Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 2, 305–307

ON THE DIFFERENTIATION OF RANDOM MEASURES WITH RESPECT TO HOMOTHECY INVARIANT BASES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. For every homothecy invariant convex density differentiation basis B in $\mathbb{R}^d$, there are characterized sequences of weights $w = (w_j)_{j\in\mathbb{N}}$ for which the random measures $\mu_{w,\theta} = \sum\limits_{j=1}^{\infty} w_j\delta_{\theta_j}$ are differentiable with respect to the basis B for almost every selection of a sequence of points $\theta_1, \theta_2, \dots$ from the unit cube $[0,1]^d$.

1. Definitions and Notation

Let $w = (w_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ and $\sum\limits_{j=1}^{\infty} w_j < \infty$. The *random measure* generated by the sequence of weights $w = (w_j)_{j\in\mathbb{N}}$ and corresponding to a selection of points $\theta_1, \theta_2, \dots$ from the unit cube $[0,1]^d$ is defined as the discrete Lebesgue–Stieltjes measure $\mu_{w,\theta} = \sum\limits_{j=1}^{\infty} w_j\delta_{\theta_j}$. Here and below, $\delta_x = \delta_x^X$ denotes the *Dirac measure* on a non-empty set X supported on a point $x \in X$.

We denote by $\boldsymbol{m}_d$, $\boldsymbol{\mu}_d$ and $\boldsymbol{\mu}$ the Lebesgue measures in $\mathbb{R}^d$, $[0,1]^d$ and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, respectively.

A mapping B defined on $\mathbb{R}^d$ is said to be a *differentiation basis* (briefly, *basis*) if for every $x \in \mathbb{R}^d$, $B(x)$ is a collection of bounded open subsets of $\mathbb{R}^d$ which contain the point x, and there exists a sequence (R_j) of sets from $B(x)$ with $\lim\limits_{j\to\infty} \operatorname{diam} R_j = 0$.

For a Lebesgue–Stieltjes measure μ and a basis B, the numbers

$$\overline{D}_B(\mu,x) = \limsup_{R\in B(x),\operatorname{diam}R\to 0} \frac{\mu(R)}{\boldsymbol{m}_d(R)}, \quad \underline{D}_B(\mu,x) = \liminf_{R\in B(x),\operatorname{diam}R\to 0} \frac{\mu(R)}{\boldsymbol{m}_d(R)}$$

are called respectively the *upper* and *lower derivatives* of μ at the point x with respect to B. If the upper and lower derivatives coincide, then their common value is called the *derivative* of μ at a point x with respect to B and denoted by $D_B(\mu,x)$. Replacing $\mu(R)$ by $\int\limits_R f\mathrm{d}\boldsymbol{m}_d$ in the above expressions, we define $\overline{D}_B(\int f, x)$, $\underline{D}_B(\int f, x)$ and $D_B(\int f, x)$ for a function $f \in L_{\mathrm{loc}}(\mathbb{R}^d)$. The *maximal operator* M_B corresponding to the basis B is defined as follows:

$$M_B(\mu)(x) = \sup_{R\in B(x)} \frac{\mu(R)}{\boldsymbol{m}_d(R)},$$

where μ is a Lebesgue–Stieltjes measure on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$. Replacing $\mu(R)$ in the above supremum by $\int\limits_R |f|\mathrm{d}\boldsymbol{m}_d$, we get the definition of the maximal operator M_B for a function $f \in L_{\mathrm{loc}}(\mathbb{R}^d)$. Denote by B^r $(r > 0)$ the *truncation* of a basis B at the level r, i.e., $B^r(x) = \{R \in B(x) : \operatorname{diam}R < r\}$ $(x \in \mathbb{R}^d)$. The operator M_{B^r} $(r > 0)$ is called the *truncated maximal operator*.

A basis B is said:

- to *differentiate a Lebesgue–Stieltjes measure* μ if $D_B(\mu,x)$ exists for almost all $x \in \mathbb{R}^d$;
- to *differentiate the integral of a function* $f \in L_{\mathrm{loc}}(\mathbb{R}^d)$ if $D_B(\int f, x) = f(x)$ for almost all $x \in \mathbb{R}^d$;

2020 *Mathematics Subject Classification.* 28A15, 42B25.
Key words and phrases. Random measures; Differentiation; Homothecy invariant basis.

- to *differentiate a class* $\Omega \subset L_{\mathrm{loc}}(\mathbb{R}^d)$ if B differentiates $\int f$ for every $f \in \Omega$;
- to be *translation invariant* (*homothecy invariant*) if for every $x \in \mathbb{R}^d, R \in B(x)$ and a translation (homothecy) $M : \mathbb{R}^d \to \mathbb{R}^d$, we have $M(R) \in B(M(x))$;
- to be *convex* if each set $R \in \bigcup\limits_{x\in\mathbb{R}^d} B(x)$ is convex;
- to be a *density basis* if B differentiates the integral of the characteristic function of every bounded measurable set;
- to be a *Busemann–Feller basis* if $(x \in \mathbb{R}^d, R \in B(x), y \in R) \Rightarrow R \in B(y)$.

Note that each homothecy invariant basis is also translation invariant.

In what follows, the dimension of the space $\mathbb{R}^d$ is assumed to be greater than one.

Denote by $\mathbf{I}$ the bases for which $\mathbf{I}(x)$ $(x \in \mathbb{R}^d)$ consists of all d-dimensional open intervals containing x. The differentiation with respect to $\mathbf{I}$ is called the *strong differentiation.*

For a basis B in $\mathbb{R}^d$, we denote by φ_B the function defined by

$$\varphi_B(\lambda) = \boldsymbol{m}_d(\{M_{B^1}(\delta_0) > \lambda\}) \quad (0 < \lambda < \infty),$$

where δ_0 is the Dirac measure supported at the origin.

For an arbitrary homothecy invariant convex density basis B, the estimate $\varphi_B(\lambda) \geq \frac{c}{\lambda}$ $(\lambda > 1)$ is valid, where c is a positive constant, not depending on λ. If, additionally, it is known that B does not differentiate $L(\mathbb{R}^d)$, then $\limsup\limits_{\lambda\to\infty} \lambda\varphi_B(\lambda) = \infty$. These two estimates for φ_B will be checked later.

For the basis $\mathbf{I}$ in [2] (see Lemma 1), it is shown that there exist the constants $0 < c_d < C_d$ for which

$$\frac{c_d}{\lambda}(1 + \ln^{d-1}\lambda) \leq \varphi_{\mathbf{I}}(\lambda) \leq \frac{C_d}{\lambda}(1 + \ln^{d-1}\lambda) \quad (\lambda > 1).$$

Let (X, μ) be a measure space. By $\mu^{\mathbb{N}}$ we denote the measure in $X^{\mathbb{N}}$ which is the product of a countable number of copies of the measure μ.

2. Results

Kahane (see [1, Chapter X]) proved the following alternative for the Fourier series of random measures on the torus: If a sequence of weights $w = (w_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ is such that $\sum\limits_{j=1}^{\infty} w_j < \infty$ and $\sum\limits_{j=1}^{\infty} w_j \ln\frac{1}{w_j} < \infty$, then for a.e. sequence $\theta = (\theta_j)_{j\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$ (in the sense of the measure $\boldsymbol{\mu}^{\mathbb{N}}$), the sequence of the partial sums of the Fourier series for the random measure $\mu_{w,\theta}$ is bounded for a.e. $x \in \mathbb{T}$; and if $w = (w_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ is such that $\sum\limits_{j=1}^{\infty} w_j < \infty$ and $\sum\limits_{j=1}^{\infty} w_j \ln\frac{1}{w_j} = \infty$, then for a.e. sequence $\theta = (\theta_j)_{j\in\mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$, the Fourier series of the random measure $\mu_{w,\theta}$ diverges unboundedly for a.e. $x \in \mathbb{T}$.

Similar result for random measures on $[0,1]^d$ in the context of the strong differentiation was shown by Karagulyan [2]. Namely, in [2], the following theorem is proved: If $w = (w_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ is such that $\sum\limits_{j=1}^{\infty} w_j < \infty$ and $\sum\limits_{j=1}^{\infty} w_j \ln^{d-1}\frac{1}{w_j} < \infty$, then for a.e. sequence $\theta = (\theta_j)_{j\in\mathbb{N}} \in ([0,1]^d)^{\mathbb{N}}$ (in the sense of the measure $\boldsymbol{\mu}_d^{\mathbb{N}}$), the random measure $\mu_{w,\theta}$ is strongly differentiable, moreover, $D_{\mathbf{I}}(\mu_{w,\theta}, x) = 0$ for a.e. $x \in [0,1]^d$; and if $w = (w_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ is such that $\sum\limits_{j=1}^{\infty} w_j < \infty$ and $\sum\limits_{j=1}^{\infty} w_j \ln^{d-1}\frac{1}{w_j} = \infty$, then for a.e. sequence $\theta = (\theta_j)_{j\in\mathbb{N}} \in ([0,1]^d)^{\mathbb{N}}$, the random measure $\mu_{w,\theta}$ is not strongly differentiable, moreover, $\overline{D}_{\mathbf{I}}(\mu_{w,\theta}, x) = \infty$ for a.e. $x \in [0,1]^d$.

In the theorem given below, for every homothecy invariant convex density basis B in $\mathbb{R}^d$, there are characterized sequences of weights $w = (w_j)_{j\in\mathbb{N}}$ for which the random measures $\mu_{w,\theta} = \sum\limits_{j=1}^{\infty} w_j\delta_{\theta_j}$ are differentiable with respect to the basis B for almost every selection of a sequence of points $\theta_1, \theta_2, \ldots$ from the unit cube $[0,1]^d$.

Theorem 2.1. *Let B be a homothecy invariant convex density basis in $\mathbb{R}^d$. If $w = (w_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ is such that $\sum_{j=1}^{\infty} w_j < \infty$ and $\sum_{j=1}^{\infty} \varphi_B\big(\frac{1}{w_j}\big) < \infty$, then for a.e. sequence $\theta = (\theta_j)_{j\in\mathbb{N}} \in ([0,1]^d)^{\mathbb{N}}$ the random measure $\mu_{w,\theta} = \sum_{j=1}^{\infty} w_j\delta_{\theta_j}$ is differentiable with respect to the basis B, moreover, $D_B(\mu_{w,\theta}, x) = 0$ for a.e. $x \in [0,1]^d$; and if $w = (w_j)_{j\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ is such that $\sum_{j=1}^{\infty} w_j < \infty$ and $\sum_{j=1}^{\infty} \varphi_B\big(\frac{1}{w_j}\big) = \infty$, then for a.e. sequence $\theta = (\theta_j)_{j\in\mathbb{N}} \in ([0,1]^d)^{\mathbb{N}}$ the random measure $\mu_{w,\theta} = \sum_{j=1}^{\infty} w_j\delta_{\theta_j}$ is not differentiable with respect to the basis B, moreover, $\overline{D}_B(\mu_{w,\theta}, x) = \infty$ for a.e. $x \in [0,1]^d$.*

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(Received 08.01.2023)

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