omega^k_{p,p}(K)} &\leq \sum_i (1 + O(\varepsilon))\|\operatorname{res}_{\Sigma_i, K}\omega\|_{\Omega^k_{p,p}(\Sigma_i)} \\ &\leq \frac{(1+O(\varepsilon))}{n}\|\omega\|_{\Omega^k_{p,p}(K)}\end{aligned}$$

In the light of what we have just said, $\mathscr{R}$ is a bounded map $\mathscr{R} \colon \Omega^*_{p,p}(K) \to \Omega^*_{p,p}(K)$. Moreover, it is not hard to see that $\mathscr{R} \colon \Omega^*_{p,p}(K) \to S\mathscr{L}_p^*(K)$. Indeed, we know that every $\mathcal{R}_i = (\varphi_i^{-1})^*\mathcal{R}_\varepsilon\varphi_i^*$, then $\mathcal{R}_\varepsilon\varphi_i^* \colon \Omega^*_{p,p}(\Sigma_i) \to \Omega^*_{\text{smooth}}(U)$, and $(\varphi_i^{-1})^*$ is a Lipschitz piecewise smooth map. □

Lemma 7. *For every m-dimensional skeleton $K[m]$ of K, the operator $\operatorname{res}_{K[m], K} \circ \mathscr{R}$ is a morphism $\Omega^*_{p,p}(K) \to S\mathscr{L}_p^k(K[m])$ in Ban_∞ (a bounded operator).*

Proof. From now on, we will follow the notation stated in the proof of Lemma 2. In particular, let a star Σ_i be spanned by points $\{e_{j_1}, \dots, e_{j_n}\}$. Every n-dimensional simplex σ can be covered by stars of K': $\sigma \subset \bigcup_{k=0}^{n-1} \Sigma'_{j_k}$ applying Lemma 3 and the argument from the proof of Lemma 2, we arrive at

$$\|\operatorname{res}_{\sigma, K} \circ \mathscr{R}\omega\|_{\Omega^*_\infty(\sigma)} \leq \sum_{k=0}^n \|\operatorname{res}_{\Sigma'_{j_k}, K} \circ \mathscr{R}\omega\|_{\Omega^*_\infty(\Sigma'_{j_k})} \leq C\sum_{k=0}^n \|\operatorname{res}_{\Sigma_{j_k}, K}\omega\|_{\Omega^*_p(\Sigma_{j_k})}$$

and

$$\begin{aligned}\sum_i \|\operatorname{res}_{\sigma_i, K} \circ \mathscr{R}\omega\|_{\Omega^*_\infty(\sigma_i)} &\leq C\sum_i \sum_{k=0}^n \|\operatorname{res}_{\Sigma^i_{j_k}, K}\omega\|_{\Omega^*_p(\Sigma^i_{j_k})} \\ \leq CN\sum_j \|\operatorname{res}_{\Sigma_j, K}\omega\|_{\Omega^*_p(\Sigma_j)} &\leq CNn\sum_i \|\operatorname{res}_{\sigma_i, K}\omega\|_{\Omega^*_p(\sigma_i)} = C'\|\omega\|_{\Omega^*_p(K)}.\end{aligned}$$

Hence

$$\begin{aligned}\|\operatorname{res}_{K[m], K} \circ \mathscr{R}\omega\|_{\Omega^k_p(K[m])} &= \sum_i \|\operatorname{res}_{\tau_i, K} \circ \mathscr{R}\omega\|_{\Omega^k_p(\tau_i)} \\ &\leq \sum_i (\operatorname{mes}\tau_i)^{\frac{1}{p}}\|\operatorname{res}_{\tau_i, K} \circ \mathscr{R}\omega\|_{\Omega^k_\infty(\tau_i)}.\end{aligned}$$

It is not hard to see that $\|\mathrm{res}_{\tau_i, K} \circ \mathscr{R}\omega\|_{\Omega^k_\infty(\tau_i)} \leq \sum_{\sigma \in K[n],\, \tau_i \hookrightarrow \sigma} \|\mathrm{res}_{\sigma, K} \circ \mathscr{R}\omega\|_{\Omega^k_\infty(\sigma)}$. As a result, we have

$$\|\mathrm{res}_{K[m], K} \circ \mathscr{R}\omega\|_{\Omega^k_p(K[m])} \leq \left(\frac{\sqrt{m+1}}{m!\sqrt{2^m}}\right)^{\frac{1}{p}} \sum_i \sum_{\sigma \in K[n],\, \tau_i \hookrightarrow \sigma} \|\mathrm{res}_{\sigma, K} \circ \mathscr{R}\omega\|_{\Omega^k_\infty(\sigma)}$$

$$\leq \left(\frac{\sqrt{m+1}}{m!\sqrt{2^m}}\right)^{\frac{1}{p}} \binom{n+1}{m+1} C' \|\omega\|^p_{\Omega^*_p(K)}.$$

Here, we again make a note that $d\mathscr{R} = \mathscr{R}d$ and the above argument holds for $d\omega$. ☐

Lemma 8. *The arrow* $\Omega^k_{p,\,p}(K) \xrightarrow{\mathscr{A}} \Omega^{k-1}_{p,\,p}(K)$ *is a morphism in the category* Ban_∞.

Proof. Let $\omega \in \Omega^k_{p,\,p}(M)$, consider $\mathcal{A}_i\omega$. It is not hard to see that $\mathrm{supp}(\mathcal{A}_i\omega) \subset \varphi^{-1}(\mathbf{B}_1)$. Indeed, $\mathfrak{s}^*_{tv}$ acts on the complement $K \setminus \varphi^{-1}(\mathbf{B}_1)$ leaving its points fixed. It follows that $\mathfrak{X}_v = 0$ and, consequently, $\iota_{\mathfrak{X}_v}$ is a zero map at every point belonging to $K \setminus \varphi^{-1}(\mathbf{B}_1)$.

Let Σ_i be spanned by points $\{e_{j_1}, \dots, e_{j_n}\}$. For every form, compactly supported inside $\varphi^{-1}(\mathbf{B}_1) \subset \Sigma_i$, there are only a fixed number of operators, namely, $\mathcal{R}_{j_1}, \dots, R_{j_n}$, which are distinct from the identity. Similarly to the above, we can choose sufficiently small ε for each operator R_{j_k} in the composition $\mathcal{R}_1 \dots \mathcal{R}_{i-1}\mathcal{A}_i$, that allows us to preserve the support of a form derived by every partial composition inside $\mathrm{Int}\,\Sigma_i$. Then we have $\mathcal{R}_1 \dots \mathcal{R}_{i-1}\mathcal{A}_i\omega = \mathcal{R}_{j_1} \dots \mathcal{R}_{j_n}\mathcal{A}_i\omega$ and $\mathcal{R}_{j_1} \dots R_{j_n}\mathcal{A}_i\omega = 0$ outside Σ_i. Now, we can estimate the norm $\|\mathscr{A}\omega\|_{\Omega^{k-1}_p(K)}$:

$$\|\mathcal{R}_{j_1} \dots R_{j_n}\mathcal{A}_i\omega\|_{\Omega^{k-1}_p(\Sigma_i)} \leq C^n_p M_p \|\omega\|_{\Omega^k_p(\Sigma_i)}.$$

So, we have

$$\|\mathscr{A}\omega\|^p_{\Omega^{k-1}_{p,\,p}(K)} = \|\mathscr{A}\omega\|^p_{\Omega^{k-1}_{p,\,p}(K)} + \|d\mathscr{A}\omega\|^p_{\Omega^{k-1}_{p,\,p}(K)}$$

$$= \int_K \left|\sum_i \mathcal{R}_1 \dots R_{i-1}\mathcal{A}_i\omega\right|^p dx + \int_K \left|\sum_i d\mathcal{R}_1 \dots R_{i-1}\mathcal{A}_i\omega\right|^p dx$$

$$= \sum_{\sigma \in K[n]} \left(\int_\sigma \left|\sum_i \mathrm{res}_{\sigma, K}\mathcal{R}_1 \dots R_{i-1}\mathcal{A}_i\omega\right|^p dx\right.$$

$$\left. + \int_\sigma \left|\sum_i \mathrm{res}_{\sigma, K} d\mathcal{R}_1 \dots R_{i-1}\mathcal{A}_i\omega\right|^p dx\right).$$

Every simplex $\sigma = \{e_0, \dots, e_n\}$ is the intersection of stars assigned to its vertices $\sigma = \bigcap_{e_i \in \{e_0, \dots, e_n\}} \Sigma_i$ and, moreover, σ lies in no other star. It follows that $\sum_i \mathrm{res}_{\sigma, K}\mathcal{R}_1 \dots R_{i-1}\mathcal{A}_i\omega = \sum_{e_j \in \{e_0, \dots, e_n\}} \mathrm{res}_{\sigma, K}\mathcal{R}_1 \dots R_{j-1}\mathcal{A}_j\omega$ and $\sum_i \mathrm{res}_{\sigma, K} d\mathcal{R}_1 \dots R_{i-1}\mathcal{A}_i\omega = \sum_{e_j \in \{e_0, \dots, e_n\}} \mathrm{res}_{\sigma, K} d\mathcal{R}_1 \dots R_{j-1}\mathcal{A}_j\omega$. Hence

$$\|\mathscr{A}\omega\|^p_{\Omega^{k-1}_{p,\,p}(K)} = \sum_{\sigma \in K[n]} \int_\sigma \left|\sum_{e_j \in \{e_0, \dots, e_n\}} \mathrm{res}_{\sigma, K}\mathcal{R}_1 \dots R_{j-1}\mathcal{A}_j\omega\right|^p dx$$

$$+ \sum_{\sigma \in K[n]} \int_\sigma \left|\sum_{e_j \in \{e_0, \dots, e_n\}} \mathrm{res}_{\sigma, K} d\mathcal{R}_1 \dots R_{j-1}\mathcal{A}_j\omega\right|^p dx$$

$$= (n+1)^{p-1} \sum_{\sigma \in K[n]} \int_\sigma \sum_{e_j \in \{e_0, \dots, e_n\}} |\mathrm{res}_{\sigma, K}\mathcal{R}_1 \dots R_{j-1}\mathcal{A}_j\omega|^p dx$$

$$+ (n+1)^{p-1} \sum_{\sigma \in K[n]} \int_\sigma \sum_{e_j \in \{e_0, \dots, e_n\}} |\mathrm{res}_{\sigma, K} d\mathcal{R}_1 \dots R_{j-1}\mathcal{A}_j\omega|^p dx.$$

Let us check the following:

$$\int_\sigma \sum_{e_j \in \{e_0, \dots, e_n\}} |\mathrm{res}_{\sigma, K}\mathcal{R}_1 \dots R_{j-1}\mathcal{A}_j\omega|^p dx$$

$$
\begin{aligned}
&= \sum_{e_j \in \{e_0, \dots, e_n\}} \int_{\sigma} |\mathrm{res}_{\sigma, K} \mathcal{R}_1 \dots R_{j-1} \mathcal{A}_j \omega|^p dx \\
&\leq \sum_{e_j \in \{e_0, \dots, e_n\}} \int_{\Sigma_j} |\mathcal{R}_1 \dots R_{j-1} \mathcal{A}_j \omega|^p dx \\
&= \sum_{e_j \in \{e_0, \dots, e_n\}} \int_{\Sigma_j} |\mathcal{R}_{j_0} \dots R_{j_n} \mathcal{A}_j \omega|^p dx \\
&= \sum_{e_j \in \{e_0, \dots, e_n\}} \|\mathcal{R}_{j_0} \dots R_{j_n} \mathcal{A}_j \omega\|^p_{\Omega^{k-1}_p(\Sigma_j)} \\
&\leq \sum_{e_j \in \{e_0, \dots, e_n\}} C^n_p M_p \|\omega\|_{\Omega^k_p(\Sigma_j)}.
\end{aligned}
$$

That implies

$$
\begin{aligned}
&\sum_{\sigma \in K[n]} \int_{\sigma} \sum_{e_j \in \{e_0, \dots, e_n\}} |\mathrm{res}_{\sigma, K} \mathcal{R}_1 \dots R_{j-1} \mathcal{A}_j \omega|^p dx \\
&\leq \sum_{\sigma \in K[n]} \sum_{e_j \in \{e_0, \dots, e_n\}} C' \|\omega\|_{\Omega^k_p(\Sigma_j)} \leq C'(n+1) \|\omega\|_{\Omega^k_p(K)}.
\end{aligned}
$$

Similarly, we can estimate $\|d\mathscr{A}\omega\|^p_{\Omega^{k-1}_{p,\,p}(K)}$. As a result, we have $\|\mathscr{A}\omega\|^p_{\Omega^{k-1}_{p,\,p}(K)} \leq C'' \|\omega\|^p_{\Omega^k_{p,\,p}(K)}$. □

Summing up the results of preceding lemmas, we can conclude that Theorem 4 holds.

Theorem 6. *There is an isomorphism $H^k(S\mathscr{L}^*_p(K)) \cong H^k(\Omega^*_{p,\,p}(K))$ in the category $\mathsf{Vec}_{\mathbb{R}}$.*

Proof. Let ω be a form lying in $\Omega^*_{p,\,p}(K)$ such that $\omega \in \ker d$. Due to the existence of homotopy between $\mathscr{R}$ and $\mathrm{Id}_{\Omega^*_{p,\,p}(K)}$, we can see that $\mathscr{R}\omega \in S\mathscr{L}^*_p(K)$ and ω belongs to the same cohomologycal class in $H^k(\Omega^*_{p,\,p}(K))$. □

6. De Rham Theorem

Let K be an n-dimension simplicial complex and $S\mathscr{L}^*_p(K)$ be a chain complex of differential forms of Lipschitz-Sullivan's type on K. Put the following $\mathscr{I} \colon \omega \mapsto f_\omega(\sigma) = \int_\sigma \omega$. In order to prove the lemma below, we introduce an $\mathbb{R}$-linear map $\mathscr{W} : C^k_p(K) \to S\mathscr{L}^k_p(K)$. Due to Hassler Whitney's works, where such transformation was introduced (see e.g., [10]), an object $\mathrm{im}\,\mathscr{W}$ is usually called the set of Whitney's forms.

Lemma 9. *Let K be a simplicial complex, there exists a short strictly exact sequence of the following type in the category of Banach spaces Ban_∞:*

$$
0 \longrightarrow \ker^k(\mathscr{I}) \longrightarrow S\mathscr{L}^k_p(K) \xrightarrow{\mathscr{I}} C^k_p(K) \longrightarrow 0 .
$$

In other words, the map $\mathscr{I}$ is a split epimorphism $S\mathscr{L}^k_p(K) \cong C^k_p(K) \oplus \ker^k(\mathscr{I})$.

Proof. First of all, we confirm that $\mathscr{I}$ is, in fact, a morphism between L_p-spaces. Indeed,

$$
\begin{aligned}
\|\mathscr{I}\omega\|^p_{C^k_p(K)} &= \sum_{\tau \in K[k]} \left| \int_\tau \omega \right|^p \leq \sum_{\tau \in K[k]} \left(\int_\tau |\omega| \right)^p \\
&\leq \{\mathrm{mes}\,\tau\}^{\mathrm{p}-1} \left(\sum_{\tau \in K[k]} \int_\tau |\omega|^p \right) \leq \left(\frac{\sqrt{k+1}}{k!\sqrt{2^k}} \right)^{p-1} \|\omega\|^p_{S\mathscr{L}^k_p(K)}.
\end{aligned}
$$

Assume $c \in C_p^k(K)$, let us represent c as a sum in the Banach space which converges absolutely $c = \sum_{\sigma\in K([k])} c(\sigma)\chi_\sigma$, where

$$\chi_\sigma(\sigma') = \begin{cases} 1, & \text{if } \sigma' = \sigma; \\ 0, & \text{otherwise,} \end{cases}$$

that is, $\sum_{\sigma\in K([k])} |c(\sigma)|^p < \infty$. In other words, we can define $C_p^k(K)$ as a closure of the $\mathbb{R}$-vector space $C_c^k(K)$ generated by elements $\{\chi_\sigma\}_{\sigma\in K([k])}$, or equivalently a closure of $\mathbb{R}$-vector space of compactly supported cochains.

Define $\mathscr{W}$ as follows. First, let us write it down for elements of basis $\mathscr{W}(\chi_\sigma) = k!\sum_{i=0}^k (-1)^i t_i dt_0\wedge\cdots\wedge dt_{i-1}\wedge dt_{i+1}\wedge\cdots\wedge dt_k$, where $\{t_i\}$ is a set of barycentric coordinates assigned to vertices of the simplex σ. The $\operatorname{supp}(\mathscr{W}(\chi_\sigma))$ contains only one simplex belonging k-skeleton of K, namely, σ. It implies that any sum $\mathscr{W}(\chi_\sigma)+\mathscr{W}(\chi_{\sigma'})$ can be uniquely restricted to every k-simplex. Then we can extend our map to the space $C_c^k(K)$ by linearity. It is clear that $\mathscr{W}(C_c^k(K))$ is a set of compactly supported differential forms and, moreover, the support of every form $\mathscr{W}(\chi_\sigma)$ contains at most N simplexes due to the star-boundedness of the complex. Let us check that $\mathscr{W}\colon C_c^k(K) \to S\mathscr{L}_p^k(K)$ is a bounded map. Assume that $c = \sum_{i=1}^m c(\tau_i)\chi_{\tau_i}$ then $\|\mathscr{W}(c)\|^p_{\Omega_{\infty,\infty}(\sigma)} \le \binom{n+1}{k+1}^{p-1}\sum_{i=1}^m |c(\tau_i)|^p \|\mathscr{W}(\chi_{\tau_i})\|^p_{\Omega_{\infty,\infty}(\sigma)}$. As mentioned above, there are only a finite number q of simplexes belonging to the support $\bigcup_{i=1}^m \operatorname{supp}(\mathscr{W}(\chi_{\tau_i}))$ of $\mathscr{W}(c)$.

$$\begin{aligned} \|\mathscr{W}(c)\|^p_{SL_p(K)} &= \sum_{j=1}^q \|\mathscr{W}(c)\|^p_{\Omega_{\infty,\infty}(\sigma_j)} \\ &\le \binom{n+1}{k+1}^{p-1} \sum_{j=1}^q\sum_{i=1}^m |c(\tau_i)|^p \|\mathscr{W}(\chi_{\tau_i})\|^p_{\Omega_{\infty,\infty}(\sigma_j)} \end{aligned}$$

It is not hard to see that there is a constant C such that $\|\mathscr{W}(\chi_{\tau_i})\|^p_{\Omega_{\infty,\infty}(\sigma_j)} \le C$, if τ_i is a k-face of σ_j and $\|\mathscr{W}(\chi_{\tau_i})\|^p_{\Omega_{\infty,\infty}(\sigma_j)} = 0$, otherwise. As a result, we have

$$\begin{aligned} \sum_{j=1}^q\sum_{i=1}^m |c(\tau_i)|^p \|\mathscr{W}(\chi_{\tau_i})\|^p_{\Omega_{\infty,\infty}(\sigma_j)} &= \sum_{i=1}^m |c(\tau_i)|^p \sum_{j=1}^q \|\mathscr{W}(\chi_{\tau_i})\|^p_{\Omega_{\infty,\infty}(\sigma_j)} \\ &\le NC\sum_{i=1}^m |c(\tau_i)|^p. \end{aligned}$$

Hence the map $\mathscr{W}\colon C_c^k(K) \to S\mathscr{L}_p^k(K)$ possesses the following property: $\|\mathscr{W}(c)\|^p_{S\mathscr{L}_p(K)} \le NC\binom{n+1}{k+1}^{p-1} \|c\|_{C_p^k(K)}$. Now, we can extend $\mathscr{W}$ to the closure $\overline{(C_c^k(K))}_{\|\|_{C_p^k}} = C_p^k(K)$ by

$$\mathscr{W}(c) = \sum_{\sigma\in K([k])} c(\sigma)\mathscr{W}(\chi_\sigma) = \lim_{i\to\infty} \mathscr{W}(c_i),$$

as $c_i \xrightarrow[i\to\infty]{} c$, $\{c_i\} \subset C_c^k(K)$, and it follows that $\mathscr{W}\colon C_p^k(K) \to S\mathscr{L}_p^k(K)$.

It remains to verify that $\mathscr{I}$ is a retraction. First, let $\sigma \in K[k]$. It is clear that $\operatorname{supp}(\mathscr{W}(\chi_\sigma)) \cap K[k] = \{\sigma\}$ and so, $\langle \mathscr{I}\circ\mathscr{W}(\chi_\sigma)\rangle(\sigma') = 0$, if $\sigma' \ne \sigma$. Let $\mathscr{W}(\chi_\sigma) = k!\sum_{i=0}^k(-1)^i t_i dt_0\wedge\cdots\wedge dt_{i-1}\wedge dt_{i+1}\wedge\cdots\wedge dt_k$ so, we have $t_0 = 1 - \sum_{i=1}^k t_i$ inside σ. As a result, we can write

$$\begin{aligned} \frac{1}{k!}\int_\sigma \mathscr{W}(\chi_\sigma) &= \int_\sigma (1 - \sum_{i=1}^k t_i) dt_1\wedge\cdots\wedge dt_k \\ &+ \int_\sigma \sum_{i=1}^k (-1)^i t_i d(1 - \sum_{j=1}^k t_j)\wedge\cdots\wedge dt_{i-1}\wedge dt_{i+1}\wedge\cdots\wedge dt_k \end{aligned}$$

$$
\begin{aligned}
&= \int\limits_\sigma (1 - \sum_{i=1}^{k} t_i) dt_1 \wedge \cdots \wedge dt_k \\
&+ \int\limits_\sigma \sum_{i=1}^{k} (-1)^{i+1} t_i d(\sum_{j=1}^{k} t_j) \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_k \\
&= \int\limits_\sigma (1 - \sum_{i=1}^{k} t_i) dt_1 \wedge \cdots \wedge dt_k \\
&+ \int\limits_\sigma \sum_{i=1}^{k} (-1)^{i+1} t_i dt_i \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_k \\
&= \int\limits_\sigma dt_1 \wedge \cdots \wedge dt_k = \frac{\sqrt{k+1}}{k!\sqrt{2^k}},
\end{aligned}
$$

since

$$dt_i \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \wedge \cdots \wedge dt_k = (-1)^{i-1} dt_1 \wedge \cdots \wedge dt_{i-1} \wedge dt_i \wedge dt_{i+1} \wedge \cdots \wedge dt_k.$$

Hence we have $\mathscr{I} \circ \mathscr{W}(\chi_\sigma) = \frac{\sqrt{k+1}}{\sqrt{2^k}} \chi_\sigma$ and so, $\mathscr{I} \circ \mathscr{W}(c) = \frac{\sqrt{k+1}}{\sqrt{2^k}} c$ holds for $c \in C_p^k(K)$.

Now we can denote $\tilde{\mathscr{W}} = \frac{\sqrt{2^k}}{\sqrt{k+1}} \mathscr{W}$ that implies that the following diagram

$$
\begin{array}{ccc}
C_p^k(K) & \xrightarrow{\tilde{\mathscr{W}}} & S\mathscr{L}_p^k(K) \\
 & \searrow^{\mathrm{Id}_{C_p^k(K)}} & \downarrow \mathscr{I} \\
 & & C_p^k(K)
\end{array}
$$

commutes and $\mathscr{I}$ is a retraction of the morphism $\tilde{\mathscr{W}}$. □

Lemma 10. *In the following diagram in the category* $\mathsf{Vec}_{\mathbb{R}}$

$$
\begin{array}{ccccc}
 & & \overset{0}{\frown} & & \\
\mathrm{im}(d^k) & \longrightarrow & S\mathscr{L}_p^{k+1}(K) & \xrightarrow{d^{k+1}} & S\mathscr{L}_p^{k+2}(K) \\
 & \nwarrow & \uparrow & & \\
 & & \ker(\mathscr{I}) & &
\end{array}
$$

the embedding of $\ker(\mathscr{I})$ *into* $S\mathscr{L}_p^{k+1}(K)$ *factors through* $\mathrm{im}(d^k)$.

Proof. Assume that $\omega \in \ker(\mathscr{I})$ is k-form. Let σ be a k-dimensional simplex. Consider a complex

$$0 \longrightarrow S\mathscr{L}_p^0(\sigma, \partial\sigma) \xrightarrow{d} \dots \xrightarrow{d} S\mathscr{L}_p^{k-1}(\sigma, \partial\sigma) \xrightarrow{d} S\mathscr{L}_p^k(\sigma, \partial\sigma) \longrightarrow 0 .$$

It is known that

$$H^i(S\mathscr{L}_p^*(\sigma,\, \partial\sigma)) = \begin{cases} \mathbb{R} & \text{for} \quad i = k, \\ 0 & \text{for} \quad i < k. \end{cases}$$

In particular, we can say that the cohomology group $H^k(\Omega_p^*(\sigma,\, \partial\sigma))$ consists of elements of the type $[c\omega]$, where ω is a cocycle such that $\int_\sigma \omega \neq 0$ and $c \in \mathbb{R}$. To put it otherwise, every closed form θ can be presented as below $\theta = c\omega + d\eta$, $c = \frac{\int_\sigma \theta}{\int_\sigma \omega}$.

Hence we can say that $\mathrm{im}(d^{k-1})$ is precisely a set of forms which have zero integral over σ. Since those forms constitute a closed set, so do the elements of $\mathrm{im}(d^{k-1})$. Thereby the following map $d\colon S\mathscr{L}_p^{k-1}(\sigma,\, \partial\sigma) \to \mathrm{im}(d^{k-1})$ is an epimorphism in the category Ban_∞ and by the Banach inverse operator theorem, there exists a constant C such that for $d\colon \eta \mapsto \mathrm{res}_{\sigma,\,\mathrm{K}}\omega$, we have the estimation $\|\eta\|_{\Omega_\infty} \le C \|\mathrm{res}_{\sigma,\,\mathrm{K}}\omega\|_{\Omega_\infty}$.

To summarize, for every $\sigma \in K[k]$, there exists $\eta_\sigma \in S\mathscr{L}_p^{k-1}(\sigma, \partial\sigma)$ such that $\mathrm{res}_{\sigma, K}\omega = d\eta_\sigma$. Moreover, for the set of forms $\{\eta_\sigma\}_{\sigma \in K[k]}$, holds $\mathrm{res}_{\sigma'\cap\sigma, \sigma}\eta_\sigma = \mathrm{res}_{\sigma'\cap\sigma, \sigma'}\eta_{\sigma'} = 0$ for any couple of simplexes $\sigma, \sigma' \in K[k]$. As a result, we have a $\mathrm{res}_{K[k], K}\omega = d\eta$, where $\mathrm{res}_{\sigma, K[k]}\eta = \eta_\sigma$ and

$$\|\eta\|^p_{S\mathscr{L}_p^*(K[k])} = \sum_{\sigma \in K[k]} \|\eta_\sigma\|^p_{\Omega^*_\infty(\sigma)} \leq C \sum_{\sigma \in K[k]} \|\mathrm{res}_{\sigma, K}\omega\|^p_{\Omega^*_\infty(\sigma)} \leq C\|\omega\|^p_{S\mathscr{L}_p^k(K)}.$$

Then we have a bounded map $\gamma\colon S\mathscr{L}_p^k(K) \to S\mathscr{L}_p^{k-1}(K[k], K[k-1])$. It is known that for every simplex $\delta \in K[k+1]$ and every form $\alpha \in S\mathscr{L}_p^k(\partial\delta)$, there is a morphism of normed spaces $s\colon S\mathscr{L}_p^k(\partial\delta) \to S\mathscr{L}_p^k(\delta)$ continuing the forms off the boundary of simplex to its interior.

In the light of previous steps, it was, in fact, established that $s\gamma\colon S\mathscr{L}_p^k(K) \to S\mathscr{L}_p^{k-1}(K[k+1])$ is a bounded map. Let us look at $\omega' = \mathrm{res}_{K[k+1], K}\omega - d(s\gamma\omega)$. It is clear that ω' is a closed form, as well as ω, and $\omega' \in S\mathscr{L}_p^k(K[k+1], K[k])$. The restriction of ω' to every simplex $\sigma \in K[k+1]$ is an exact form, since $H^k(S\mathscr{L}_p^*(\sigma, \partial\sigma)) = 0$. In other words, $\mathrm{im}(d^k) = \ker(d^{k+1})$ and $d\colon S\mathscr{L}_p^{k-1}(\sigma, \partial\sigma) \to \mathrm{Im}(d^{k-1})$ is an epimorphism in the category Ban_∞. Using the Banach inverse operator theorem and the above argument, we can see that there exists a bounded map $\gamma^1\colon S\mathscr{L}_p^k(K) \longrightarrow S\mathscr{L}_p^{k-1}(K[k+1], K[k])$. Then we can take $s\gamma^1\colon S\mathscr{L}_p^k(K) \longrightarrow S\mathscr{L}_p^{k-1}(K[k+2])$ and so on. In essence, repeating this construction for every dimension, as a result, we obtain a finite composition of bounded operators which can be presented as follows: $\ker(\mathscr{I}) \to S\mathscr{L}_p^{k-1}(K)$. Moreover, the map $\ker(\mathscr{I})$ factors through $\mathrm{im}(d^k)$. □

Theorem 7. *Assume K is a simplicial complex of bounded geometry, then the following cohomology groups are isomorphic in the category $\mathsf{Vec}_\mathbb{R}$:*

$$H^k(C_p^*(K)) \cong H^k(S\mathscr{L}_p^*(K)).$$

Proof.

$$H^k(S\mathscr{L}_p^*(K)) \cong H^k(C_p^*(K) \oplus \ker^* \mathscr{I})$$
$$\cong H^k(C_p^*(K)) \oplus H^k(\ker^* \mathscr{I}) \cong H^k(C_p^*(K)).$$ □

References

1. O. Attie, Quasi-isometry classification of some manifolds of bounded geometry. *Math. Z.* **216** (1994), no. 4, 501–527.
2. C. Druţu, M. Kapovich, *Geometric Group Theory.* With an appendix by Bogdan Nica. American Mathematical Society Colloquium Publications, 63. American Mathematical Society, Providence, RI, 2018.
3. V. Gol'dshtein, V. Kuz'minov, I. Shvedov, Differential forms on a Lipschitz manifold. (Russian) *Sibirsk. Mat. Zh.* **23** (1982), no. 2, 16–30, 215.
4. V. Gol'dshtein, V. Kuz'minov, I. Shvedov, A property of de Rham regularization operators. (Russian) *Sibirsk. Mat. Zh.* **25** (1984), no. 2, 104–111.
5. V. Gol'dshtein, V. Kuz'minov, I. Shvedov, The de Rham isomorphism of the L_p-cohomology of noncompact Riemannian manifolds. (Russian) *translated from Sibirsk. Mat. Zh.* **29** (1988), no. 2, 34–44, 216 *Siberian Math. J.* **29** (1988), no. 2, 190–197.
6. G. de Rham, *Differentiable Manifolds.* Springer-Verlag, Berlin, 1984.
7. W. Rudin, *Functional Analysis.* Second edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.
8. D. Sullivan, Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 269–331.
9. F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups.* vol. 94. Springer Science & Business Media, 1983.
10. H. Whitney, *Geometric Integration Theory.* Princeton University Press, Princeton, N. J., 1957.

(Received 15.01.2023)

Department of Mathematics, Ben Gurion University of the Negev, P. O. Box 653, Beer Sheva, Israel
Email address: vladimir@bgu.ac.il
Email address: panenkora@gmail.com