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ON SOME SPACES WITH MIXED NORMS

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. We present Rubio de Francía's extrapolation theorem in grand function spaces generated by $L^{p(\cdot)}$ and L^q_u, where $L^{p(\cdot)}$ is the variable exponent Lebesgue space and L^q_u is the weighted constant exponent Lebesgue space. We study diagonal and off-diagonal cases. As a consequence, we have the boundedness of operators of Harmonic Analysis in these spaces.

In recent years, it was understood that classical function spaces are no longer appropriate for solving a number of contemporary problems arising naturally in various mathematical models of applied sciences. It thus became necessary to introduce and study quite new nonstandard function spaces from various viewpoints. We emphasize that in recent years the following function spaces were studied: variable exponent Lebesgue and variable exponent Sobolev spaces, grand function spaces, Morrey type spaces, mixed-normed function spaces, etc. (see, e.g., the monographs [3,4,18,19,22,23], the survey paper [12] and references cited therein).

In this note, we deal with the non-standard function spaces $\big(L^{p(\cdot)}, L^q_u\big)^{\psi(\cdot),\varphi(\cdot)}$ and $\big(L^{p(\cdot),\psi(\cdot)}, L^{q),\varphi(\cdot)}_u\big)$ defined on the base of the variable exponent Lebesgue space $L^{p(\cdot)}$ and the constant exponent weighted Lebesgue space L^q_u. The space $\big(L^{p(\cdot)}, L^q_u\big)^{\psi(\cdot),\varphi(\cdot)}$ is the grand mixed-normed space and $\big(L^{p(\cdot),\psi(\cdot)}, L^{q),\varphi(\cdot)}_u\big)$ is the mixed-normed space generated by the grand spaces $L^{p(\cdot),\psi(\cdot)}$ and $L^{q),\varphi(\cdot)}_u$. In particular, we give Rubio de Francia's extrapolation theorem in these spaces. As a consequence of the extrapolation results, we have the boundedness of operators of Harmonic Analysis in the mentioned spaces.

In 1961, the mixed Lebesgue space $L^{\overrightarrow{p}}$ with $\overrightarrow{p} \in (0,\infty]^n$, as a natural generalization of the classical Lebesgue space L^p via replacing the constant exponent p by an exponent vector $\overrightarrow{p}$, was investigated by Benedek and Panzone [1]. Indeed, the origin of these mixed Lebesgue spaces can be traced back to the interesting article of Hörmander [11] on the estimates for translation invariant operators.

For grand mixed-normed Lebesgue spaces $\big(L^{p_0}_v, L^{q_0}_u\big)^{\theta_1,\theta_2}$ and the boundedness of operators of Harmonic Analysis in these spaces we refer to the papers [14,16,17]. Extrapolation in mixed-normed Banach function spaces was studied in [8,10,17].

The boundedness problem of the strong Hardy–Littlewood maximal operator $M^{(S)}$ in the space $\big(L^{p_1(\cdot)}, L^{p_2(\cdot)}\big)$ is still open (see, e.g., [8,10]); however, it is known (see [21]) that the strong maximal operator is bounded in $L^{p(\cdot)}$ unless $p(\cdot)$ is constant.

Grand Lebesgue spaces $L^{p_0),\theta}(\Omega)$, where p_0 is a constant, $1 < p_0 < \infty$, and Ω is a bounded open set in $\mathbb{R}^n$, were introduced in T. Iwaniec and C. Sbordone [13] for $\theta = 1$, regarding the Jacobian integrability problem, and in L. Greco, T. Iwaniec and C. Sbordone [7] for $\theta > 0$ when studying the solvability problem of certain non-linear PDEs.

Grand variable exponent Lebesgue spaces were introduced in [15] (see also [5] for more precise spaces). In that paper, the mapping properties of operators of Harmonic Analysis were studied, as well. Finally, we mention that Rubio de Francía's extrapolation problem in mixed-normed space $(L^{p(\cdot)}, L^q)$ with a variable $p(\cdot)$ and constant q was studied in [10]. Our aim is to study similar problems for the spaces $\big(L^{p(\cdot)}, L^q_u\big)^{\psi(\cdot),\varphi(\cdot)}$ and $\big(L^{p(\cdot),\psi(\cdot)}, L^{q),\varphi(\cdot)}_u\big)$.

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1. Preliminaries

Let Ω be an open set in $\mathbb{R}^n$, L^0 be the space (equivalence classes) of measurable real-valued functions on Ω. A Banach space $E := E(\Omega)$ is said to be a Banach function space (BFS, briefly) on Ω if the following properties are satisfied:

(i) $\|f\|_E = 0$ if and only if $f = 0$ *a.e.*;

(ii) $|g| \le |f|$ *a.e.* implies that $\|g\|_E \le \|f\|_E$;

(iii) if $0 \le f_j \uparrow f$ *a.e.*, then $\|f_j\|_E \uparrow \|f\|_E$;

(iv) if $\chi_F \in L^0$ is such that $\mu(F) < \infty$, then $\chi_F \in E$;

(v) if $\chi_F \in L^0$ is such that $|F| < \infty$, then $\int_F f(x)dx \le C_F \|f\|_E$ for all $f \in E$ and with some positive constant C_F.

For a BFS, E, the Köthe dual (or associate) space E', is defined to be the set of all $f \in L^0(\mu)$ for which

$$\|f\|_{E'} = \sup\left\{ \int\limits_\Omega f(x)g(x)dx : \|g\|_E \le 1 \right\} < \infty.$$

It is known that the space E' is a Banach function space. For examples and properties of BFSs, we refer to [2].

Let E_1 and E_2 be BFSs defined on the open sets $\Omega_1 \subseteq \mathbb{R}^n$ and $\Omega_2 \subseteq \mathbb{R}^m$, respectively. The mixed-norm space, denoted by $(E_1(\Omega_1), E_2(\Omega_2))$, (or simply (E_1, E_2)) is defined with respect to the norm defined for a measurable function $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ as follows:

$$\|f\|_{(E_1,E_2)} = \|\|f\|_{E_1}\|_{E_2}.$$

It can be checked that for its associate space,

$$(E_1, E_2)' = (E_1', E_2')$$

holds (see, e.g., [10]).

For a Banach space E and a constant $0 < r < \infty$, the r-convexification of E is defined as follows:

$$E^r = \{f : |f|^r \in E\}.$$

E^r can be equipped with the quasi-norm $\|f\|_{E^r} = \||f|^r\|_E^{1/r}$. It can be observed that if $1 \le r < \infty$, then E^r is a Banach space, as well. For $1 \le r < \infty$ and BFSs E_1 and E_2 we have

$$(E_1, E_2)^r = (E_1^r, E_2^r).$$

For a survey on the recent developments of function spaces with mixed norms on $\mathbb{R}^n$, including mixed Lebesgue spaces, iterated weak Lebesgue spaces, weak mixed-norm Lebesgue spaces, etc., we refer, e.g., to [9, 12].

One of the examples of a BFS is a variable exponent Lebesgue space.

Let Ω be an open set in $\mathbb{R}^n$. We denote by $P(\Omega)$ the family of all real-valued measurable functions $p(\cdot)$ on Ω such that

$$1 < p_- \le p_+ < \infty,$$

where

$$p_- := p_-(\Omega) := \inf_\Omega p(x), \quad p_+ := p_+(\Omega) := \sup_\Omega p(x).$$

Let $p(\cdot) \in P(\Omega)$. The variable exponent Lebesgue space denoted by $L^{p(\cdot)}(\Omega)$ is the class of all μ-measurable functions f on Ω for which

$$S_p(f) := \int\limits_\Omega |f(x)|^{p(x)} dx < \infty.$$

The norm in $L^{p(\cdot)}(\Omega)$ is defined as follows:

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\big\{\lambda > 0 : S_p\big(f/\lambda\big) \le 1\big\}.$$

If $p(\cdot) \equiv p_0$ is constant, then $L^{p(\cdot)}(\Omega)$ is the classical Lebesgue space $L^{p_0}(\Omega)$.

Let $p(\cdot) \in P(\Omega)$. We write that $p(\cdot) \in \mathcal{B}(\Omega)$ if the Hardy-Littlewood maximal operator M_Ω defined on Ω:

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{B(x,r)} \int\limits_{B(x,r)\cap\Omega} |f(y)|dy, \ x \in \Omega,$$

is bounded in $L^{p(\cdot)}(\Omega)$.

If $\Omega = \mathbb{R}^n$, then we denote M_Ω by M.

We say that a function $p(\cdot) \in P(\Omega)$ belongs to the class $\mathcal{P}_0^{\log}(\Omega)$ (or $p(\cdot)$ satisfies the log-Hölder continuity condition on Ω) if there is a positive constant C_0 such that for all $x, y \in \Omega$ with $|x-y| \le 1/2$,

$$|p(x) - p(y)| \le \frac{C_0}{-\log|x-y|}.$$

If Ω is unbounded, then we denote by $\mathcal{P}_\infty^{\log}(\Omega)$ the class of exponents $p(\cdot)$ satisfying the condition

$$|p(x) - p(y)| \le \frac{C_\infty}{\log(e+|x|)}, \quad |y| \ge |x|, \quad x, y \in \Omega,$$

with the positive constant C_∞, independent of x and y.

If $p(\cdot) \in \mathcal{P}_0^{\log}(\Omega) \cap \mathcal{P}_\infty^{\log}(\Omega)$, then we say that $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$.

The class of exponents $\mathcal{P}^{\log}$ plays an important role in the theory of integral operators in $L^{p(\cdot)}$ spaces. For example, maximal, fractional and singular integral operators are bounded in $L^{p(\cdot)}$ under the condition $p(\cdot) \in \mathcal{P}^{\log}$ (see, e.g., the monographs [3, 4, 18] and references cited therein).

Criterion governing the boundedness of the maximal operator M in $L^{p(\cdot)}(\mathbb{R}^n)$, provided that $p(\cdot)$ is constant outside some large ball, was found in [20]. In particular, in that paper the author proved that M is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ for $p(\cdot) \in P(\mathbb{R}^n)$ if and only if $dx \in A_{p(\cdot)}$, i.e.,

$$\sup_Q |Q|^{-1} \|\chi_Q\|_{L^{p(\cdot)}} \|\chi_Q\|_{L^{p'(\cdot)}} < \infty, \ \ p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1},$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ with sides, parallel to the coordinate axes.

Let $|\Omega| < \infty$ and $p \in P(\Omega)$. In [15], the authors introduced the space $L^{p(\cdot),\psi(\cdot)}(\Omega)$ (resp., $\mathcal{L}^{p(\cdot),\psi(\cdot),\sigma(\cdot)}(\Omega)$) called grand variable exponent Lebesgue space ($GVELS$ briefly) which is defined with respect to the norm

$$\|f\|_{L^{p(\cdot),\psi(\cdot)}(\Omega)} = \sup_{0<\varepsilon<p_- -1} \big(\psi(\varepsilon)\big)^{\frac{1}{p_- - \varepsilon}} \|f\|_{L^{p(\cdot)-\varepsilon}(\Omega)}$$

$$\Big(\text{resp. } \|f\|_{\mathcal{L}^{p(\cdot),\psi(\cdot),\sigma(\cdot)}(\Omega)} = \sup_{\eta(\cdot)\in\mathcal{P}_0(\sigma(\cdot),\Omega)} \varphi(\eta_+)^{\frac{1}{p_- - \eta_+}} \|f\|_{L^{p(\cdot)-\eta(\cdot)}(\Omega)} \Big),$$

where $\psi(\cdot)$ is a non-decreasing function on $(0, p_- - 1)$ such that $\lim\limits_{t\to 0+} \psi(t) = 0$, and $\mathcal{P}_0(\sigma(\cdot),\Omega)$ is the class of exponents $\eta(\cdot)$ such that $\eta(\cdot) \in \mathcal{P}_0(\Omega)$ and $0 < \eta_- \le \eta(x) \le \sigma(x) \le \sigma_+ \le p_- - 1$ with a given positive function $\sigma(\cdot)$ on Ω.

If $\psi(t) = t^\theta$ with the positive constant θ, then $GVELS$s $L^{p(\cdot),\psi(\cdot)}(\Omega)$ and $\mathcal{L}^{p(\cdot),\psi(\cdot),\sigma(\cdot)}(\Omega)$ will be denoted by $L^{p(\cdot),\theta}(\Omega)$ and $\mathcal{L}^{p(\cdot),\theta,\sigma(\cdot)}(\Omega)$, respectively.

For a constant q, $1 < q < \infty$, an open bounded set Ω on $\mathbb{R}^m$ and a weight (i.e., a.e. positive locally integrable function) u on Ω, we denote by $L_u^{q),\varphi(\cdot)}(\Omega)$ weighted grand Lebesgue space which is determined by the norm

$$L_u^{q),\varphi(\cdot)}(\Omega) := \sup_{0<\varepsilon<q-1} \Bigg(\varphi(\varepsilon) \int\limits_\Omega |f(x)|^{q-\varepsilon} u(x) dx \Bigg)^{1/(q-\varepsilon)}.$$

For $\psi(t) = t^\theta$, $\theta > 0$, and $u \equiv$ const, $L_u^{q),\varphi(\cdot)}(\Omega)$ is the Iwaniec-Sbordone space denoted by $L^{q),\theta}(\Omega)$. In [6], the authors proved that for the boundedness of the Hardy–Littlewood maximal operator in $L_u^{q),\theta}(\Omega)$ it is necessary and sufficient that u belongs to the Muckenhoupt class A_q. Later, it turned out that a similar result holds for other operators of Harmonic Analysis (see [19] and references therein).

Let Ω_1 and Ω_2 be open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Suppose that $p(\cdot)\in P(\Omega)$ and let q be a constant such that $1<q<\infty$. Let u be a weight function on $\mathbb{R}^m$. Let us introduce a mixed-normed space (see [10] for unweighted case) defined with respect to the norm

$$\|f\|_{\left(L^{p(\cdot)}(\Omega_1),L_u^{q)}(\Omega_2)\right)}:=\|\|f(x,\cdot)\|_{L^{p(x)}(\Omega_1)}\|_{L_u^q(\Omega_2)}=\Bigg(\int\limits_{\Omega_2}\|f(\cdot,y)\|^q_{L^{p(\cdot)}(\Omega_1)}u(y)dy\Bigg)^{\frac{1}{q}}.$$

Let $p(\cdot)\in P(\Omega)$ and $1<q<\infty$. Suppose that $\psi(\cdot)$ and $\varphi(\cdot)$ are positive non-decreasing functions on the intervals $(0,p_--1)$ and $(0,q-1)$, respectively such that $\lim\limits_{\lambda\to 0+}\psi(\lambda)=\lim\limits_{\lambda\to 0+}\varphi(\lambda)=0$. In this case, we say that $(\psi(\cdot),\varphi(\cdot))\in\mathcal{A}_{p(\cdot),q}$.

Let now Ω_1 and Ω_2 be bounded open sets in $\mathbb{R}^n$ and $\mathbb{R}^n$, respectively. Suppose that $p(\cdot)\in P(\Omega_1)$ and let q be a constant such that $1<q<\infty$. Let u be a weight function on Ω_2 and $(\psi(\cdot),\varphi(\cdot))\in\mathcal{A}_{p(\cdot),q}$. We say that a measurable function f on $\Omega_1\times\Omega_2$ belongs to $\left(L^{p(\cdot)}(\Omega_1),L_u^q(\Omega_2)\right)^{\psi(\cdot),\varphi(\cdot)}$ if

$$\|f\|_{\left(L^{p(\cdot)}(\Omega_1),L_u^q(\Omega_2)\right)^{\psi(\cdot),\varphi(\cdot)}}$$
$$=\sup_{0<\varepsilon_1<p_--1}\sup_{0<\varepsilon_2<q-1}\big(\psi(\varepsilon_1)\big)^{\frac{1}{p_--\varepsilon_1}}\big(\varphi(\varepsilon_2)\big)^{\frac{1}{q-\varepsilon_2}}\|f\|_{\left(L^{p(\cdot)-\varepsilon_1}(\Omega_1),L_u^{q-\varepsilon_2}(\Omega_2)\right)}<\infty.$$

A measurable function $f:\Omega_1\times\Omega_2\mapsto\mathbb{R}$ belongs to $\left(\mathcal{L}^{p(\cdot)}(\Omega_1),L_u^q(\Omega_2)\right)^{\psi(\cdot),\varphi(\cdot)}_{\sigma(\cdot)}$ if

$$\|f\|_{\left(\mathcal{L}^{p(\cdot)}(\Omega_1),L_u^q(\Omega_2)\right)^{\psi(\cdot),\varphi(\cdot)}_{\sigma(\cdot)}}$$
$$=\sup_{\eta(\cdot)\in\mathcal{P}_0(\sigma(\cdot),\Omega)}\sup_{0<\varepsilon<q-1}\big(\psi(\eta_+)\big)^{\frac{1}{p_--\eta_+}}\big(\varphi(\varepsilon)\big)^{\frac{1}{q-\varepsilon}}\|f\|_{\left(L^{p(\cdot)-\eta(\cdot)}(\Omega_1),L_u^{q-\varepsilon}(\Omega_2)\right)}<\infty.$$

Further, we are also interested in the investigation of the mixed-normed grand function spaces

$$\left(L^{p(\cdot),\psi(\cdot)},L_u^{q),\varphi(\cdot)}\right)\ \text{ and }\ \left(\mathcal{L}^{p(\cdot),\psi(\cdot),\sigma(\cdot)},L_u^{q),\varphi(\cdot)}\right).$$

It is clear that

$$\left(L^{p(\cdot),\psi(\cdot)},L_u^{q),\varphi(\cdot)}\right)\hookrightarrow\left(L^{p(\cdot)},L_u^q\right)^{\psi(\cdot),\varphi(\cdot)}$$

and

$$\left(\mathcal{L}^{p(\cdot),\psi(\cdot),\sigma(\cdot)},L_u^{q),\varphi(\cdot)}\right)\hookrightarrow\left(\mathcal{L}^{p(\cdot)},L_u^q\right)^{\psi(\cdot),\varphi(\cdot)}_{\sigma(\cdot)}.$$

Define a strong Hardy–Littlewood maximal operator on $\Omega\times\Omega$ as follows:

$$M^{(S)}_{\Omega_1,\Omega_2}g(x,y)=\sup_{B_1\ni x,B_2\ni y}\frac{1}{|B_1||B_2|}\int\limits_{\overline{B}_1\times\overline{B}_2}|g(t,s)|\,dtds,\quad g\in L_{loc}(\Omega_1\times\Omega_2),$$

where $\overline{B}_1:=B_1\cap\Omega_1$, $\overline{B}_2:=B_2\cap\Omega_2$, B_1 and B_2 are the balls in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

If $\Omega_1=\mathbb{R}^n$ and $\Omega_2=\mathbb{R}^m$, then $M^{(S)}_{\Omega_1,\Omega_2}$ is denoted by $M^{(S)}$.

We say that a weight function u on $\mathbb{R}^n$ belongs to the Muckenhoupt class A_{p_0}, $1<p_0<\infty$, if

$$\Bigg(\frac{1}{|B|}\int\limits_B w(x)dx\Bigg)\Bigg(\frac{1}{|B|}\int\limits_B w^{1-p_0'}(x)dx\Bigg)^{p_0-1}\le C,\quad p_0'=\frac{p_0}{p_0-1},$$

for all balls B in $\mathbb{R}^n$. Further, a weight function w on $\mathbb{R}^n\times\mathbb{R}^m$ belongs to the strong Muckenhoupt class $A^{(S)}_{p_0}$ if

$$\big[w\big]_{A^{(S)}_{p_0}}:=\sup_{B_1,B_2}\Bigg(\frac{1}{|B_1\times B_2|}\int\limits_{B_1\times B_2}w(x,y)dxdy\Bigg)\Bigg(\frac{1}{|B_1\times B_2|}\int\limits_{B_1\times B_2}w^{1-p_0'}(x,y)dxdy\Bigg)^{p_0-1}<\infty,$$

where the supremum is taken over all balls $B_1\subset\mathbb{R}^n$ and $B_2\subset\mathbb{R}^m$. We say that a weight w on $\mathbb{R}^n\times\mathbb{R}^m$ belongs to the class $A_1^{(S)}$ if there is a positive constant C such that for a.e. $(x,y)\in\mathbb{R}^n\times\mathbb{R}^m$,

$$M^{(S)}w(x,y)\le Cw(x,y).\tag{1}$$

The best possible constant in (1) is called $A_1^{(S)}-$ characteristic of a weight w and is denoted by $[w]_{A_1^{(S)}}$.

It is easy to see that the following relation

$$[w]_{A_{p_0}^{(S)}} \le [w]_{A_{p_1}^{(S)}}, \qquad 1 \le p_1 \le p_0 < \infty$$

holds.

Further, we will need also the Muckenhoupt–Wheeden class of weights A_{p_0,q_0}. A weight u on $\mathbb{R}^m$ belongs to the class $\in A_{p_0,q_0}$, $1 < p_0, q_0 < \infty$, if

$$[u]_{A_{p_0,q_0}} = \sup_B \left(\frac{1}{|B|}\int\limits_B u^{q_0}(x)dx\right)\left(\frac{1}{|B|}\int\limits_B u^{-p_0'}(x)dx\right)^{q_0/p_0} < \infty.$$

Our first statement reads as follows.

Theorem 1.1 (Diagonal case)**.** *Let Ω_1 and Ω_2 be bounded domains in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $\mathcal{F}$ be a family of pairs (f,g) of measurable functions f and g defined on $\Omega_1\times\Omega_2$. Suppose that for some $1 < p_0 < \infty$, for every $w \in A_{p_0}^{(S)}$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\left(\int\limits_{\Omega_1\times\Omega_2} g^{p_0}(x,y)w(x,y)\,dxdy\right)^{\frac{1}{p_0}} \le CN\big([w]_{A_{p_0}^{(S)}}\big)\left(\int\limits_{\Omega_1\times\Omega_2} f^{p_0}(x,y)w(x,y)\,dxdy\right)^{\frac{1}{p_0}}$$

holds with some positive constant C, independent of (f,g), and the constant $N\big([w]_{A_{p_0}^{(S)}}\big)$ such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Then for any exponent $p(\cdot) \in \mathcal{B}(\Omega_1)$, the constant q, $1 < q < \infty$, for all $(f,g) \in \mathcal{F}$, $u \in A_q$, and every $\big(\psi(\cdot),\varphi(\cdot)\big) \in \mathcal{A}_{p(\cdot),q}$, the inequality

$$\|g\|_{(E_1,E_2)} \le \widetilde{C}\|f\|_{(E_1,E_2)}$$

holds with a positive constant $\widetilde{C}$, independent of (f,g), where

$$(E_1,E_2) := \big(L^{p(\cdot),\psi(\cdot)}(\Omega_1), L_u^{q),\varphi(\cdot)}(\Omega_2)\big),$$

$$\big(resp.,\ (E_1,E_2) := \big(L^{p(\cdot)}(\Omega_1), L_u^{q}(\Omega_2)\big)^{\psi(\cdot),\varphi(\cdot)}\big).$$

In the off-diagonal case, we have

Theorem 1.2 (Off-diagonal Case)**.** *Let Ω_1 and Ω_2 be bounded domains in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $\mathcal{F}$ be a family of pairs (f,g) of measurable functions f and g defined on $\Omega_1\times\Omega_2$. Suppose that for some $1 < p_0 \le q_0 < \infty$, for every $w \in A_{1+q_0/p_0'}^{(S)}(\Omega_1\times\Omega_2)$ and all $(f,g) \in \mathcal{F}$, the one-weight inequality*

$$\left(\int\limits_{\Omega_1\times\Omega_2} g^{q_0}(x,y)w(x,y)\,dxdy\right)^{\frac{1}{q_0}} \le CN\big([w]_{A_{1+\frac{q_0}{p_0'}}^{(S)}}\big)\left(\int\limits_{\Omega_1\times\Omega_2} f^{p_0}(x,y)w^{\frac{p_0}{q_0}}(x,y)\,dxdy\right)^{\frac{1}{p_0}}$$

holds with the positive constant C, independent of (f,g), and the constant $N\big([w]_{A_{1+\frac{q_0}{p_0'}}^{(S)}}\big)$ such that the mapping $s \to N(s)$ is non-decreasing. Then for any variable exponents $p_1(\cdot), q_1(\cdot) \in P(\Omega_1)$, with $q_1(\cdot) \in \mathcal{B}(\Omega_1)$, constant exponents $1 < p_2, q_2 < \infty$ satisfying the condition

$$\frac{1}{p_1(\cdot)} - \frac{1}{q_1(\cdot)} = \frac{1}{p_2} - \frac{1}{q_2} = \frac{1}{p_0} - \frac{1}{q_0},$$

for all $0 < \theta_1,\ \theta_2 < \infty$, for all $(f,g) \in \mathcal{F}$ and any $u \in A_{p_2,q_2}(\mathbb{R}^m)$, the inequality

$$\|u(y)\|g(\cdot,y)\|_{\mathcal{L}^{q_1(\cdot),\frac{\theta_1(q_1)_-}{(p_1)_-},\sigma(\cdot)}(\Omega_1)}\|_{L^{q_2),\frac{\theta_2 q_2}{p_2}}(\Omega_2))} \le \widetilde{C}\|u(y)\|f(\cdot,y)\|_{\mathcal{L}^{p_1(\cdot),\theta_1,\eta(\cdot)}(\Omega_1)}\|_{L^{p_2),\theta_2}(\Omega_2))},$$

$$\left(resp.\ \|u(y)g(x,y)\|_{\left(\mathcal{L}^{q_1(x)}(\Omega_1),L^{q_2}(\Omega_2)\right)_{\sigma(\cdot)}^{\frac{\theta_1(q_1)_-}{(p_1)_-},\frac{\theta_2 q_2}{p_2}}} \le \widetilde{C}\|u(y)f(x,y)\|_{\left(\mathcal{L}^{p_1(x)}(\Omega_1),L^{p_2}(\Omega_2)\right)_{\eta(\cdot)}^{\theta_1,\theta_2}}\right)$$

is fulfilled, where $\sigma(\cdot)$, $\eta(\cdot)$ are defined as follows:

$$\frac{1}{p_1(x)-\sigma(x)}-\frac{1}{q_1(x)-\eta(x)}=\frac{1}{p_2}-\frac{1}{q_2}=\frac{1}{p_0}-\frac{1}{q_0},$$

and the positive constant $\widetilde{C}$ *is independent of* (f,g).

Further, we have

Theorem 1.3. *Let* $\mathcal{F}$ *be a family of pairs* (f,g) *of measurable functions* f *and* g *defined on* $\mathbb{R}^n\times\mathbb{R}^m$. *Suppose that for some* $1<p_0<\infty$ *and for every* $w\in A^{(S)}_{p_0}$ *and all* $(f,g)\in\mathcal{F}$, *the one-weight inequality*

$$\bigg(\int\limits_{\mathbb{R}^n\times\mathbb{R}^m} g^{p_0}(x,y)w(x,y)\,dxdy\bigg)^{\frac{1}{p_0}}\le CN\big([w]_{A^{(S)}_{p_0}}\big)\bigg(\int\limits_{\mathbb{R}^n\times\mathbb{R}^m} f^{p_0}(x,y)w(x,y)\,dxdy\bigg)^{\frac{1}{p_0}}$$

holds with some positive constant C, *independent of* (f,g), *and the constant* $N\big([w]_{A^{(S)}_{p_0}}\big)$ *such that the mapping* $\cdot\mapsto N(\,\cdot\,)$ *is non-decreasing. Then for any variable exponent* $p(\cdot)$ *with* $p(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, *a constant* q, $1<q<\infty$, *and every* $u\in A_q(\mathbb{R}^m)$, *the inequality*

$$\|g\|_{\big(L^{p(\cdot)}(\mathbb{R}^n),L^q_u(\mathbb{R}^m)\big)}\le\widetilde{C}\|f\|_{\big(L^{p(\cdot)}(\mathbb{R}^n),L^q_u(\mathbb{R}^m)\big)},$$

where the positive constant $\widetilde{C}$ *is independent of* (f,g).

Theorem 1.3 in the unweighted case ($u\equiv 1$) was derived in [10, Theorem 1].

Theorem 1.4 (Off-diagonal Case). *Let* Ω_1 *and* Ω_2 *be the bounded domains in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$, *respectively. Let* $\mathcal{F}$ *be a family of pairs* (f,g) *of measurable functions* f *and* g *defined on* $\mathbb{R}^n\times\mathbb{R}^m$. *Suppose that for some* $1<p_0\le q_0<\infty$ *and for every* $w\in A^{(S)}_{1+q_0/p_0'}$ *and* $(f,g)\in\mathcal{F}$, *the one-weight inequality*

$$\bigg(\int\limits_{\mathbb{R}^n\times\mathbb{R}^m} g^{q_0}(x,y)w(x,y)\,dxdy\bigg)^{\frac{1}{q_0}}\le CN\big([w]_{A^{(S)}_{1+\frac{q_0}{p_0'}}}\big)\bigg(\int\limits_{\mathbb{R}^n\times\mathbb{R}^m} f^{p_0}(x,y)w^{\frac{p_0}{q_0}}(x,y)\,dxdy\bigg)^{\frac{1}{p_0}}$$

holds with the positive constant C *independent of* (f,g), *and some non-decreasing mapping* $s\to N(s)$. *Then for any exponents* $p_1(\cdot),q_1(\cdot)\in P(\mathbb{R}^n)$ *with* $q_1(\cdot)\in\mathcal{B}(\mathbb{R}^n)$, *constant exponents* $1<p_2,q_2<\infty$ *satisfying the condition*

$$\frac{1}{p_1(x)}-\frac{1}{q_1(x)}=\frac{1}{p_2}-\frac{1}{q_2}=\frac{1}{p_0}-\frac{1}{q_0},$$

for all $(f,g)\in\mathcal{F}$ *and any* $u\in A_{p_2,q_2}(\mathbb{R}^m)$,

$$\|u(y)\|g(\cdot,y)\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|_{L^{q_2}(\mathbb{R}^m)}\le\widetilde{C}\|u(y)\|f(\cdot,y)\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|_{L^{p_2}(\mathbb{R}^m)},$$

with the positive constant $\widetilde{C}$, *independent of* $(f,g)\in\mathcal{F}$.

Remark 1.1. As a consequence of these statements, we have the boundedness of operators of Harmonic Analysis for which the one-weight inequality holds under the strong Muckenhoupt condition $A^{(S)}_{p_0}$, $1<p_0<\infty$. Such operators are, for example, Hardy–Littlewood maximal operator $M^{(S)}$, multiple Calderón–Zygmund and multiple fractional integral operators, etc.

Remark 1.2. In Theorems 1.1 and 1.2, we may assume that the Muckenhoupt classes are defined for weights on domains Ω_1 and Ω_2, where Ω_1 and Ω_2 satisfy the conditions: $|B(x,R_1)|\ge CR_1^n$, $|B(y,R_2)|\ge CR_2^m$, $x\in\Omega_1$, $y\in\Omega_2$, $R_1\in(0,\,\mathrm{diam}(\Omega_1))$, $R_2\in(0,\,\mathrm{diam}(\Omega_2))$.

In the sequel, we write $(E_1,E_2)\in\mathbb{M}_{\mathbb{S}}$ if the operator $M^{(S)}$ is bounded in (E_1,E_2).

Theorems 1.1, 1.2, 1.3, 1.4 follow from more general statements formulated in quantitative form (see also [10] for the diagonal case).

Theorem 1.5 (Diagonal Case)**.** *Let $\mathcal{F}$ be a family of pairs (f,g) of measurable functions f and g defined on $\Omega_1\times\Omega_2$. Suppose that for some $1<p_0<\infty$ and for every $w\in A^{(S)}_{p_0}$ and all $(f,g)\in\mathcal{F}$, the one-weight inequality*

$$\bigg(\int\limits_{\Omega_1\times\Omega_2} g^{p_0}(x,y)w(x,y)\,dxdy\bigg)^{\frac{1}{p_0}}\le CN\big([w]_{A^{(S)}_{p_0}}\big)\bigg(\int\limits_{\Omega_1\times\Omega_2} f^{p_0}(x,y)w(x,y)\,dxdy\bigg)^{\frac{1}{p_0}} \tag{2}$$

holds with some positive constant C, independent of (f,g), and some non-decreasing mapping $s\to N(s)$. Suppose that there exists $1<q_0<\infty$ such that E_1^{1/q_0} and E_2^{1/q_0} are again BFSs. If $\big((E_1^{1/q_0})',(E_2^{1/q_0})'\big)\in\mathbb{M}_{\mathbb{S}}$, then for any $(f,g)\in\mathcal{F}$,

$$\|g\|_{(E_1,E_2)}\le J\big(\|M^{(S)}\|_{(E_1^{1/q_0},E_2^{1/q_0})'},p_0,q_0\big)\|f\|_{(E_1,E_2)},$$

where $J\big(\|M^{(S)}\|_{(E_1^{1/q_0},E_2^{1/q_0})'},p_0,q_0\big)$ is the constant such that the mapping $t\mapsto J\big(t,p_0,q_0\big)$ is non-decreasing in t for fixed p_0 and q_0.

Theorem 1.6 (Off-diagonal Case)**.** *Let $\mathcal{F}$ be a family of pairs (f,g) of measurable functions f,g on $\Omega_1\times\Omega_2$. Suppose that for some $1<p_0,q_0<\infty$ and for every $w\in A^{(S)}_{1+q_0/p_0'}$ and $(f,g)\in\mathcal{F}$, the one-weight inequality*

$$\bigg(\int\limits_{\Omega_1\times\Omega_2} g^{q_0}(x,y)w(x,y)\,dxdy\bigg)^{\frac{1}{q_0}}\le CN\big([w]_{A^{(S)}_{1+\frac{q_0}{p_0'}}}\big)\bigg(\int\limits_{\Omega_1\times\Omega_2} f^{p_0}(x,y)w^{\frac{p_0}{q_0}}(x,y)\,dxdy\bigg)^{\frac{1}{p_0}} \tag{3}$$

holds with some positive constant C, independent of (f,g), and non-decreasing mapping $s\to N(s)$. Suppose that $\overline{E}_1:=\overline{E}_1(\Omega_1)$, $E_1:=E_1(\Omega_1)$, $\overline{E}_2:=\overline{E}_2(\Omega_2)$ and $E_2:=E_2(\Omega_2)$ are BFSs. Let there exist $1<\widetilde{p}_0<\infty$, $1<\widetilde{q}_0<\infty$ such that

$$\frac{1}{\widetilde{p}_0}-\frac{1}{\widetilde{q}_0}=\frac{1}{p_0}-\frac{1}{q_0},$$

and $\overline{E}_1^{1/\widetilde{q}_0}$, $E_1^{1/\widetilde{p}_0}$, $\overline{E}_2^{1/\widetilde{q}_0}$, $E_2^{1/\widetilde{p}_0}$ are again BFSs satisfying the conditions

$$\big(\overline{E}_1^{1/\widetilde{q}_0}\big)'=\big[\big(E_1^{1/\widetilde{p}_0}\big)'\big]^{\widetilde{p}_0/\widetilde{q}_0};\qquad \big(\overline{E}_2^{1/\widetilde{q}_0}\big)'=\big[\big(E_2^{1/\widetilde{p}_0}\big)'\big]^{\widetilde{p}_0/\widetilde{q}_0}.$$

If $\big((\overline{E}_1^{1/\widetilde{q}_0})',(\overline{E}_2^{1/\widetilde{q}_0})'\big)\in\mathbb{M}_{\mathbb{S}}$, then for any $(f,g)\in\mathcal{F}$,

$$\|g\|_{(\overline{E}_1,\overline{E}_2)}\le\overline{J}\Big(\|M^{(S)}\|_{\left((\overline{E}_1^{1/\widetilde{q}_0})',(\overline{E}_2^{1/\widetilde{q}_0})'\right)},p_0,q_0,\widetilde{p}_0,\widetilde{q}_0\Big)\|f\|_{(E_1,E_2)},$$

where the constant $\overline{J}\big(\|M^{(S)}\|_{\left((\overline{E}_1^{1/\widetilde{q}_0})',(\overline{E}_2^{1/\widetilde{q}_0})'\right)},p_0,q_0,\widetilde{p}_0,\widetilde{q}_0\big)$ is such that the mapping $t\mapsto\overline{J}\big(t,p_0,q_0,\widetilde{p}_0,\widetilde{q}_0\big)$ is non-decreasing in t for fixed $p_0,q_0,\widetilde{p}_0,\widetilde{q}_0$.

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