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THE FORGOTTEN PARAMETER IN GRAND LEBESGUE SPACES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. Let $1 < p < \infty$, $\varepsilon_0 \in]0, p-1]$, $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set of positive, finite measure, and let $\delta : (0, p-1] \to (0, \infty)$ be such that $\widehat{\delta}(\cdot) := \delta(\cdot)^{\frac{1}{p-\cdot}}$ is nondecreasing and bounded. We show that the linear set of functions

$$\left\{ f \text{ Lebesgue measurable on } \Omega \ : \ \sup_{0<\varepsilon\le\varepsilon_0} \left(\delta(\varepsilon) \fint_{\Omega} |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} < \infty \right\}$$

does not depend on small values of ε_0 if and only if $\widehat{\delta} \in \Delta_2(0+)$ (i.e., $\widehat{\delta}(2\varepsilon) \le c\widehat{\delta}(\varepsilon)$ for ε small, for some $c > 1$), which is equivalent to say that $\delta \in \Delta_2(0+)$. This means that in the case $\widehat{\delta} \notin \Delta_2(0+)$, the parameter ε_0 plays a crucial role in the definition of a generalized grand Lebesgue space, namely, different values of ε_0 define different Banach function spaces.

1. Introduction

It is well known that the *Banach function spaces,* defined as in the work by Bennett and Sharpley ([2], see also [15, 16]), consist of sets of functions defined through the finiteness of a so-called Banach function norm. Properties of norms (in particular, homogeneity and triangle inequality) guarantee that the set of functions such that a norm is finite is, in fact, a linear space. When such properties are missing, as in the case of modulars, the finiteness does not allow, in general, to get a linear space, and one has to use the trick to consider Luxemburg–Nakano functionals (see, e.g., the survey by the second author [9], where the question is analyzed in detail). The finiteness of a norm allows to define a linear set of functions and, of course, equivalent norms identifying the same set of functions. This remark is implicitly and frequently used in the Function Spaces theory, when estimates can be done through the change of a norm into another, equivalent one. The function spaces which apply a variety of different, but equivalent norms are the grand Lebesgue spaces along with their associate spaces: the reader can consult, e.g., the survey [11], which contains an overview on a recent, fruitful topic of research. Grand Lebesgue spaces have been originally defined through the finiteness of the norm (here we consider $(0,1) \subset \mathbb{R}$ equipped with the Lebesgue measure as an underlying measure space; moreover, we allow, equivalently, $0 < \varepsilon \le p-1$ instead of $0 < \varepsilon < p-1$)

$$\|f\|_{L^{p)}(0,1)} = \sup_{0<\varepsilon\le p-1} \left(\varepsilon \int_0^1 |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}} \qquad (1 < p < \infty)$$

introduced by Iwaniec and Sbordone in 1992 (see [14]) which then have been generalized to

$$\|f\|_{L^{p),\delta}(0,1)} = \operatorname*{ess\,sup}_{0<\varepsilon\le p-1} \left(\delta(\varepsilon) \int_0^1 |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}},$$

where δ is a nonnegative measurable bounded function on $]0, p-1]$ (see Remark 5.7 of [4], where this expression of the norm appears for the first time; see also [5], where a deep study has been carried out, and finally, we mention [1], where a kind of maximal generalization of the norm made either with

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respect to δ, or with respect to the exponent p; here both of them are assumed to be measurable, coinciding with a norm considered in [5]).

Recently, this class of spaces has been considered in the framework of the study of the boundedness of singular integral operators, also in their weighted version (see, e.g., [3]), but also they have been studied in their own: in particular, a special attention has been payed on how growth properties of δ influence the properties of the resulting function space $L^{p),\delta}(0,1)$. In [8], a characterization of the norm in terms of the decreasing rearrangement has been discovered, which holds if and only if the function

$$\widehat{\delta}(\varepsilon) := \delta(\varepsilon)^{\frac{1}{p-\varepsilon}}, \qquad \varepsilon \in]0, p-1],$$

assumed nondecreasing, is bounded and $\Delta_2(0+)$ (i.e., $\widehat{\delta}(2\varepsilon) \le c\widehat{\delta}(\varepsilon)$ for ε small, for some $c > 1$). In [10], it has been found that with the "price" of the replacement of δ with an equivalent function, a characterization of the norm in terms of the decreasing rearrangement holds also in the weaker assumption that δ itself is nondecreasing, bounded and $\Delta_2(0+)$. In [7], the characterization has been extended to a class of δ's, not satisfying the $\Delta_2(0+)$ property.

From the time of the first appearance of the norm of grand Lebesgue spaces, in several papers (see, e.g., [5, 6, 8, 12, 13, 17]) it has been used, and sometimes explicitly observed, that the parameter $p-1$ "under the supremum" has no special importance. In fact, the essence of the matter in the norm of grand Lebesgue spaces (either the original ones, or in the generalized version when considered with $\delta \in \Delta_2(0+)$) is played by the *small* ε's: the replacement of $p-1$ by a different, smaller parameter ε_0 makes the norm smaller, but equivalent. Hence it is natural to ask whether the same phenomenon happens to the generalized grand Lebesgue spaces. Actually, in this case this is not true, and we are going to characterize the functions δ for which the phenomenon is preserved.

2. The Main Result

Let $1 < p < \infty$, and let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set of positive, finite measure $|\Omega|$. Given $0 < \varepsilon_0 \le p-1$ and $\widehat{\delta} :]0, p-1] \to]0, +\infty[$ nondecreasing, for any measurable function f on Ω, we set

$$[f]_{\widehat{\delta},\varepsilon_0} = \sup_{0<\varepsilon\le\varepsilon_0} \widehat{\delta}(\varepsilon)\left(\fint_\Omega |f(x)|^{p-\varepsilon}dx \right)^{\frac{1}{p-\varepsilon}}.$$

The aim of this section is to prove that the linear set of functions

$$\left\{ f \text{ Lebesgue measurable on } \Omega \ : \ [f]_{\widehat{\delta},\varepsilon_0} < \infty \right\}$$

does not depend on small values of ε_0 if and only if $\widehat{\delta} \in \Delta_2(0+)$ (i.e., $\widehat{\delta}(2\varepsilon) \le c\widehat{\delta}(\varepsilon)$ for ε small, for some $c > 1$).

First, we need to prove the following preliminary

Proposition 2.1. *If $1 < p < \infty$, $0 < |\Omega| < \infty$, and $\widehat{\delta} :]0, p-1] \to]0, +\infty[$ is nondecreasing, then*

$$\begin{aligned} &\exists c = c(\widehat{\delta}) > 1 : [f]_{\widehat{\delta},2\varepsilon_0} \le c[f]_{\widehat{\delta},\varepsilon_0} \ \forall \varepsilon_0 \in \left]0, \frac{p-1}{2}\right] \\ &\Leftrightarrow \exists c = c(\widehat{\delta}) > 1 : \widehat{\delta}(2\varepsilon_0) \le c\widehat{\delta}(\varepsilon_0) \ \forall \varepsilon_0 \in \left]0, \frac{p-1}{2}\right]. \end{aligned} \tag{2.1}$$

Proof. First of all, we show $\Rightarrow$ in (2.1).

Setting $f \equiv 1$ in

$$\sup_{0<\varepsilon\le 2\varepsilon_0} \widehat{\delta}(\varepsilon)\left(\fint_\Omega |f(x)|^{p-\varepsilon}dx \right)^{\frac{1}{p-\varepsilon}} \le c \sup_{0<\varepsilon\le\varepsilon_0} \widehat{\delta}(\varepsilon)\left(\fint_\Omega |f(x)|^{p-\varepsilon}dx \right)^{\frac{1}{p-\varepsilon}} \ \forall \varepsilon_0 \in \left]0, \frac{p-1}{2}\right],$$

we have $\left(\fint_\Omega |f(x)|^{p-\varepsilon}dx \right)^{\frac{1}{p-\varepsilon}} = 1$ for all ε's, hence

$$\sup_{0<\varepsilon\le 2\varepsilon_0} \widehat{\delta}(\varepsilon) \le c \sup_{0<\varepsilon\le\varepsilon_0} \widehat{\delta}(\varepsilon) \quad \forall \varepsilon_0 \in \left]0, \frac{p-1}{2}\right]$$

and, since $\widehat{\delta}$ is nondecreasing, the implication follows.

On the other hand, let us assume that

$$\widehat{\delta}(2\varepsilon_0) \le c\widehat{\delta}(\varepsilon_0) \quad \forall \varepsilon_0 \in \left]0, \frac{p-1}{2}\right].$$

Fix $\varepsilon_0 \in \left]0, \frac{p-1}{2}\right]$. Since $\widehat{\delta}$ is nondecreasing, we have

$$\varepsilon_0 < \varepsilon \le 2\varepsilon_0 \;\Rightarrow\; \widehat{\delta}(\varepsilon) \le \widehat{\delta}(2\varepsilon_0) \le c\widehat{\delta}(\varepsilon_0)$$

and therefore, by Hölder's inequality,

$$\begin{aligned}
\sup_{\varepsilon_0<\varepsilon\le 2\varepsilon_0} \widehat{\delta}(\varepsilon)\left(\fint_\Omega |f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} &\le \sup_{\varepsilon_0<\varepsilon\le 2\varepsilon_0} c\widehat{\delta}(\varepsilon_0)\left(\fint_\Omega |f(x)|^{p-\varepsilon_0}dx\right)^{\frac{1}{p-\varepsilon_0}}\\
&= c\widehat{\delta}(\varepsilon_0)\left(\fint_\Omega |f(x)|^{p-\varepsilon_0}dx\right)^{\frac{1}{p-\varepsilon_0}}\\
&\le c[f]_{\widehat{\delta},\varepsilon_0},
\end{aligned}$$

from which, since $c > 1$, it follows that

$$[f]_{\widehat{\delta},2\varepsilon_0} = \max\left\{[f]_{\widehat{\delta},\varepsilon_0}, \sup_{\varepsilon_0<\varepsilon\le 2\varepsilon_0} \widehat{\delta}(\varepsilon)\left(\fint_\Omega |f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}\right\} \le c[f]_{\widehat{\delta},\varepsilon_0}. \qquad \square$$

In the next statement, we adopt the standard equivalence symbol $\approx$.

Corollary 2.2. *If $1 < p < \infty$, $0 < |\Omega| < \infty$, and $\widehat{\delta} :]0, p-1] \to]0, +\infty[$ is nondecreasing, then*

$$[f]_{\widehat{\delta},2\varepsilon_0} \approx [f]_{\widehat{\delta},\varepsilon_0} \;\text{ for }\; \varepsilon_0 \;\text{ small } \;\Leftrightarrow\; \widehat{\delta} \in \Delta_2(0+).$$

Proof. The statement follows simply by using the obvious inequality $[f]_{\widehat{\delta},\varepsilon_0} \le [f]_{\widehat{\delta},2\varepsilon_0}$ and that in the proposition above the number $\frac{p-1}{2}$ can be replaced by any smaller (positive) number. $\square$

Now, we are in a position to prove our main result.

Theorem 2.3. *If $1 < p < \infty$, $\varepsilon_0 \in (0, p-1]$, $0 < |\Omega| < \infty$, and $\widehat{\delta} :]0, p-1] \to]0, +\infty[$ is nondecreasing, then the linear set of functions*

$$X_{\widehat{\delta},\varepsilon_0} := \left\{f \text{ Lebesgue measurable on } \Omega \;:\; \sup_{0<\varepsilon\le\varepsilon_0} \widehat{\delta}(\varepsilon)\left(\fint_\Omega |f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}} < \infty\right\}$$

does not depend on small values of ε_0 if and only if $\widehat{\delta} \in \Delta_2(0+)$.

Proof. First, we observe that since

$$[f]_{\widehat{\delta},\varepsilon_0} = \sup_{0<\varepsilon\le\varepsilon_0} \widehat{\delta}(\varepsilon)\left(\fint_\Omega |f(x)|^{p-\varepsilon}dx\right)^{\frac{1}{p-\varepsilon}}$$

is a norm, then $X_{\widehat{\delta},\varepsilon_0}$ is a linear set of functions.

Let us start with assuming that $X_{\widehat{\delta},\varepsilon_0}$ does not depend on small values of ε_0. Since $[f]_{\widehat{\delta},\varepsilon_0}$ is a Banach function norm, by a classical result (see, e.g., [2, Corollary 1.9]),

$$X_{\widehat{\delta},2\varepsilon_0} = X_{\widehat{\delta},\varepsilon_0} \text{ for } \varepsilon_0 \text{ small } \;\Leftrightarrow\; [f]_{\widehat{\delta},2\varepsilon_0} \approx [f]_{\widehat{\delta},\varepsilon_0} \text{ for } \varepsilon_0 \text{ small},$$

and therefore, using Corollary 2.2, we get $\widehat{\delta} \in \Delta_2(0+)$.

For the converse implication let us assume $\widehat{\delta} \in \Delta_2(0+)$ and, by the contradiction, let $\varepsilon_1, \varepsilon_2 \in \left]0, \frac{p-1}{2}\right]$ be such that $X_{\widehat{\delta},\varepsilon_1} \ne X_{\widehat{\delta},\varepsilon_2}$. Without loss of generality, assume that $\varepsilon_1 < \varepsilon_2$, which implies

$[f]_{\widehat{\delta},\varepsilon_1} \leq [f]_{\widehat{\delta},\varepsilon_2}$, which in its turn gives $X_{\widehat{\delta},\varepsilon_2} \subseteq X_{\widehat{\delta},\varepsilon_1}$. Therefore there exists f_0 such that $[f_0]_{\widehat{\delta},\varepsilon_1} < \infty$ and $[f_0]_{\widehat{\delta},\varepsilon_2} = \infty$.

By Proposition 2.1,

$$[f_0]_{\widehat{\delta},\varepsilon_2} \leq c[f_0]_{\widehat{\delta},\frac{\varepsilon_2}{2}} \leq c^2[f_0]_{\widehat{\delta},\frac{\varepsilon_2}{4}} \leq \cdots \leq c^k[f_0]_{\widehat{\delta},\frac{\varepsilon_2}{2^k}}.$$

For k sufficiently large, $\dfrac{\varepsilon_2}{2^k} < \varepsilon_1$ and therefore $[f_0]_{\widehat{\delta},\varepsilon_2} \leq c^k[f_0]_{\widehat{\delta},\frac{\varepsilon_2}{2^k}} \leq c^k[f_0]_{\widehat{\delta},\varepsilon_1} < \infty$. The contradiction proves the assertion. □

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