Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 2, 275–288

# EXTRAPOLATION THEOREMS IN LEBESGUE AND GRAND LEBESGUE SPACES FOR QUASI-MONOTONE FUNCTIONS

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*Dedicated to the memory of Academician Vakhtang Kokilashvili*

**Abstract.** We prove Rubio de Francia extrapolation results in Lebesgue and grand Lebesgue spaces for quasi-monotone functions with $QB_{\beta,p}$ weights. The extrapolation in Lebesgue spaces with the weight class $QB_{\beta,\infty}$ has also been investigated. As an application, we characterize the boundedness of the Hardy averaging operator for quasi- monotone functions in the grand Lebesgue spaces.

## 1. Introduction

By a weight function, we shall mean a function which is measurable, non-negative, finite almost everywhere (a.e.) and locally integrable on the specified domain. A weight $w$ is said to belong to the class $B_p$ $(p > 0)$ if there exists a constant $C > 0$ such that the inequality

$$\int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \le C \int_0^r w(x)dx$$

holds for every $r > 0$. The weight class $B_p$ is an important class of weights. It characterizes the boundedness of the Hardy averaging operator

$$Hf(x) := \frac{1}{x}\int_0^x f(t)dt$$

for non-increasing functions in Lebesgue $L^p_w$ spaces [2,30], as well as in grand Lebesgue spaces (defined in Section 3), see [16, 25]. These characterizations are, in fact, equivalent to the boundedness of the maximal operator, respectively, in the Lorentz space $\Lambda^p(w)$ [2] and in the grand Lorentz space $\Lambda^{p)}(w)$ [16].

Let us write

$$[w]_{B_p} = \inf\left\{C > 0 : \int_0^r w(x)dx + \int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \le C\int_0^r w(x)dx,\ r > 0\right\}.$$

One of the important properties of $B_p$ class of weights (see [7]) is that: if $w \in B_p$ $(p > 0)$, there exists $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$. Moreover,

$$[w]_{B_{p-\varepsilon}} \le \frac{C[w]_{B_p}}{1-\varepsilon\alpha^p[w]_{B_p}}, \tag{1.1}$$

where $C$ and $0 < \alpha < 1$ are universal constants and $\varepsilon$ is such that $1 - \varepsilon\alpha^p[w]_{B_p} > 0$.

In 2010, Carro and Lorente [6] made a remarkable use of the $B_p$-class of weights to prove the following extrapolation result.

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2020 *Mathematics Subject Classification.* 26D10, 26D15, 46E35.

*Key words and phrases.* Rubio de Francia extrapolation; Grand Lebesgue space; $QB_{\beta,p}$-weights; Hardy averaging operator; Quasi-monotone functions.

**Theorem A** ([6])**.** *Let $\varphi$ be an increasing ($\uparrow$) function defined on $(0,\infty)$, $(f,g)$ be a pair of positive decreasing ($\downarrow$) functions defined on $(0,\infty)$ and $0 < p_0 < \infty$. Suppose that for every $w \in B_{p_0}$, the inequality*

$$\int_0^\infty f^{p_0}(x)w(x)dx \le \varphi([w]_{B_{p_0}}) \int_0^\infty g^{p_0}(x)w(x)dx$$

*holds. Then for all $0 < p < \infty$ and all $w \in B_p$, the inequality*

$$\int_0^\infty f^{p}(x)w(x)dx \le \tilde{\varphi}([w]_{B_{p}}) \int_0^\infty g^{p}(x)w(x)dx,$$

*holds, where*

$$\tilde{\varphi}([w]_{B_p}) = \inf_{0<\varepsilon<\frac{p_0}{p\alpha^p[w]_{B_p}}} \psi(p_0/\varepsilon)^{p/p_0} \frac{C[w]_{B_p}}{1-\varepsilon(p/p_0)\alpha^p[w]_{B_p}}$$

*with $C$ as in* (1.1).

The genesis of the above result lies in the excellent extrapolation result of J.L. Rubio de Francia [29] (also, see [9] and the references therein) who proved it for another important class of weights, the so-called Muckenhoupt class, or $A_p$-class of weights. A weight $w$ is said to be in the Muckenhoupt class $A_p$, $1 < p < \infty$, if

$$[w]_{A_p} := \sup_J \frac{W(J)}{|J|}\left(\frac{1}{|J|}\int_J w^{-p'/p}\right)^{p-1} < \infty,$$

and in class $A_1$, if

$$[w]_{A_1} := \operatorname*{ess\,sup}_{x\in J} \frac{W(J)}{w(x)|J|} < \infty,$$

where the supremum is taken over all non-degenerate intervals $J \subset \mathbb{R}^+$, $\frac{1}{p}+\frac{1}{p'}=1$ and $W(J) := \int_J w(x)dx$.

The weight class $A_p$ is found to be useful in many ways. It characterizes the boundedness of the maximal operator [27] and Riesz potential [12] in Lebesgue spaces. Moreover, this class also characterizes the boundedness of these operators in grand Lebesgue spaces [10,25]. The extrapolation theory has been generalized to $A_\infty$-weights as well (see [8]).

We denote by $\mathcal{M}$ the set of all measurable functions, definite and finite a.e. on $\mathbb{R}^+$. Also, $\mathcal{M}^+ \subset \mathcal{M}$ and $\mathcal{M}^+_\downarrow \subset \mathcal{M}^+$ denote, respectively, the cones of non-negative and non-negative non-increasing ($\downarrow$) functions in $\mathcal{M}$. In this paper, we consider quasi-non-increasing functions, the class of such functions being denoted by $Q_\beta$ : A function $f \in \mathcal{M}^+$ is said to belong to $Q_\beta$, $\beta \in \mathbb{R}$, if $x^{-\beta}f(x)$ is non-increasing. Clearly, $\mathcal{M}^+_\downarrow = Q_0$. The quasi-monotone functions are believed to be defined in [4,5]. For a more later reference, we mention [28]. For the functions $f \in Q_\beta$, Bergh, Burenkov and Persson [3] investigated Hardy's inequality with power type weights, while for general weights, it has been proved in [19] that the inequality

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p w(x)dx\right) \le C\int_0^\infty f^p(x)w(x)dx, \quad 1 \le p < \infty,$$

holds for all $f \in Q_\beta$ if and only if $w \in QB_{\beta,p}$, $\beta > -1$, i.e.,

$$\int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \le C\int_0^r \left(\frac{x}{r}\right)^{\beta p} w(x)dx, \quad r > 0. \tag{1.2}$$

Note that for $\beta = 0$, the weight class $QB_{\beta,p}$ reduces to the class $B_p$.

In the present paper, we define a variant of the class $QB_{\beta,p}$, to be denoted by $\widehat{Q}B_{\beta,p}$, and prove the extrapolation results for this class of weights, as well as for the weight class

$$QB_{\beta,\infty} := \bigcup_{p>0} QB_{\beta,p} \ .$$

Further, we prove the extrapolation result for quasi-monotone functions in the frame of grand Lebesgue spaces. As an application, we prove the boundedness of the Hardy averaging operator for quasi-monotone functions in the grand Lebesgue spaces. Our results generalize the extrapolation results of Carro and Lorente [6] and Meskhi [24]. Throughout, all the functions used in this paper are assumed to be non-negative and measurable.

We close this section by mentioning that for a weight $w$ and $1 \le p < \infty$, we shall denote by $L^p_w$, the weighted Lebesgue space consisting of all $f \in \mathcal{M}$ such that

$$\|f\|_{L^p_w} := \left( \int_0^\infty |f|^p w \right)^{1/p} < \infty.$$

## 2. Extrapolation Results in Lebesgue Spaces

For $p > 0$, we say that a weight $w \in QB_{\beta,\psi,p}$ if

$$\int_r^\infty \left( \frac{\Psi(r)}{\Psi(x)} \right)^p w(x)\, dx \le C \int_0^r \left( \frac{\Psi(x)}{\Psi(r)} \right)^{\beta p} w(x)\ dx, \quad r > 0 \tag{2.1}$$

for some constant $C > 0$ and $\Psi(x) := \int_0^x \psi(t)dt$, where $\psi$ is a non-negative, non-increasing locally integrable function, i.e., $\psi \in L^1_{\text{loc}}$. For $\psi \equiv 1$, the weight class $QB_{\beta,\psi,p}$ reduces to the class $QB_{\beta,p}$.

In [19], the class $QB_{\beta,\psi,p}$ was used to characterize the boundedness of the operator

$$S_\psi f(x) := \frac{1}{\Psi(x)} \int_0^x f(t)\psi(t)dt$$

on the cone of functions $f \in Q_\beta$. Precisely, the following was proved.

**Theorem B** ([19])**.** *Let $p \ge 1$ and $-1 < \beta \le 0$. Then the inequality*

$$\int_0^\infty \left(S_\psi f\right)^p(x) w(x)dx \le C' \int_0^\infty f^p(x)w(x)dx$$

*holds for all $f \in Q_\beta$ if and only if $w \in QB_{\beta,\psi,p}$, where $C' = \frac{C+1}{(\beta+1)^p}$ and $C$ is as in* (2.1).

We define $QB_{\beta,\psi,p}$-constant for a weight $w \in QB_{\beta,\psi,p}$ as follows:

$$[w]_{QB_{\beta,\psi,p}} := \inf \left\{ D : \int_r^\infty \left( \frac{\Psi(r)}{\Psi(x)} \right)^p w(x)\, dx \le (D-1) \int_0^r \left( \frac{\Psi(x)}{\Psi(r)} \right)^{\beta p} w(x)dx, r > 0 \right\}. \tag{2.2}$$

**Remark 2.1.** Note that

(1) $[w]_{QB_{\beta,\psi,p}} > 1$.
(2) For $-1 < \beta \le 0$ and $p \le q$, we have $QB_{\beta,\psi,p} \subset QB_{\beta,\psi,q}$.

We begin with the following

**Lemma 2.2.** *Let the function $\varphi$ be non-decreasing ($\uparrow$) defined on $(0,\infty)$, $f,g \in Q_\beta$ ($\beta > -1$), $0 < p_0 < \infty$, $\psi \in L^1_{\mathrm{loc}}$ be $\downarrow$ and $\lim_{x\to\infty} \Psi(x) = \infty$. Suppose that for each $w \in QB_{\beta,\psi,p_0}$, the inequality*

$$\int_0^\infty f(x)w(x)dx \le \varphi\big([w]_{QB_{\beta,\psi,p_0}}\big) \int_0^\infty g(x)w(x)dx$$

*holds. Then for every $0 < \varepsilon < p_0(\beta+1)$ and $t > 0$, the following inequality*

$$\int_0^t f(s)(\Psi(s))^{p_0-1-\varepsilon}\psi(s)ds \le \varphi\left(\frac{p_0(\beta+1)}{\varepsilon}\right) \int_0^t g(s)(\Psi(s))^{p_0-1-\varepsilon}\psi(s)ds$$

*holds.*

*Proof.* Let $v \in \mathcal{M}^+_\downarrow$. Set $w(x) = v(x)(\Psi(x))^{p_0-1-\varepsilon}\psi(x)$ so that $w \in L^1_{\mathrm{loc}}$. We claim that $w \in QB_{\beta,\psi,p_0}$. Indeed, we have

$$\begin{aligned}
\Big(\Psi(r)^{\beta+1}\Big)^{p_0} \int_r^\infty \frac{w(x)}{\Psi(x)^{p_0}}dx &= \Big(\Psi(r)^{\beta+1}\Big)^{p_0} \int_r^\infty v(x)(\Psi(x))^{-1-\varepsilon}\psi(x)dx \\
&\le \frac{v(r)}{\varepsilon}(\Psi(r))^{p_0(\beta+1)-\varepsilon} \\
&= \frac{(p_0(\beta+1)-\varepsilon)}{\varepsilon} v(r) \int_0^r (\Psi(x))^{p_0(\beta+1)-1-\varepsilon}\psi(x)dx \\
&\le \left(\frac{p_0(\beta+1)}{\varepsilon} - 1\right) \int_0^r v(x)(\Psi(x))^{p_0(\beta+1)-1-\varepsilon}\psi(x)dx \\
&\le \frac{p_0(\beta+1)}{\varepsilon} \int_0^r (\Psi(x))^{\beta p_0} v(x)(\Psi(x))^{p_0-1-\varepsilon}\psi(x)dx \\
&= \frac{p_0(\beta+1)}{\varepsilon} \int_0^r (\Psi(x))^{\beta p_0} w(x)dx.
\end{aligned}$$

The assertion now follows on taking $v(x) = \chi_{(0,s]}(x)$ and using the fact that $[w]_{QB_{\beta,\psi,p_0}} \le \frac{p_0(\beta+1)}{\varepsilon}$. $\square$

**Definition 2.3.** For a given $\beta > -1$, a weight function $w$ is said to be in the class $\widehat{Q}B_{\beta,p}$ if
(i) $w \in QB_{\beta,p}$; and
(ii) there exists $0 < \varepsilon < p(\beta+1)$ such that $w \in QB_{\beta,p-\varepsilon}$.

**Remark 2.4.** The class $\widehat{Q}B_{\beta,p}$ in Definition 2.3 is reasonably defined. In view of Lemma 2.3 [19], it is clear that for $\beta \ge 0$, $\widehat{Q}B_{\beta,p} = QB_{\beta,p}$. We prove below that for $-1 < \beta < 0$, the power weights belong to the class $\widehat{Q}B_{\beta,p}$. It is of interest if the same can be proved for general weights as well.

**Lemma 2.5.** *Let $1 \le p < \infty$, $-1 < \beta < 0$ and $\alpha \in \mathbb{R}$. If $x^\alpha \in QB_{\beta,p}$, then there exists $0 < \varepsilon < p(\beta+1)$ such that $x^\alpha \in QB_{\beta,p-\varepsilon}$.*

*Proof.* Since $x^\alpha \in QB_{\beta,p}$, we have

$$\int_r^\infty \left(\frac{r}{x}\right)^p x^\alpha dx \le C \int_0^r \left(\frac{x}{r}\right)^{\beta p} x^\alpha dx, \quad r > 0 \tag{2.3}$$

which holds if and only if

$$-\beta p - 1 < \alpha < p - 1. \tag{2.4}$$

Choose $\varepsilon > 0$ such that $0 < \varepsilon < p - \alpha - 1$. Clearly, $0 < \varepsilon < p(\beta + 1)$. Now, using estimates (2.3) and (2.4) at the appropriate places, we obtain

$$\begin{aligned}\int_r^\infty \left(\frac{r}{x}\right)^{p-\varepsilon} x^\alpha dx &= \frac{r^{\alpha+1}}{p-\varepsilon-\alpha-1} \\ &= \frac{p-\alpha-1}{p-\varepsilon-\alpha-1} \int_r^\infty \left(\frac{r}{x}\right)^p x^\alpha dx \\ &\le C\left(\frac{p-\alpha-1}{p-\varepsilon-\alpha-1}\right) \int_0^r \left(\frac{x}{r}\right)^{\beta p} x^\alpha dx \\ &= \frac{K}{(\alpha+\beta p+1) r^{\beta p}} r^{\beta p+\alpha+1} \\ &= K\left(\frac{\beta(p-\varepsilon)+\alpha+1}{\alpha+\beta p+1}\right) \int_0^r \left(\frac{x}{r}\right)^{\beta(p-\varepsilon)} x^\alpha dx,\end{aligned}$$

i.e., $x^\alpha \in QB_{\beta,p-\varepsilon}$ with the constant

$$C^* := K\left(\frac{\beta(p-\varepsilon)+\alpha+1}{\alpha+\beta p+1}\right),$$

where $K = C\left(\frac{p-\alpha-1}{p-\varepsilon-\alpha-1}\right)$ and $C$ is as in (2.3). □

**Remark 2.6.** For $-\beta p - 1 < \alpha < p - 1$, from Lemma 2.5 and (2.2), it follows that

$$[x^\alpha]_{QB_{\beta,p-\varepsilon}} \le C^* + 1 = C\left(\frac{p-\alpha-1}{p-\varepsilon-\alpha-1}\right)\left(\frac{\beta(p-\varepsilon)+\alpha+1}{\alpha+\beta p+1}\right) + 1.$$

We now prove the first main extrapolation theorem.

**Theorem 2.7.** *Let $\varphi \uparrow$ be defined on $(0,\infty)$, $(f,g)$ be a pair of functions such that $f, g \in Q_\beta$, $-1 < \beta \le 0$ and $1 \le p_0 < \infty$. Suppose that for every $w \in QB_{\beta,p_0}$, the inequality*

$$\int_0^\infty f^{p_0}(x) w(x) dx \le \varphi\Big([w]_{QB_{\beta,p_0}}\Big) \int_0^\infty g^{p_0}(x) w(x) dx$$

*holds. Then for all $p_0 \le p < \infty$ and all $w \in \widehat{Q}B_{\beta,p}$, the following inequality*

$$\int_0^\infty f^p(x) w(x) dx \le C \int_0^\infty g^p(x) w(x) dx$$

*holds, where*

$$C = \inf_{0<\varepsilon<p_0(\beta+1)} [w]_{QB_{\beta,(p_0-\varepsilon)\frac{p}{p_0}}} \left[\frac{1}{\beta+1}\left(\frac{p_0(\beta+1)-\varepsilon}{p_0-\varepsilon}\right)\varphi\left(\frac{p_0(\beta+1)}{\varepsilon}\right)\right]^{p/p_0}.$$

*Proof.* The case $\beta = 0$ is just Theorem A. So, we assume that $-1 < \beta < 0$.

Let $p_0 \le p < \infty$, $w \in \widehat{Q}B_{\beta,p}$ and $0 < \varepsilon < p_0(\beta+1)$. Clearly, the function $h(x) := x^{-\beta} f(x)$ is $\downarrow$. Note that

$$\int_0^\infty f^p(x) w(x) dx = \int_0^\infty h^p(x) w(x) x^{\beta p} dx. \tag{2.5}$$

Since $h$ is decreasing, we have

$$h^{p_0}(x) \le \frac{p_0(\beta+1)-\varepsilon}{x^{p_0(\beta+1)-\varepsilon}} \int_0^x h^{p_0}(s) s^{p_0(\beta+1)-\varepsilon-1} ds$$

which together with (2.5) and Lemma 2.2 (for $\psi \equiv 1$) gives

$$\begin{aligned}
&\int_0^\infty f^p(x)w(x)dx \\
&\le \left(\frac{p_0(\beta+1)-\varepsilon}{p_0-\varepsilon}\right)^{p/p_0} \int_0^\infty \left(\frac{p_0-\varepsilon}{x^{p_0-\varepsilon}} \int_0^x f^{p_0}(s)s^{p_0-1-\varepsilon}ds\right)^{p/p_0} w(x)dx \\
&\le \left(\frac{p_0(\beta+1)-\varepsilon}{p_0-\varepsilon}\right)^{p/p_0} \varphi\left(\frac{p_0(\beta+1)}{\varepsilon}\right)^{p/p_0} \int_0^\infty \left(\frac{p_0-\varepsilon}{x^{p_0-\varepsilon}} \int_0^x g^{p_0}(s)s^{p_0-1-\varepsilon}ds\right)^{p/p_0} w(x)dx \\
&= \gamma \int_0^\infty \left(\frac{p_0-\varepsilon}{x^{p_0-\varepsilon}} \int_0^x g^{p_0}(s)s^{p_0-1-\varepsilon}ds\right)^{p/p_0} w(x)dx \\
&= \gamma \int_0^\infty \left(S_\psi g^{p_0}(x)\right)^{p/p_0} w(x)dx, \qquad (2.6)
\end{aligned}$$

where $\psi(s) = s^{p_0-1-\varepsilon}$ and

$$\gamma = \left(\frac{p_0(\beta+1)-\varepsilon}{p_0-\varepsilon}\right)^{p/p_0} \varphi\left(\frac{p_0(\beta+1)}{\varepsilon}\right)^{p/p_0}.$$

Now, since $w \in \widehat{Q}B_{\beta,p}$, by the definition, there exists $\widetilde{\varepsilon} > 0$ such that $w \in QB_{\beta,p-\widetilde{\varepsilon}}$. It suffices to take $\varepsilon$ so that $p - \widetilde{\varepsilon} = (p_0-\varepsilon)\frac{p}{p_0}$, or $\varepsilon = \frac{p_0}{p}\widetilde{\varepsilon}$. Then $w \in QB_{\beta,(p_0-\varepsilon)\frac{p}{p_0}}$, for all $r > 0$, i.e., the following inequality

$$\int_r^\infty \left(\frac{r}{x}\right)^{(p_0-\varepsilon)\frac{p}{p_0}} w(x)dx \le (A-1)\int_0^r \left(\frac{x}{r}\right)^{\beta(p_0-\varepsilon)\frac{p}{p_0}} w(x)dx$$

holds, or

$$\int_r^\infty \left(\frac{\Psi(r)}{\Psi(x)}\right)^{p/p_0} w(x)dx \le (A-1)\int_0^r \left(\frac{\Psi(x)}{\Psi(r)}\right)^{\beta(p/p_0)} w(x)dx$$

with $\psi(s) = s^{p_0-1-\varepsilon}$, which by Theorem B holds if and only if

$$\int_0^\infty \left(S_\psi g^{p_0}(x)\right)^{p/p_0} w(x)dx \le \frac{A}{(\beta+1)^{p/p_0}} \int_0^\infty g^p(x)w(x)dx,$$

where $A = [w]_{QB_{\beta,(p_0-\varepsilon)\frac{p}{p_0}}} = [w]_{QB_{\beta,p-\widetilde{\varepsilon}}}$.

Consequently, (2.6) results in

$$\begin{aligned}
\int_0^\infty f^p(x)w(x)dx &\le \frac{\gamma A}{(\beta+1)^{p/p_0}} \int_0^\infty g^p(x)w(x)dx \\
&= K\int_0^\infty g^p(x)w(x)dx,
\end{aligned}$$

where

$$K = [w]_{QB_{\beta,(p_0-\varepsilon)\frac{p}{p_0}}} \left[\left(\frac{p_0(\beta+1)-\varepsilon}{(\beta+1)(p_0-\varepsilon)}\right)\varphi\left(\frac{p_0(\beta+1)}{\varepsilon}\right)\right]^{p/p_0}.$$

Since $\varepsilon \in (0, p_0(\beta+1))$ is arbitrary, taking infimum over all such $\varepsilon$, the assertion follows. □

In view of Remark 2.1 (for $\psi \equiv 1$), following the definition of the class $B_\infty$ [6], we define the class $QB_{\beta,\infty}$ as

$$QB_{\beta,\infty} := \bigcup_{p>0} QB_{\beta,p}$$

and we also define

$$[w]_{QB_{\beta,\infty}} := \inf\left\{[w]_{QB_{\beta,p}} : w \in QB_{\beta,p},\ p>0\right\}.$$

Similarly, we define

$$QB_{\beta,\psi,\infty} := \bigcup_{p>0} QB_{\beta,\psi,p}$$

and

$$1 \le [w]_{QB_{\beta,\psi,\infty}} := \inf\left\{[w]_{QB_{\beta,\psi,p}} : w \in QB_{\beta,\psi,p},\ p>0\right\}.$$

We prove the following

**Lemma 2.8.** *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be $\uparrow$, $v \in L^1_{\mathrm{loc}}$ be $\downarrow$, $-1 < \beta \le 0$ and $\alpha > -1$. Then the function $w$ defined by*

$$w(x) = \Psi^\alpha(x)\psi(x)v(x)$$

*belongs to the class $QB_{\beta,\psi,\infty}$.*

*Proof.* Let $0 < r < \infty$ be arbitrary and choose $p_0$ such that $\alpha+1 < p_0 < -\frac{1}{\beta}(\alpha+1)$. Then we have

$$\begin{aligned}\int_r^\infty \left(\frac{\Psi^{\beta+1}(r)}{\Psi(x)}\right)^{p_0} w(x)dx &= (\Psi(r))^{(\beta+1)p_0}\int_r^\infty (\Psi(x))^{\alpha-p_0}\psi(x)v(x)dx\\ &\le \frac{1}{(p_0-\alpha-1)}(\Psi(r))^{\beta p_0+\alpha+1}v(r)\\ &\le \left(\frac{\beta p_0+\alpha+1}{p_0-\alpha-1}\right)\int_0^r (\Psi(x))^{\beta p_0+\alpha}\psi(x)v(x)dx\\ &= \left(\frac{\beta p_0+\alpha+1}{p_0-\alpha-1}\right)\int_0^r (\Psi(x))^{\beta p_0}w(x)dx\end{aligned}$$

and the assertion follows. Moreover, $[w]_{QB_{\beta,\psi,\infty}} \le \frac{\beta p_0+\alpha+1}{p_0-\alpha-1} + 1$. □

Below, we prove an extrapolation result for the $QB_{\beta,\infty}$-class of weights.

**Theorem 2.9.** *Let $\varphi$ be $\uparrow$ defined on $(0,\infty)$, $-1 < \beta \le 0$, $(f,g)$ be a pair of functions such that $f, g \in Q_\beta$ and $0 < p_0 < \infty$. Suppose that for every weight $w \in QB_{\beta,\infty}$, the inequality*

$$\int_0^\infty f^{p_0}(t)w(t)dt \le \varphi([w]_{QB_{\beta,\infty}})\int_0^\infty g^{p_0}(t)w(t)dt \tag{2.7}$$

*holds. Then for every $p_0 \le p < \infty$ and $w \in QB_{\beta,\infty}$, the inequality*

$$\int_0^\infty f^p(t)w(t)dt \le K\int_0^\infty g^p(t)w(t)dt$$

*with*

$$K = \inf_{\alpha>-1}[w]_{QB_{\beta,\frac{(\alpha+1)p}{p_0}}}\left(\frac{\varphi(1)}{\beta+1}\right)^{p/p_0}$$

*holds.*

*Proof.* For $s > 0$ and $\alpha > -1$, consider the following:

$$\widetilde{w}(t) = \chi_{(0,s)}(t)t^{\alpha}.$$

Clearly, by Lemma 2.8, $\widetilde{w} \in QB_{\beta,\infty}$. Then, in view of Remark 2.1 and Lemma 2.8, we have

$$1 \leq [\widetilde{w}]_{QB_{\beta,\psi,\infty}} \leq \lim_{p_0\to\infty} \frac{\beta p_0 + \alpha + 1}{p_0 - \alpha - 1} + 1 = \beta + 1 \leq 1$$

and consequently, in view of (2.7), the following inequality

$$\int_0^s f^{p_0}(t)t^{\alpha}dt \leq \varphi(1) \int_0^s g^{p_0}(t)t^{\alpha}dt \tag{2.8}$$

holds. Further, since $t^{-\beta}f(t)$ is $\downarrow$, we find that

$$\begin{aligned} f^{p_0}(t) &= \frac{\alpha+1}{t^{\alpha+1}} \int_0^t f^{p_0}(t)s^{\alpha}ds \\ &= \frac{\alpha+1}{t^{\alpha+1}} \int_0^t (t^{-\beta}f(t))^{p_0}t^{\beta p_0}s^{\alpha}ds \\ &\leq \frac{\alpha+1}{t^{\alpha+1}} \int_0^t (s^{-\beta}f(s))^{p_0}t^{\beta p_0}s^{\alpha}ds \\ &= \frac{\alpha+1}{t^{\alpha+1}} \int_0^t f^{p_0}(s)\left(\frac{t}{s}\right)^{\beta p_0} s^{\alpha}ds \\ &\leq \frac{\alpha+1}{t^{\alpha+1}} \int_0^t f^{p_0}(s)s^{\alpha}ds \end{aligned}$$

which in view of (2.8) gives

$$\begin{aligned} \int_0^\infty f^p(t)w(t)dt &\leq \int_0^\infty \left(\frac{\alpha+1}{t^{\alpha+1}} \int_0^t f^{p_0}(s)s^{\alpha}ds\right)^{p/p_0} w(t)dt \\ &\leq \varphi(1)^{p/p_0} \int_0^\infty \left(\frac{\alpha+1}{t^{\alpha+1}} \int_0^t g^{p_0}(s)s^{\alpha}ds\right)^{p/p_0} w(t)dt \\ &= \varphi(1)^{p/p_0} \int_0^\infty \Big(S_{\psi}g^{p_0}(t)\Big)^{p/p_0} w(t)dt, \end{aligned} \tag{2.9}$$

with $\psi(s) = s^{\alpha}$.

Now, let $w \in QB_{\beta,\infty}$. Then there exists $q > 0$ such that $w \in QB_{\beta,q}$. We can choose $\alpha > -1$ such that $q = (\alpha+1)\frac{p}{p_0}$. Then $w \in QB_{\beta,(\alpha+1)\frac{p}{p_0}}$, which in view of (1.2) implies that for all $r > 0$, the inequality

$$\int_r^\infty \left(\frac{r}{t}\right)^{(\alpha+1)\frac{p}{p_0}} w(t)dt \leq (C-1) \int_0^r \left(\frac{t}{r}\right)^{\beta(\alpha+1)\frac{p}{p_0}} w(t)dt,$$

or, equivalently,

$$\int_r^\infty \left(\frac{\Psi(r)}{\Psi(t)}\right)^{p/p_0} w(t)dt \leq (C-1) \int_0^r \left(\frac{\Psi(t)}{\Psi(r)}\right)^{\beta p/p_0} w(t)dt$$

with $\psi(s) = s^\alpha$ holds. But the last inequality, in view of Theorem B, holds if and only if

$$\int_0^\infty \Big(S_\psi g^{p_0}(t)\Big)^{p/p_0} w(t)dt \le \frac{C}{(\beta+1)^{p/p_0}} \int_0^\infty g^p(t)w(t)dt, \tag{2.10}$$

where $C = [w]_{QB_{\beta,(\alpha+1)\frac{p}{p_0}}}$. Now, (2.9) and (2.10) give

$$\int_0^\infty f^p(t)w(t)dt \le [w]_{QB_{\beta,(\alpha+1)\frac{p}{p_0}}} \left(\frac{\varphi(1)}{\beta+1}\right)^{p/p_0} \int_0^\infty g^p(t)w(t)dt,$$

so that on taking the infimum over all $\alpha > -1$, the assertion follows. □

## 3. Extrapolation Results in Grand Lebesgue Spaces

In this section, we shall prove a version of the extrapolation result (Theorem 2.7) in the framework of grand Lebesgue spaces defined on finite intervals which, without any loss of generality, are taken as $I = (0, 1)$.

Let $0 < p < \infty$ and $-1 < \beta < \infty$. We say that a weight function $w$ on $I$ belongs to the class $QB_{\beta,p}(I)$ if there exists a constant $C > 0$ such that the inequality

$$\int_r^1 \left(\frac{r}{t}\right)^p w(t)dt \le C \int_0^r \left(\frac{t}{r}\right)^{\beta p} w(t)dt$$

holds for all $0 < r \le 1$. Also, for $0 < r \le 1$, we set

$$[w]_{QB_{\beta,p}(I)} := \inf\left\{C > 1 : \int_r^1 \left(\frac{r}{t}\right)^p w(t)dt \le (C-1) \int_0^r \left(\frac{t}{r}\right)^{\beta p} w(t)dt\right\}.$$

It can be seen that if $0 < p < \infty$ and $w \in QB_{\beta,p}(I)$, then the function $\widetilde{w} = w\chi_I \in QB_{\beta,p}$ and

$$[w]_{QB_{\beta,p}(I)} = [\widetilde{w}]_{QB_{\beta,p}}.$$

Following the arguments used in Lemma 2.5, we can prove the following

**Lemma 3.1.** *Let $-1 < \beta \le 0$ and $1 \le p < \infty$. If $x^\alpha \in QB_{\beta,p}(I)$, then there exists $0 < \varepsilon < p(\beta+1)$ such that $x^\alpha \in QB_{\beta,p-\varepsilon}(I)$.*

**Definition 3.2.** For a given $-1 < \beta < \infty$, a weight function $w \in \widehat{Q}B_{\beta,p}(I)$ if
(i) $w \in QB_{\beta,p}(I)$; and
(ii) there exists $0 < \varepsilon < p(\beta+1)$ such that $w \in QB_{\beta,p-\varepsilon}(I)$.

**Remark 3.3.** It can be checked that for $-1 < \beta \le 0$, the power weights $x^\alpha \in QB_{\beta,p}(I)$ if and only if $-\beta p - 1 < \alpha < p - 1$. Then, in view of Lemma 3.1, the class $\widehat{Q}B_{\beta,p}(I)$ is reasonably defined.

It is seen that Theorem 2.7 can be modified for the interval $I$. We state it formally for later purpose.

**Theorem 3.4.** *Let $\varphi \uparrow$ be defined on $\mathbb{R}^+$ and $(f, g)$ be a pair of functions such that $f, g \in Q_\beta(I)$, $-1 < \beta \le 0$. Let $1 \le p_0 < \infty$ and for every weight function $w \in QB_{\beta,p_0}(I)$, the inequality*

$$\int_0^1 f^{p_0}(x)w(x)dx \le \varphi\Big([w]_{QB_{\beta,p_0}(I)}\Big) \int_0^1 g^{p_0}(x)w(x)dx$$

*holds. Then for every $p_0 \le p < \infty$ and every $w \in \widehat{Q}B_{\beta,p}(I)$, the inequality*

$$\int_0^1 f^p(x)w(x)dx \le K'(p) \int_0^1 g^p(x)w(x)dx$$

*holds, where*

$$K'(p) := \inf_{0<\delta<p_0(\beta+1)} [w]_{QB_{\beta,(p_0-\delta)\frac{p}{p_0}}(I)} \left[\frac{1}{\beta+1}\left(\frac{p_0(\beta+1)-\delta}{p_0-\delta}\right)\varphi\left(\frac{p_0(\beta+1)}{\delta}\right)\right]^{p/p_0}.$$

In this section, we prove Theorem 2.7 in the framework of grand Lebesgue spaces $L^{p),\theta}(I)$ ($\theta > 0$, $p > 1$) which consist of all measurable functions $f$, finite a.e. on $I$ for which

$$\|f\|_{L^{p),\theta}(I)} := \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int_0^1 |f(t)|^{p-\varepsilon} dt\right)^{1/(p-\varepsilon)} < \infty.$$

These spaces without weight have been defined in [13], which were, in fact, initially defined for $\theta = 1$ by Iwaniec and Sbordone [15] and later generalized, studied and applied by several researches in different directions. We refer to [18] and the references therein. For some very recent updates on grand Lebesgue spaces, we mention [11,14,17,20–22,26].

We now prove the following

**Theorem 3.5.** *Let $\theta > 0$, $\varphi$ be a non-negative $\uparrow$ function defined on $(0,\infty)$, $-1 < \beta \le 0, 1 < p_0 < \infty$ and $(f,g)$ be a pair of functions such that $f,g \in Q_\beta(I)$. Suppose that for every $w \in QB_{\beta,p_0}(I)$, the following inequality*

$$\int_0^1 f^{p_0}(x)w(x)dx \le \varphi\big([w]_{QB_{\beta,p_0}(I)}\big) \int_0^1 g^{p_0}(x)w(x)dx$$

*holds. Then for every $p:\ p_0 \le p < \infty$ and every $w \in \widehat{Q}B_{\beta,p}(I)$, the inequality*

$$\|f\|_{L^{p),\theta}_w(I)} \le C^* \|g\|_{L^{p),\theta}_w(I)}$$

*holds with*

$$C^* = \inf_{0<\sigma<p-1} \left[\max\left\{1, p^\theta \sigma^{-\frac{\theta}{p-\sigma}} \Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}}\right\} \sup_{0<\varepsilon\le\sigma} \left(K'(p-\varepsilon)\right)^{\frac{1}{p-\varepsilon}}\right].$$

*Proof.* Let $w \in \widehat{Q}B_{\beta,p}(I)$, then by definition $w \in QB_{\beta,p}(I)$, and there exists $0 < \xi < p(\beta+1)$ such that $w \in QB_{\beta,p-\xi}(I)$. Take $\sigma = \min\{\xi, p-p_0\}$. Clearly, $0 < \sigma < p-1$, so, by Remark 2.1, $w \in QB_{\beta,p-\sigma}(I)$. Let $\varepsilon \in (0,\sigma)$. Then, in view of the fact that $\widehat{Q}B_{\beta,p} \subset \widehat{Q}B_{\beta,q}$ for $p < q$, we have $w \in \widehat{Q}B_{\beta,p-\varepsilon}(I)$. Therefore, by Theorem 3.4, we have

$$\int_0^1 f^{p-\varepsilon}(x)w(x)dx \le K'(p-\varepsilon) \int_0^1 g^{p-\varepsilon}(x)w(x)dx, \tag{3.1}$$

where

$$K'(p-\varepsilon) := \inf_{0<\delta<p_0(\beta+1)} [w]_{QB_{\beta,(p_0-\delta)\frac{p-\varepsilon}{p_0}}(I)} \left[\frac{1}{\beta+1}\left(\frac{p_0(\beta+1)-\delta}{p_0-\delta}\right)\varphi\left(\frac{p_0(\beta+1)}{\delta}\right)\right]^{p-\varepsilon/p_0}.$$

Now, for $\sigma < \varepsilon < p-1$, using Hölder's inequality with the indices $\frac{p-\sigma}{p-\varepsilon}$ and $\frac{p-\sigma}{\varepsilon-\sigma}$, we obtain

$$\begin{aligned}\|f\|_{L^{p-\varepsilon}_w} &= \left(\int_0^1 f^{p-\varepsilon}(x)w(x)dx\right)^{1/p-\varepsilon} \\ &\le \left(\int_0^1 f^{p-\sigma}(x)w(x)dx\right)^{\frac{1}{p-\sigma}} \Big(W(I)\Big)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}} \\ &\le \left(\int_0^1 f^{p-\sigma}(x)w(x)dx\right)^{\frac{1}{p-\sigma}} \Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}}. \end{aligned} \tag{3.2}$$

Now, in view of (3.1) and (3.2), we get

$$\begin{aligned}
\|f\|_{L_w^{p),\theta}(I)} &= \max\left\{\sup_{0<\varepsilon\le\sigma}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_{L_w^{p-\varepsilon}},\ \sup_{\sigma<\varepsilon<p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_{L_w^{p-\varepsilon}}\right\}\\
&\le \max\left\{\sup_{0<\varepsilon\le\sigma}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_{L_w^{p-\varepsilon}},\ \sup_{\sigma<\varepsilon<p-1}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_{L_w^{p-\sigma}}\Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}}\right\}\\
&\le \max\left\{1, p^\theta\sigma^{-\frac{\theta}{p-\sigma}}\Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}}\right\}\sup_{0<\varepsilon\le\sigma}\varepsilon^{\frac{\theta}{p-\varepsilon}}\|f\|_{L_w^{p-\varepsilon}}\\
&\le \max\left\{1, p^\theta\sigma^{-\frac{\theta}{p-\sigma}}\Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}}\right\}\sup_{0<\varepsilon\le\sigma}\varepsilon^{\frac{\theta}{p-\varepsilon}}(K'(p-\varepsilon))^{\frac{1}{p-\varepsilon}}\|g\|_{L_w^{p-\varepsilon}}\\
&\le \max\left\{1, p^\theta\sigma^{-\frac{\theta}{p-\sigma}}\Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}}\right\}\sup_{0<\varepsilon\le\sigma}(K'(p-\varepsilon))^{\frac{1}{p-\varepsilon}}\|g\|_{L_w^{p),\theta}(I)}\\
&= C^*\|g\|_{L_w^{p),\theta}(I)},
\end{aligned}$$

where $C^* = c(p,\theta,\sigma)\sup\limits_{0<\varepsilon\le\sigma}(K'(p-\varepsilon))^{\frac{1}{p-\varepsilon}}$ and

$$c(p,\theta,\sigma) = \max\left\{1,\ p^\theta\sigma^{-\frac{\theta}{p-\sigma}}\Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}}\right\}.$$

The proof is completed. □

## 4. Application

We provide an application of the extrapolation result proved in the previous section to characterize the boundedness of the Hardy averaging operator $H$ between the weighted grand Lebesgue spaces $L_w^{p),\theta}(I)$ for quasi-monotone functions. We prove the following

**Theorem 4.1.** *Let $1<p<\infty$, $-1<\beta\le 0$ and $\theta>0$. The inequality*

$$\|Hf\|_{L_w^{p),\theta}(I)} \le C\|f\|_{L_w^{p),\theta}(I)} \tag{4.1}$$

*holds for all $f\in Q_\beta(I)$ if and only if $w\in\widehat{Q}B_{\beta,p}(I)$.*

*Proof.* Let us first assume that $w\in\widehat{Q}B_{\beta,p}(I)$. Note that if $f\in Q_\beta(I)$, then for $0<t\le s$ and $\alpha\in I$, we have

$$t^{-\beta}f\Big(\frac{\alpha t}{s}\Big)\ge s^{-\beta}f(\alpha)$$

by using which we get

$$\begin{aligned}
s^{-\beta}Hf(s) &\le \frac{1}{s}\int_0^s t^{-\beta}f\Big(\frac{\alpha t}{s}\Big)d\alpha\\
&= t^{-\beta-1}\int_0^t f(z)dz\\
&= t^{-\beta}Hf(t),
\end{aligned}$$

i.e., $Hf\in Q_\beta(I)$.

Further, on taking $\psi\equiv 1$ in a modified form of Theorem B and considering the functions $f$ defined on $I$ instead of $(0,\infty)$, we see that the inequality

$$\int_0^1\Big(Hf(x)\Big)^p w(x)dx \le C\int_0^1 f^p(x)w(x)dx$$

holds. Now, in view of Theorem 3.5, inequality (4.1) holds.

Conversely, assume that inequality (4.1) holds. Consider the test function $f_r(x) = x^\beta \chi_{(0,r)}(x)$ for $0 < r < 1$. Then

$$\|f_r\|_{L_w^{p),\theta}(I)} = \sup_{0<\varepsilon<p-1} \left( \varepsilon^\theta \int_0^r x^{\beta(p-\varepsilon)} w(x)dx \right)^{\frac{1}{p-\varepsilon}}$$
$$= \max\left\{ \sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f_r\|_{L_w^{p-\varepsilon}}, \sup_{\sigma<\varepsilon<p-1} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f_r\|_{L_w^{p-\varepsilon}} \right\},$$

where $\sigma$ is chosen such that $0 < \sigma < \min\{(\beta+1)p,\ p-1\}$. Now, for $\sigma < \varepsilon < p-1$, taking the conjugate indices $\frac{p-\sigma}{p-\varepsilon}$ and $\frac{p-\sigma}{\varepsilon-\sigma}$, on using Hölder's inequality, we obtain

$$\|f_r\|_{L_w^{p-\varepsilon}} \le \left( \int_0^r x^{\beta(p-\sigma)} w(x)dx \right)^{\frac{1}{p-\sigma}} \Big(W(I)\Big)^{\frac{\varepsilon-\sigma}{(p-\sigma)(p-\varepsilon)}}$$
$$\le \left( \int_0^r x^{\beta(p-\sigma)} w(x)dx \right)^{\frac{1}{p-\sigma}} \Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}}. \tag{4.2}$$

Thus, using (4.2) and an argument from [24, Theorem 3.1], we have

$$\|f_r\|_{L_w^{p),\theta}(I)} \le \max\left\{1, p^\theta \sigma^{-\frac{\theta}{p-\sigma}} \Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}} \right\} \sup_{0<\varepsilon\le\sigma} \varepsilon^{\frac{\theta}{p-\varepsilon}} \|f_r\|_{L_w^{p-\varepsilon}}$$
$$= \max\left\{1, p^\theta \sigma^{-\frac{\theta}{p-\sigma}} \Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}} \right\} \varepsilon_r^{\frac{\theta}{p-\varepsilon_r}} \|f_r\|_{L_w^{p-\varepsilon_r}}$$
$$= C_1 \left( \varepsilon_r^\theta \int_0^r x^{\beta(p-\varepsilon_r)} w(x)dx \right)^{\frac{1}{p-\varepsilon_r}} \tag{4.3}$$

for some $0 < \varepsilon_r \le \sigma$, where $C_1 := \inf_{0<\sigma<(\beta+1)p} \max\left\{1, p^\theta \sigma^{-\frac{\theta}{p-\sigma}} \Big(W(I)+1\Big)^{\frac{p-1-\sigma}{p-\sigma}} \right\}$. Further, note that

$$\int_0^1 \Big(Hf_r(x)\Big)^{p-\varepsilon} w(x)dx \ge \int_r^1 \Big(Hf_r(x)\Big)^{p-\varepsilon} w(x)dx$$
$$= \left( \frac{r^{\beta+1}}{\beta+1} \right)^{p-\varepsilon} \int_r^1 \frac{w(x)}{x^{p-\varepsilon}} dx,$$

so that

$$\|Hf_r\|_{L_w^{p),\theta}(I)} \ge \frac{r^{\beta+1}}{\beta+1} \sup_{0<\varepsilon<p-1} \left( \varepsilon^\theta \int_r^1 \frac{w(x)}{x^{p-\varepsilon}} dx \right)^{\frac{1}{p-\varepsilon}}$$
$$\ge \frac{r^{\beta+1}}{\beta+1} \left( {\varepsilon_r}^\theta \int_r^1 \frac{w(x)}{x^{p-\varepsilon_r}} dx \right)^{\frac{1}{p-\varepsilon_r}}.$$

The above estimate together with (4.3), and the assumption that (4.1) holds, yield

$$\frac{r^{\beta+1}}{\beta+1} \left( {\varepsilon_r}^\theta \int_r^1 \frac{w(x)}{x^{p-\varepsilon_r}} dx \right)^{\frac{1}{p-\varepsilon_r}} \le CC_1 \left( {\varepsilon_r}^\theta \int_0^r x^{\beta(p-\varepsilon_r)} w(x)dx \right)^{\frac{1}{p-\varepsilon_r}}.$$

Therefore

$$\int_r^1 \left(\frac{r}{x}\right)^{p-\varepsilon_r} w(x)dx \leq \Big(CC_1(\beta+1)\Big)^{p-\varepsilon_r} \int_0^r \left(\frac{x}{r}\right)^{\beta(p-\varepsilon_r)} w(x)dx$$

$$\leq \Big(CC_1(\beta+1)+1\Big)^{p} \int_0^r \left(\frac{x}{r}\right)^{\beta(p-\varepsilon_r)} w(x)dx.$$

Thus $w \in QB_{\beta,p-\varepsilon_r}(I)$, where $0 < \varepsilon_r < (\beta+1)p$. Consequently, $w \in QB_{\beta,p}(I)$ and hence $w \in \widehat{Q}B_{\beta,p}(I)$. □

## Acknowledgement

The first author acknowledges the MATRICS Research Grant No. MTR/2019 /000783 of SERB, Department of Science and Technology (DST), India. Also, the second author acknowledges the research fellowship award No.: 09/045(1716)/2019-EMR-I of Council of Scientific and Industrial Research (CSIR), INDIA.

*Conflict of interest statement.* On behalf of all authors, the corresponding author states that there is no conflict of interests.

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(Received 20.12.2022)

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