Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 2, 169–175

AN L_p INEQUALITY AND GENERALISED TRIGONOMETRIC FUNCTIONS

DAVID E. EDMUNDS[1] AND JAN LANG[2,3,4]

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The paper explores the consequences of a new L_p inequality for generalised trigonometric functions.

1. Introduction

Let (S, Σ, μ) be a measure space; given any real-valued, $\mu-$measurable function f on S and any $p \in \mathbb{R}\backslash \{0\}$, we write

$$\|f\|_p = \left(\int\limits_S |f|^p \, d\mu \right)^{1/p}.$$

In a striking paper [2], it is shown that for all $\mu-$measurable functions f, g on S,

$$\int\limits_S |f+g|^p \, d\mu \leq \left(1 + \frac{2^{2/p} \|fg\|_{p/2}}{\left(\|f\|_p^p + \|g\|_p^p \right)^{2/p}} \right)^{p-1} \int\limits_S (|f|^p + |g|^p) \, d\mu, \tag{1.1}$$

when $p \in (0,1] \cup [2, \infty)$. The inequality is reversed if $p \in (-\infty, 0) \cup (1, 2)$, assuming that when $p \in (1, 2)$, the functions f and g are positive $\mu-$ almost everywhere. As remarked in [2], (1.1) (and its analogue for other values of p) may be regarded as a refinement of Minkowski's inequality; the connection with Hanner's inequality (see [1, 5] and [6]) is also discussed.

The method of proof adopted in [2] is to show that (1.1) is equivalent to the inequality

$$1 \leq \left(1 + \left(\frac{2\alpha^{p/2} (1-\alpha)^{p/2}}{\alpha^p + (1-\alpha)^p} \right)^{2/p} \right)^{p-1} (\alpha^p + (1-\alpha)^p) \quad \text{for all } \ \alpha \in [0,1] \tag{1.2}$$

when $p \in (0,1] \cup [2, \infty)$, with the reverse inequality when $p \in (-\infty, 0) \cup (1, 2)$. This inequality is then established by an ingenious and intricate real-variable argument.

In the present paper we make an elementary observation that an equivalent form of (1.2) when $p \in [2, \infty)$ (and its reverse when $p \in (1, 2)$) is a deceptively simple real-variable inequality, the sharpness of which is illustrated by various pictures. This inequality can be viewed in terms of generalised trigonometric functions, the growing importance of which is now well documented. After a brief summary of some of the basic facts and background information concerning such functions we again illustrate the sharpness of the corresponding inequality by means of various pictures.

It is a pleasure to express indebtedness to Peter Bushell for drawing attention to [2].

2. The Basic Inequality

Suppose that $p \in (0, \infty)$ and let (X, Y) lie in the positive quadrant of the $p-$circle, so that

$$X^p + Y^p = 1, \quad X \geq 0, \ \ Y \geq 0.$$

Then $X = x^{1/p}$, $Y = (1-x)^{1/p}$ for some $x \in [0, 1]$, and

$$\alpha := \frac{x^{1/p}}{x^{1/p} + (1-x)^{1/p}} \in [0,1];$$

2020 *Mathematics Subject Classification.* 33B10, 26D05, 26D07, 34A34.
Key words and phrases. Generalized Trigonometric functions; Minkowski's inequality; Hanner's inequality.

note that

$$1-\alpha = \frac{(1-x)^{1/p}}{x^{1/p}+(1-x)^{1/p}}\,.$$

Conversely, any $\alpha \in [0,1]$ can be written in this form for some $x \in [0,1]$. It follows from [2] that when $p \in (0,1) \cup [2,\infty)$, (1.2) holds if and only if

$$x^{1/p}+(1-x)^{1/p} \le \Big(1+2^{2/p}x^{1/p}(1-x)^{1/p}\Big)^{1/p'} \quad \text{for all } \ x\in[0,1/2]. \tag{2.1}$$

The reverse inequality holds if and only if $p \in (1,2)$. Obviously, the inequality is an equation for $p=1$ and $p=2$.

We therefore have

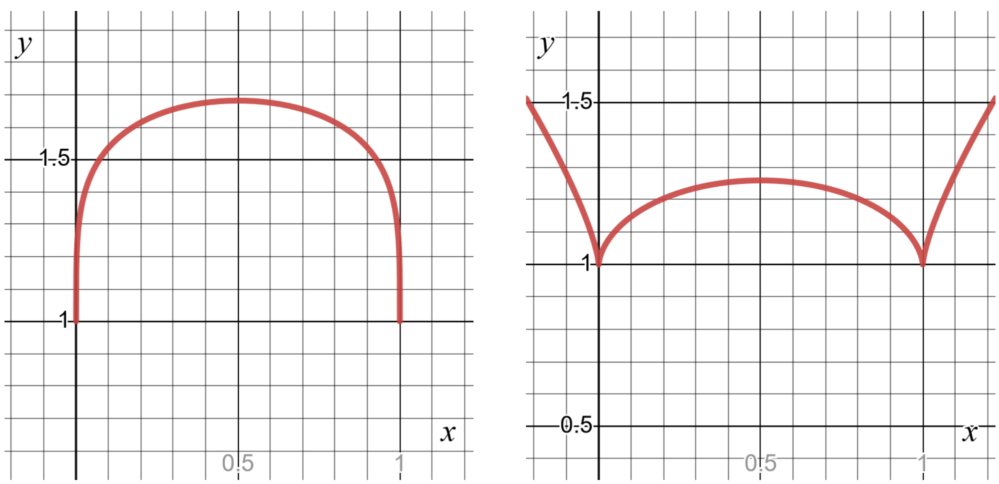
Theorem 2.1. *Inequality* (2.1) *holds for all* $p \in (0,1)\cup[2,\infty)$; *the reverse inequality holds if* $p \in (1,2)$.

Let us denote

$$L(x) := x^{1/p}+(1-x)^{1/p} \text{ and}$$
$$R(x) := \big(1+2^{2/p}x^{1/p}(1-x)^{1/p}\big)^{1/p'}, \quad \text{for all } \ x\in[0,1].$$

Figures 1, 2, 3 below may help to understand why a direct proof of Theorem 2.1 seems challenging.

The first couple of pictures corresponds to the graphs of $L(x)$ and $R(x)$ for $p=4$ and $p=1.5$ (the graphs $L(x)$ and $R(x)$ are indistinguishable in these cases)



(A) p=4 (B) p=1.5

Figure 1

From the graphs in Figures 1, 2, 3 it is possible to see that on $0 \le x \le 1/2$, the functions $L(x)$ and $R(x)$ are really close to each other, especially in the case when $1<p<\infty$. It is worth noting, that from the graphs, we can see that for $p=4$, we have

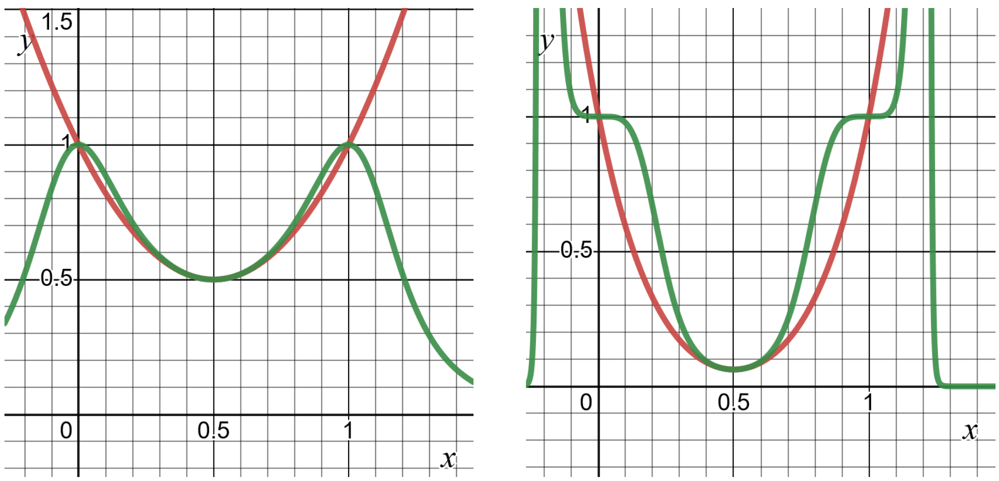
$$L(x) \le R(x) \le 1.06L(x), \quad \text{for all } \ x\in[0,1/2]$$

and for $p=1.5$, we have

$$L(x) \ge R(x) \ge 0.997L(x), \quad \text{for all } \ x\in[0,1/2].$$

Then we can observe from the numerical computations a surprising closeness of the right and left sides of (2.1) which is quite interesting.

The next couple of graphs shows $L(x)$ (in red) and $R(x)$ (in green) for $p = 0.5$ and $p = 0.2$

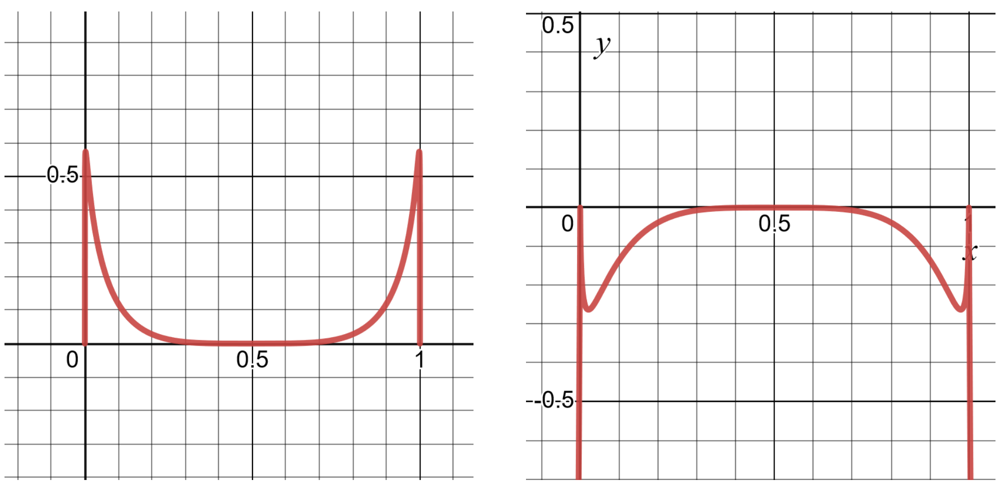


(A) p=0.5

(B) p=0.2

FIGURE 2

The next two graphs shows the difference between $L(x)$ and $R(x)$ by graphing $y = 100(R(x) - L(x))$ for $p = 4$ and $p = 1.5$.



(A) p=4

(B) p=1.5

FIGURE 3

3. Generalised Trigonometric Functions

Inequality (1.2) gives rise to inequalities involving generalised trigonometric functions that appear to be new, despite the very large literature concerning such functions. To remind the reader of some relevant basic facts, let $p, q \in (1, \infty)$ and put

$$\pi_{p,q} = 2\int_0^1 (1 - t^q)^{-1/p}\, dt = 2q^{-1} B\left(1/p', 1/q\right),$$

where B is the Beta function. By $\sin_{p,q}$ we mean the function defined on $[0, \pi_{p,q}/2]$ to be the inverse of the strictly increasing function $F_{p,q} : [0, 1] \to [0, \pi_{p,q}/2]$ given by

$$F_{p,q}(x) = \int_0^x (1 - t^q)^{-1/p} \, dt.$$

This is then extended by the evenness to $[0, \pi_{p,q}]$ and by the oddness to $[0, 2\pi_{p,q}]$ via the translations

$$\sin_{p,q} x = \sin_{p,q} (\pi_{p,q} - x), \quad x \in [\pi_{p,q}/2, \pi_{p,q}]$$

and

$$\sin_{p,q} x = -\sin_{p,q} (2\pi_{p,q} - x), \quad x \in [\pi_{p,q}, 2\pi_{p,q}],$$

respectively. Further extension to the whole real line is then achieved by the $2\pi_{p,q}-$ periodicity.

The function $\cos_{p,q}$ is defined to be to be the derivative of $\sin_{p,q}$ on $\mathbb{R}$, and it follows easily that for all $x \in \mathbb{R}$,

$$|\sin_{p,q} x|^q + |\cos_{p,q} x|^p = 1,$$

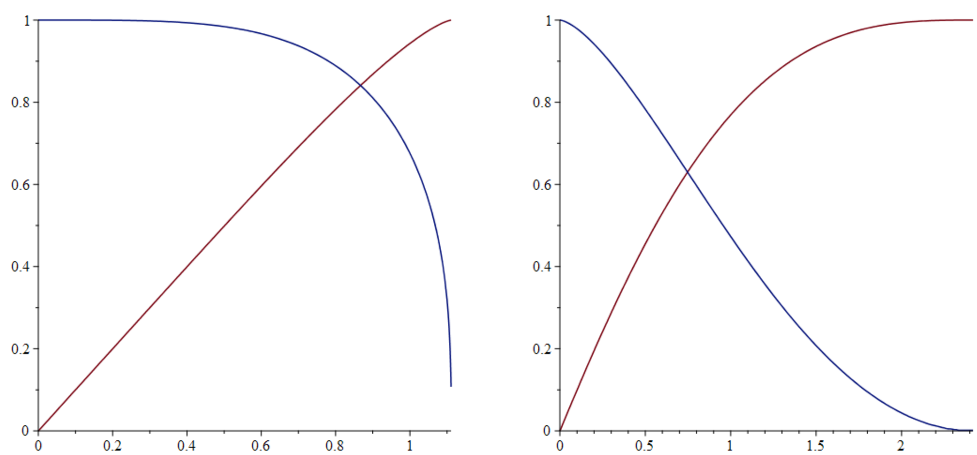
and that both $\sin_{p,q}$ and $\cos_{p,q}$ are non-negative on $[0, \pi_{p,q}/2]$. We write $\sin_p$, $\cos_p$ and π_p instead of $\sin_{p,p}$, $\cos_{p,p}$ and $\pi_{p,p}$, respectively. Further details of these functions are given in [3]. Following Lindqvist [7], the functions S_p and C_p are defined by

$$S_p = \sin_{p',p} \quad \text{and} \quad C_p = \cos_{p',p}^{p'/p}$$

where $1/p + 1/p' = 1$; note that for all $x \in [0, \pi_{p',p}/2]$, these functions are non-negative and

$$S_p^p(x) + C_p^p(x) = 1.$$

In Figure 4 we can see the graphs of $\sin_p$ (in red) and $\cos_p$ (in blue) for $p = 4$ and $p = 1.5$ on the interval $[0, \pi_p]$.



(A) p=4 (B) p=1.5

Figure 4

The generalised trigonometric functions occur naturally in many settings. For example, the eigenvalue problem for the one-dimensional $p-$Laplacian Δ_p $(1 < p < \infty)$, namely,

$$\begin{aligned} -\Delta_p u &:= -\left(|u'|^{p-2} u'\right)' = \lambda |u|^{p-2} u \text{ on } (0,1), \\ u(0) &= u(1) = 0, \end{aligned}$$

has the eigenvalues

$$\lambda_n = (p - 1)(n\pi_p)^p$$

and associated eigenvectors $\sin_p(n\pi_p t)$ $(n \in \mathbb{N})$. The details of their involvement in more complicated questions of this type, such as (p,q) −bi-Laplacian problems, are given in [4].

They also arise in connection with the notion of $p-$compactness, which we briefly recall for the convenience of the reader. Let $p \in (1,\infty)$ and suppose that X, Y are Banach spaces. A subset K of X is said to be relatively $p-$compact if there is a sequence $\{x_n\}$ in X, with $\{\|x_n\|_X\} \in l_p$, such that

$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : \sum_{n=1}^{\infty} |\alpha_n|^{p'} \leq 1 \right\}.$$

When $p = 1$, this definition takes the form

$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : \sup_n |\alpha_n| \leq 1 \right\},$$

where $\{\|x_n\|_X\} \in l_1$. These definitions stem from a classical result of Grothendieck which states that K is relatively compact if and only if there is a sequence $\{x_n\}$ in X, with $\{\|x_n\|_X\} \in c_0$, such that

$$K \subset \left\{ \sum_{n=1}^{\infty} \alpha_n x_n : \sum_{n=1}^{\infty} |\alpha_n| \leq 1 \right\}.$$

This enables us to say that relatively compact sets are relatively $\infty-$compact. It may be checked that if $1 \leq p \leq q \leq \infty$, then a relatively $p-$compact set is relatively $q-$compact. Given $p \in [1,\infty]$, a map $T \in B(X,Y)$ is said to be $p-$compact if $T(B)$ is relatively $p-$compact whenever B is a bounded subset of X.

The generalised trigonometric functions arise naturally when looking for examples of $p-$compact maps. For instance, let $p, q \in (1,\infty)$ and consider the Hardy operator $T : L_p(I) \to L_q(I)$ given by

$$(Tf)(s) = \int_0^s f(t)dt, \quad I = (0,1).$$

Using the fact that the generalised trigonometric functions $\cos_{p,p'}(n\pi_{p,p'}t)$ form a basis of $L_p(I)$ if $p \in [p_1, p_2]$, where $p_1 \in (1,2)$ and $p_2 \in (2,\infty)$ are calculable numbers, it turns out (see [4]) that if $p \in [p_1, p_2]$, there exists $r \in (1,\infty)$ such that T is $s-$compact for all $s \geq r$.

Further illustrations, such as their usefulness in connection with Sobolev embeddings, may be found in [4].

As a simple consequence of the result of [2] we now give inequalities involving the generalised trigonometric functions.

Theorem 3.1. *Let $p \in [2,\infty)$. Then*

$$\sin_p x + \cos_p x \leq \left(1 + 2^{2/p} \sin_p x \cos_p x\right)^{1/p'} \quad \text{for all} \quad x \in [0, \pi_p/2], \tag{3.1}$$

and

$$S_p(x) + C_p(x) \leq \left(1 + 2^{2/p} S_p(x) C_p(x)\right)^{1/p'} \quad \text{for all} \quad x \in [0, \pi_{p',p}/2]. \tag{3.2}$$

These inequalities are reversed if $p \in (1,2]$.

Proof. Suppose that $p \in [2,\infty)$. The choice

$$\alpha = \frac{\sin_p x}{\sin_p x + \cos_p x}, \quad x \in [0, \pi_p/2]$$

in (1.2) gives (3.1). As for (3.2), we simply take

$$\alpha = \frac{S_p(x)}{S_p(x) + C_p(x)}, \quad x \in [0, \pi_{p',p}/2]$$

in (1.2). The inequalities are reversed if $p \in (1,2)$. □

Note that by the use of the identity

$$\sin_{2,p}(2^{2/p}x) = 2^{2/p}\sin_{p',p}(x)\cos_{p',p}^{p'-1}(x)$$

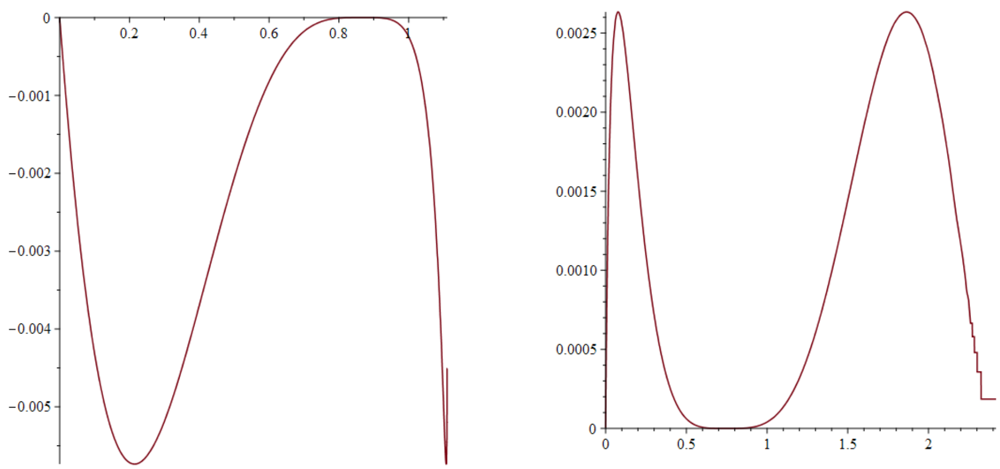
due to Takeuchi [8, Theorem 1.1], (3.2) can be written as

$$\sin_{p',p}(x) + \cos_{p',p}^{p'/p}(x) \leq \left(1 + \sin_{2,p}(2^{2/p}x)\right)^{1/p'} \quad \text{for all} \quad x \in [0, \pi_{p',p}/2].$$

To illustrate the sharpness of these inequalities, we give below some pictures. In the pictures in Figure 5, we see the graph

$$y = \sin_p x + \cos_p x - \left(1 + 2^{2/p}\sin_p x \cos_p x\right)^{1/p'}$$

on interval $x \in [0, \pi_p/2]$ when $p = 4$ and $p = 1.5$. From these graphs we can see the surprising sharpness of inequality (3.1) and similar sharpness we can expect for inequality (3.2).



(A) p=4 (B) p=1.5

Figure 5

References

1. K. Ball, E. A. Carlen, E. H. Lieb, Sharp uniform convexity and smoothness inequalities for trace norms. *Invent. Math.* **115** (1994), no. 3, 463–482.
2. E. A. Carlen, R. L. Frank, P. Ivanisvili, E. H. Lieb, Inequalities for L^p-norms that sharpen the triangle inequality and complement Hanner's inequality. *J. Geom. Anal.* **31** (2021), no. 4, 4051–4073.
3. D. E. Edmunds, J. Lang, *Eigenvalues, Embeddings and Generalised Trigonometric Functions*. Lecture Notes in Mathematics, 2016. Springer, Heidelberg, 2011.
4. D. E. Edmunds, J. Lang, Remarks on $j-$eigenfunctions of operators. *Math. Nachr.* **296** (2023), no. 3, 1071–1086.
5. O. Hanner, On the uniform convexity of L^p and l^p. *Ark. Mat.* **3** (1956), 239–244.
6. E. H. Lieb, M. Loss, *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, RI, 2001.
7. P. Lindqvist, Some remarkable sine and cosine functions. *Ricerche Mat.* **44** (1995), no. 2, 269–290 (1996).
8. S. Takeuchi, Multiple-angle formulas of generalized trigonometric functions with two parameters. *J. Math. Anal. Appl.* **444** (2016), no. 2, 1000–1014.

(Received 10.01.2023)

[1]Department of Mathematics, University of Sussex, Mantell Building, Brighton BN1 9RF, Sussex, United Kingdom

[2]Department of Mathematics, The Ohio State University, 231 West 18th Avenue, Columbus, OH 43210, USA

[3]Czech Technical University in Prague, Faculty of Electrical Engineering, Department of Mathematics, Technická 2, 166 27 Praha 6, Czech Republic

[4]Faculty of Mathematics and Physics, Charles University, 186 75 Praha 8, Czech Republic
Email address: davideedmunds@aol.com
Email address: lang@math.osu.edu