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# ON $\pi$-WEIGHTS AND EXTENSIONS OF INVARIANT MEASURES

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*Dedicated to the memory of Academician Vakhtang Kokilashvili*

**Abstract.** We consider some extensions of invariant (quasi-invariant) measures on a ground set $E$, which have a $\pi$-base of cardinality not exceeding $\mathrm{card}(E)$.

It is well known that there are many analogies between purely topological concepts and measure-theoretical concepts. The analogies of this kind are thoroughly considered and discussed, e.g., in the excellent text-book by J. C. Oxtoby [5].

For instance, the notion of a $\pi$-base (or pseudo-base) of a topological space $(E, \mathcal{T})$ is one of the main topological invariants of $(E, \mathcal{T})$ and plays an important role in set-theoretic topology (cf., for instance, [1]).

A quite similar concept of a $\pi$-base was introduced for any measure space $(E, \mu)$.

Let $(E, \mu)$ be a measure space and let $\mathcal{U}$ be a family of $\mu$-measurable subsets of $E$.

In this note we say that $\mathcal{U}$ is a $\pi$-base (or pseudo-base) of $\mu$ if for every $\mu$-measurable set $X$ with $\mu(X) > 0$, there exists a set $Y \in \mathcal{U}$ such that $Y \subset X$ and $\mu(Y) > 0$.

Similarly to the definition of the $\pi$-weight of $(E, \mathcal{T})$, the $\pi$-weight of $\mu$ is defined as the minimum of all cardinalities of $\pi$-bases of $\mu$, and denoted by $\pi(\mu)$.

In the sequel, $\mathrm{dom}(\mu)$ will stand for the family of all $\mu$-measurable subsets of $E$ and the symbol $\mathcal{I}(\mu)$ will stand for the $\sigma$-ideal in $E$ generated by the family of all $\mu$-measure zero subsets of $E$.

Recall that, by the definition, a base of $\mathcal{I}(\mu)$ is any family $\mathcal{B} \subset \mathcal{I}(\mu)$ such that, for each set $X \in \mathcal{I}(\mu)$, there exists a set $Y \in \mathcal{B}$ containing $X$.

**Lemma 1.** *If $E$ is an infinite ground set and $\mu$ is a nonzero $\sigma$-finite measure on $E$, then the $\sigma$-ideal $\mathcal{I}(\mu)$ has a base whose cardinality does not exceed $(\pi(\mu))^\omega$.*

*In particular, if $(\mathrm{card}(E))^\omega = \mathrm{card}(E)$ and $\pi(\mu) \le \mathrm{card}(E)$, then the $\sigma$-ideal $\mathcal{I}(\mu)$ has a base whose cardinality does not exceed $\mathrm{card}(E)$.*

**Remark 1.** In connection with Lemma 1, it makes sense to recall that under the Generalized Continuum Hypothesis (**GCH**), the following two assertions are equivalent:

(a) $(\mathrm{card}(E))^\omega = \mathrm{card}(E)$;

(b) $\mathrm{card}(E)$ is not cofinal with $\omega$.

At the same time, the implication (a) $\Rightarrow$ (b) is valid in **ZFC** set theory.

**Theorem 1.** *Let $(G, \cdot)$ be an infinite solvable group such that*

$$(\mathrm{card}(G))^\omega = \mathrm{card}(G)$$

*and let $\mu$ be a nonzero $\sigma$-finite left $G$-invariant (left $G$-quasi-invariant) measure on $G$ with $\pi(\mu) \le \mathrm{card}(G)$.*

*Then there exists a left $G$-invariant (left $G$-quasi-invariant) measure $\mu'$ on $G$, properly extending $\mu$ and also satisfying the inequality $\pi(\mu') \le \mathrm{card}(G)$.*

The proof of this theorem is based on the fact that there exists a countable cover of $G$ with $G$-absolutely negligible subsets of $E$ (see [3] and [4]).

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**Lemma 2.** *Let $E$ be an infinite ground set and let $\{X_i : i \in I\}$ be a family of subsets of $E$ such that* $\mathrm{card}(I) \leq \mathrm{card}(E)$ *and* $\mathrm{card}(X_i) = \mathrm{card}(E)$ *for each index* $i \in I$.

*Then there exists a family $\{Y_j : j \in J\}$ of subsets of $E$ satisfying these three relations:*

(1) $\mathrm{card}(J) > \mathrm{card}(E)$;

(2) $\{Y_j : j \in J\}$ *is almost disjoint, i.e., for any two distinct indices $j \in J$ and $k \in J$, the inequality* $\mathrm{card}(Y_j \cap Y_k) < \mathrm{card}(E)$ *holds true;*

(3) $\mathrm{card}(X_i \cap Y_j) = \mathrm{card}(E)$ *for every $i \in I$ and for every $j \in J$.*

The proof of Lemma 2 is given in [2]. Using Lemmas 1 and 2, one can establish the following statement.

**Theorem 2.** *Let $(G, \cdot)$ be an infinite group satisfying these two conditions:*

(1) $(\mathrm{card}(G))^{\omega} = \mathrm{card}(G)$;

(2) $\mathrm{card}(G)$ *is a regular cardinal number.*

*Let $\mu$ be a nonzero $\sigma$-finite left $G$-invariant (left $G$-quasi-invariant) measure on $G$ such that* $\pi(\mu) \leq \mathrm{card}(G)$ *and every subset $C$ of $G$ with* $\mathrm{card}(C) < \mathrm{card}(G)$ *is measurable with respect to $\mu$.*

*Then there exists a left $G$-invariant (left $G$-quasi-invariant) measure $\mu'$ on $G$ which properly extends $\mu$ and for which the inequality* $\pi(\mu') \leq \mathrm{card}(G)$ *is also valid.*

**Remark 2.** Furthermore, taking into account Lemma 2, it can be shown that the cardinality of the family of all measures $\mu'$ indicated in Theorem 2 is strictly greater than $\mathrm{card}(G)$.

**Lemma 3.** *Let $E$ be an infinite ground set such that*

$$(\mathrm{card}(E))^{\omega} = \mathrm{card}(E),$$

*let $G$ be a group of transformations of $E$ with* $\mathrm{card}(G) \leq \mathrm{card}(E)$, *and let $\mu$ be a nonzero $\sigma$-finite $G$-invariant ($G$-quasi-invariant) measure on $E$ satisfying the following conditions:*

(1) $\pi(\mu) \leq \mathrm{card}(E)$;

(2) *no set $Z \in \mathrm{dom}(\mu)$ with $\mu(Z) > 0$ can be covered by a family $\mathcal{F} \subset \mathcal{I}(\mu)$ whose cardinality is strictly less than* $\mathrm{card}(E)$;

(3) *all singletons in $E$ are of $\mu$-measure zero.*

*Then there exists a set $Y \subset E$ such that:*

(a) $\mathrm{card}(Y) = \mathrm{card}(E)$;

(b) *if $T$ is any $\mu$-measure zero subset of $E$, then* $\mathrm{card}(T \cap Y) < \mathrm{card}(E)$;

(c) *both sets $Y$ and $E \setminus Y$ are $\mu$-thick in $E$, i.e.,*

$$Y \cap Z \neq \emptyset, \ (E \setminus Y) \cap Z \neq \emptyset$$

*whenever $Z \in \mathrm{dom}(\mu)$ and $\mu(Z) > 0$;*

(d) *$Y$ is almost $G$-invariant in $E$, i.e., for each transformation $g \in G$, the inequality*

$$\mathrm{card}(g(Y) \triangle Y) < \mathrm{card}(E)$$

*holds true (where $\triangle$ denotes, as usual, the operation of symmetric difference of two sets).*

**Remark 3.** In connection with (a) and (b) of Lemma 3, it should be pointed out that the set $Y$ is a certain analog of a classical Sierpiński set on the real line $\mathbf{R}$ (for the definition and pivotal properties of Sierpiński sets see, e.g., [5]). Moreover, $Y$ possesses some additional properties: assertions (c) and (d) give, respectively, the $\mu$-thickness and almost $G$-invariance of $Y$. As is well known, any Sierpiński set is nonmeasurable with respect to the standard Lebesgue measure on $\mathbf{R}$. Analogously, in view of (c), the set $Y$ is nonmeasurable with respect to $\mu$.

**Remark 4.** Condition (3) in the formulation of Lemma 3 is essential for the validity of the lemma. To see this circumstance, take as $G$ a countable group of transformations of $E$ and consider the orbit $G(x)$ of some point $x \in E$. Further, for every subset $Z$ of $E$, define:

$\mu(Z) = \mathrm{card}(Z \cap G(x))$ if $\mathrm{card}(Z \cap G(x))$ is finite;

$\mu(Z) = +\infty$ if $\mathrm{card}(Z \cap G(x))$ is infinite.

It is easy to verify that the introduced functional

$$\mu : \{Z : Z \subset E\} \to [0, +\infty]$$

is a $\sigma$-finite $G$-invariant measure on $E$ satisfying conditions (1) and (2) of Lemma 3, but a set $Y$ with properties (a) and (b) cannot exist for this $\mu$.

**Theorem 3.** *Suppose that for a ground set $E$, for a group $G$ of transformations of $E$ and for a measure $\mu$ on $E$, the conditions formulated in Lemma* 3 *are fulfilled.*

*Suppose also that every set $C \subset E$ with* $\mathrm{card}(C) < \mathrm{card}(E)$ *is of $\mu$-measure zero.*

*Then there exists a $G$-invariant* ($G$*-quasi-invariant*) *measure $\mu'$ on $E$ such that:*

(a) $\pi(\mu') \leq \mathrm{card}(E)$*;*

(b) *$\mu'$ is a proper extension of $\mu$;*

(c) *there is a $\mu'$-measure zero set $X$ which almost contains any $\mu$-measure zero subset of $E$, i.e.,* $\mathrm{card}(T \setminus X) < \mathrm{card}(E)$ *whenever $T \subset E$ is of $\mu$-measure zero;*

(d) *for every $\mu'$-measurable set $A$, there exists a $\mu$-measurable set $B$ such that* $\mu'(A \triangle B) = 0$ (*in particular, the measures $\mu$ and $\mu'$ are metrically isomorphic*)*.*

The proof of Theorem 3 is as follows. Applying Marczewski's method of extending invariant (quasi-invariant) measures (see [6,7]), we can define a $G$-invariant ($G$-quasi-invariant) measure $\mu'$ on $E$ which strictly extends $\mu$ and is such that the equality $\mu'(E \setminus Y) = 0$ is valid, where $Y$ is the set indicated in Lemma 3. Further, for this $\mu'$, relations (a) and (d) are easily verified. Finally, we put $X = E \setminus Y$ and check that $X$ satisfies relation (c) of the theorem.

Let $\mathbf{c}$ denote the cardinality of the continuum and let $\lambda_n$ stand for the usual Lebesgue measure on the Euclidean space $\mathbf{R}^n$, where $n \geq 1$.

As a consequence of Theorem 3, we get the next statement.

**Theorem 4.** *Assuming Martin's Axiom* (**MA**)*, there exists a measure $\nu$ on $\mathbf{R}^n$ satisfying these five conditions:*

(1) *$\nu$ is invariant under the group of all isometries of $\mathbf{R}^n$;*

(2) *$\nu$ is a proper extension of $\lambda_n$;*

(3) $\pi(\nu) = \mathbf{c}$*;*

(4) *there is a $\nu$-measure zero set $X$ such that* $\mathrm{card}(T \setminus X) < \mathbf{c}$ *whenever $T \subset \mathbf{R}^n$ is of $\lambda_n$-measure zero;*

(5) *for every $\nu$-measurable set $A$, there exists a $\lambda_n$-measurable set $B$ such that* $\nu(A \triangle B) = 0$ (*in particular, the measures $\nu$ and $\lambda_n$ are metrically isomorphic*)*.*

**Remark 5.** Under the Continuum Hypothesis (**CH**), condition (4) of Theorem 4 means that the $\nu$-null set $X$ has the following property:

$\mathrm{card}(T \setminus X) \leq \omega$ whenever $T \subset \mathbf{R}^n$ is of $\lambda_n$-measure zero.

In some sense, one can say that $X$ is universal for the family of all $\lambda_n$-measure zero subsets of $\mathbf{R}^n$.

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