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STRONGLY LACUNARY CONVERGENCE OF ORDER α IN NEUTROSOPHIC NORMED SPACES

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Abstract. In this paper, the concept of a strongly lacunary convergence of order α in the neutrosophic normed spaces is introduced. A few fundamental properties of this new concept are investigated.

1. Introduction

The concept 'neutrosophy' implies impartial knowledge of a thought that neutrally describes the basic difference between neutral, fuzzy, intuitive fuzzy set and logic. The neutrosophic set (NS) was investigated by Smarandache [15] who defined the degree of indeterminacy (i) as an independent component. The neutrosophic logic was firstly examined in [16]. It is a logic, where each proposition is determined to have a degree of truth (T), falsity (F), and indeterminacy (I). A neutrosophic set (NS) is determined as a set in which every component of the universe has a degree of T, F and I. In intuitionistic fuzzy set (IFS) the 'degree of non-belongingness' is not independent, but it is dependent on the 'degree of belongingness'. Fuzzy sets (FSs) can be thought as a remarkable case of an IFS, where the 'degree of non-belongingness' of an element is absolutely equal to '1- degree of belongingness'. Uncertainty is based on the belongingness degree in IFSs, whereas the uncertainty in NS is considered independently of T and F values. Since there are no any limitations among the degrees of T, F, I, the neutrosophic sets (NSs) are actually more general than IFS. Neutrosophic soft linear spaces (NSLSs) were considered by Bera and Mahapatra [2]. Subsequently, in [3], the concept of a neutrosophic soft normed linear space (NSNLS) was defined and the features of NSNLS were examined.

Kirişçi and Şimşek [4] defined a new concept known as neutrosophic metric space (NMS) with continuous t-norms and continuous t-conorms. Some notable features of NMS have been examined. The neutrosophic normed space (NNS) and the statistical convergence in NNS have been investigated by Kirişçi and Şimşek [5]. Neutrosophic set and neutrosophic logic are used in applied sciences and theoretical sciences such as decision making, robotics, summability theory.

In [6], lacunary statistical convergence of sequences in NNS was examined. Also, lacunary statistically Cauchy sequence in NNS was given and lacunary statistically completeness in connection with a neutrosophic normed space was presented. Kişi [7] defined lacunary ideal convergence and gave various results about lacunary ideal convergence in [7] and [8].

Definition 1 ([9])**.** Let $* : [0,1]\times[0,1] \to [0,1]$ be an operation. When $*$ satisfies following situations, it is called continuous TN (Triangular norms (t-norms)). Take $p,q,r,s \in [0,1]$:
(i) $p * 1 = p$,
(ii) if $p \le r$ and $q \le s$, then $p * q \le r * s$,
(iii) $*$ is continuous,
(iv) $*$ is associative and commutative.

Definition 2 ([9])**.** Let $\Diamond : [0,1]\times[0,1] \to [0,1]$ be an operation. When $\Diamond$ satisfies following situations, it is said to be continuous TC (Triangular conorms (t-conorms)).
(i) $p\Diamond 0 = p$,

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(ii) if $p \leq r$ and $q \leq s$, then $p \Diamond q \leq r \Diamond s$,
(iii) $\Diamond$ is continuous,
(iv) $\Diamond$ is associative and commutative.

Definition 3 ([5]). Let F be a vector space, $N = \{\langle u, G(u), B(u), Y(u)\rangle : u \in F\}$ be a normed space (NS) such that $N : F \times \mathbb{R}^+ \to [0,1]$. While following conditions hold, $V = (F, N, *, \Diamond)$ is called to be NNS. For each $u, v \in F$ and $\lambda, \mu > 0$ and for all $\sigma \neq 0$,
(i) $0 \leq G(u,\lambda) \leq 1$, $0 \leq B(u,\lambda) \leq 1$, $0 \leq Y(u,\lambda) \leq 1$, $\forall \lambda \in \mathbb{R}^+$
(ii) $G(u,\lambda) + B(u,\lambda) + Y(u,\lambda) \leq 3$, $\forall \lambda \in \mathbb{R}^+$
(iii) $G(u,\lambda) = 1$ (for $\lambda > 0$) iff $u = 0$,
(iv) $G(\sigma u, \lambda) = G\left(u, \frac{\lambda}{|\sigma|}\right)$,
(v) $G(u,\mu) * G(v,\lambda) \leq G(u+v, \mu+\lambda)$
(vi) $G(u,.)$ is a non-decreasing continuous function,
(vii) $\lim_{\lambda\to\infty} G(u,\lambda) = 1$,
(viii) $B(u,\lambda) = 0$ (for $\lambda > 0$) iff $u = 0$,
(ix) $B(\sigma u, \lambda) = B\left(u, \frac{\lambda}{|\sigma|}\right)$,
(x) $B(u,\mu) \Diamond B(v,\lambda) \geq B(u+v, \mu+\lambda)$
(xi) $B(u,.)$ is a non-increasing continuous function,
(xii) $\lim_{\lambda\to\infty} B(u,\lambda) = 0$,
(xiii) $Y(u,\lambda) = 0$ (for $\lambda > 0$) iff $u = 0$,
(xiv) $Y(\sigma u, \lambda) = Y\left(u, \frac{\lambda}{|\sigma|}\right)$,
(xv) $Y(u,\mu) \Diamond Y(v,\lambda) \geq Y(u+v, \mu+\lambda)$
(xvi) $Y(u,.)$ is a non-increasing continuous function,
(xvii) $\lim_{\lambda\to\infty} Y(u,\lambda) = 0$,
(xviii) If $\lambda \leq 0$, then $G(u,\lambda) = 0, B(u,\lambda) = 1$ and $Y(u,\lambda) = 1$.
Then $N = (G, B, Y)$ is called Neutrosophic norm (NN).

Definition 4 ([5]). Let V be an NNS, the sequence (x_k) in V, $\varepsilon \in (0,1)$ and $\lambda > 0$. Then the squence (x_k) converges to ζ iff there is $N \in \mathbb{N}$ such that $G(x_k - \zeta, \lambda) > 1-\varepsilon, B(x_k - \zeta, \lambda) < \varepsilon, Y(x_k - \zeta, \lambda) < \varepsilon$. That is, $\lim_{k\to\infty} G(x_k - \zeta, \lambda) = 1$, $\lim_{k\to\infty} B(x_k - \zeta, \lambda) = 0$ and $\lim_{k\to\infty} Y(x_k - \zeta, \lambda) = 0$ as $\lambda > 0$. In that case, the sequence (x_k) is named a convergent sequence in V. The convergent in NNS is indicated by $N - \lim x_k = \zeta$.

Definition 5 ([5]). Let V be an NNS. For $\lambda > 0$, $w \in F$ and $\varepsilon \in (0,1)$,

$$OB(w, \varepsilon, \lambda) = \{u \in F : G(w-u, \lambda) > 1-\varepsilon, B(w-u,\lambda) < \varepsilon, Y(w-u,\lambda) < \varepsilon\}$$

is called an open ball with center w and radius ε.

Definition 6 ([5]). The set $A \subset F$ is called neutrosophic-bounded (NB) in NNS V if there exist $\lambda > 0$ and $\varepsilon \in (0,1)$ such that $G(u,\lambda) > 1-\varepsilon$, $B(u,\lambda) < \varepsilon$ and $Y(u,\lambda) < \varepsilon$ for each $u \in A$.

An increasing integer sequence $\theta = (k_r)$ is said to be a lacunary sequence such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The interval determined by θ is denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ is abbreviated by q_r, and $q_1 = k_1$, for convenience. Lacunary convergence was studied in [1, 10–13].

2. Main Results

In this section, we give the definition of a strong lacunary convergence of order α with respect to NN $(LC - NN)$, where $0 < \alpha \leq 1$ and present some results related to this concept.

Definition 7. Take an NNS V. For a lacunary sequence θ and $0 < \alpha \leq 1$, a sequence $x = (x_k)$ is said to be strongly lacunary convergent to $\zeta \in F$ of order α with respect to NN $(LC - NN)$ if for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} G(x_k - \zeta, \lambda) > 1-\varepsilon \text{ and } \frac{1}{h_r^\alpha}\sum_{k\in I_r} B(x_k - \zeta, \lambda) < \varepsilon,\ \frac{1}{h_r^\alpha}\sum_{k\in I_r} Y(x_k - \zeta, \lambda) < \varepsilon,$$

for all $r \geq r_0$. We indicate $(G,B,Y)_\theta^\alpha - \lim x = \zeta$. In case of $\theta = (2^r)$, $(G,B,Y)^\alpha - \lim x = \zeta$ is obtained.

Theorem 1. *Let V be an NNS. If x is strongly lacunary convergent with respect to NN, then $(G,B,Y)_\theta^\alpha - \lim x = \zeta$ is unique.*

Proof. Suppose that $(G,B,Y)_\theta^\alpha - \lim x = \zeta_1$, $(G,B,Y)_\theta^\alpha - \lim x = \zeta_2$ and $\zeta_1 \neq \zeta_2$. Given $\varepsilon > 0$, select $\rho \in (0,1)$ such that $(1-\rho) * (1-\rho) > 1-\varepsilon$ and $\rho \Diamond \rho < \varepsilon$. For each $\lambda > 0$, there is $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} G(x_k - \zeta_1, \lambda) > 1-\rho \text{ and } \frac{1}{h_r^\alpha}\sum_{k\in I_r} B(x_k - \zeta_1, \lambda) < \rho, \frac{1}{h_r^\alpha}\sum_{k\in I_r} Y(x_k - \zeta_1, \lambda) < \rho,$$

for all $r \geq r_1$. Also, there is $r_2 \in \mathbb{N}$ such that

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} G(x_k - \zeta_2, \lambda) > 1-\rho \text{ and } \frac{1}{h_r^\alpha}\sum_{k\in I_r} B(x_k - \zeta_2, \lambda) < \rho, \frac{1}{h_r^\alpha}\sum_{k\in I_r} Y(x_k - \zeta_2, \lambda) < \rho,$$

for all $r \geq r_2$. Assume that $r_0 = \max\{r_1, r_2\}$. Then for $r \geq r_0$, we can find a positive integer $m \in \mathbb{N}$ such that $G(\zeta_1 - \zeta_2, \lambda) \geq G(x_m - \zeta_1, \frac{\lambda}{2}) * G(x_m - \zeta_2, \frac{\lambda}{2}) > (1-\rho) * (1-\rho) > 1-\varepsilon$,

$$B(\zeta_1 - \zeta_2, \lambda) \leq B\Big(x_m - \zeta_1, \frac{\lambda}{2}\Big) \Diamond B\Big(x_m - \zeta_2, \frac{\lambda}{2}\Big) < \rho \Diamond \rho < \varepsilon$$

and

$$Y(\zeta_1 - \zeta_2, \lambda) \leq Y\Big(x_m - \zeta_1, \frac{\lambda}{2}\Big) \Diamond Y\Big(x_m - \zeta_2, \frac{\lambda}{2}\Big) < \rho \Diamond \rho < \varepsilon.$$

Since $\varepsilon > 0$ is abritrary, we get $G(\zeta_1 - \zeta_2, \lambda) = 1$, $B(\zeta_1 - \zeta_2, \lambda) = 0$ and $Y(\zeta_1 - \zeta_2, \lambda) = 0$, for all $\lambda > 0$, which gives $\zeta_1 = \zeta_2$. □

Now, we give an example to denote the sequence of strongly lacunary convergence of order α in an NNS.

Example. Let $(F, ||.||)$ be an NNS. For all $u, v, \alpha \in [0,1]$, define $u * v = uv$ and $u \Diamond v = \min\{u+v; 1\}$. For all $x \in F$ and every $\lambda > 0$, we take $G(x,\lambda) = \frac{\lambda}{\lambda + ||x||}$, $B(x,\lambda) = \frac{||x||}{\lambda + ||x||}$ and $Y(x,\lambda) = \frac{||x||}{\lambda}$. Then V is an NNS. We define a sequence (x_k) by

$$x_k = \begin{cases} 1, & \text{if } k = t^2 (t \in \mathbb{N}), \\ 0, & \text{otherwise.} \end{cases}$$

Consider

$$A = \{k \in I_r : G(x,\lambda) > 1-\varepsilon \text{ and } B(x,\lambda) < \varepsilon, Y(x,\lambda) < \varepsilon\}.$$

Then for any $\lambda > 0$ and for all $\varepsilon \in (0,1)$, the following set

$$\begin{aligned} A =& \Big\{k \in I_r : \frac{\lambda}{\lambda + ||x_k||} > 1-\varepsilon, \text{ and } \frac{||x_k||}{\lambda + ||x_k||} < \varepsilon, \frac{||x_k||}{\lambda} < \varepsilon\Big\} \\ =& \Big\{k \in I_r : ||x_k|| \leq \frac{\lambda\varepsilon}{1-\varepsilon}, \text{ and } ||x_k|| < \lambda\varepsilon\Big\} \\ \subset& \{k \in I_r : ||x_k|| = 1\} = \big\{k \in I_r : k = t^2\big\} \end{aligned}$$

i.e.,

$$A_r(\varepsilon,\lambda) = \Big\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\sum_{k\in I_r} G(x_k,\lambda) > 1-\varepsilon \text{ and } \frac{1}{h_r^\alpha}\sum_{k\in I_r} B(x_k,\lambda) < \varepsilon,\ \frac{1}{h_r^\alpha}\sum_{k\in I_r} Y(x_k,\lambda) < \varepsilon\Big\}$$

will be a finite set.

Theorem 2. *If $(G,B,Y)^{\alpha}_{\theta} - \lim x = \zeta$, then there is a subsequence (x_{ρ_k}) of x such that $(G,B,Y)^{\alpha}_{\theta} - \lim x_{\rho_k} = \zeta$.*

Proof. Take $(G,B,Y)^{\alpha}_{\theta} - \lim x = \zeta$. Then, for every $\lambda > 0$ and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} G(x_k - \zeta, \lambda) > 1 - \varepsilon \text{ and } \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} B(x_k - \zeta, \lambda) < \varepsilon, \; \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} Y(x_k - \zeta, \lambda) < \varepsilon,$$

for all $r \geq r_0$. Obviously, for each $r \geq r_0$, we choose $\rho_k \in I_r$ such that

$$G(x_{\rho_k} - \zeta, \lambda) > \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} G(x_k - \zeta, \lambda) > 1 - \varepsilon,$$
$$B(x_{\rho_k} - \zeta, \lambda) < \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} B(x_k - \zeta, \lambda) < \varepsilon,$$
$$Y(x_{\rho_k} - \zeta, \lambda) < \frac{1}{h_r^{\alpha}} \sum_{k \in I_r} Y(x_k - \zeta, \lambda) < \varepsilon.$$

It follows that $(G,B,Y)^{\alpha}_{\theta} - \lim x_{\rho_k} = \zeta$. □

Theorem 3. *Let $0 < \alpha \leq 1$. If $\liminf_r q_r > 1$, then $(G,B,Y)^{\alpha} \subset (G,B,Y)^{\alpha}_{\theta}$.*

Proof. Take $(G,B,Y)^{\alpha} - \lim x = \zeta$. Since $\frac{k_r^{\alpha}}{h_r^{\alpha}} > \frac{h_r^{\alpha}}{h_r^{\alpha}}$ for all $r \geq 1$, we can write

$$\begin{aligned}
\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} G(x_k - \zeta, \lambda) &= \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_r} G(x_k - \zeta, \lambda) - \frac{1}{h_r^{\alpha}} \sum_{k=1}^{k_{r-1}} G(x_k - \zeta, \lambda) \\
&= \frac{k_r^{\alpha}}{h_r^{\alpha}} \left(\frac{1}{k_r^{\alpha}} \sum_{k=1}^{k_r} G(x_k - \zeta, \lambda) \right) - \frac{k_{r-1}^{\alpha}}{h_r^{\alpha}} \left(\frac{1}{k_{r-1}^{\alpha}} \sum_{k=1}^{k_{r-1}} G(x_k - \zeta, \lambda) \right) \\
&> \frac{k_r^{\alpha}}{h_r^{\alpha}} \left(\frac{1}{k_r^{\alpha}} \sum_{k=1}^{k_r} G(x_k - \zeta, \lambda) \right) > \left(\frac{1}{k_r^{\alpha}} \sum_{k=1}^{k_r} G(x_k - \zeta, \lambda) \right) > 1 - \varepsilon.
\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r^{\alpha}}{h_r^{\alpha}} \leq \frac{(1+\delta)^{\alpha}}{\delta^{\alpha}} \text{ and } \frac{k_{r-1}^{\alpha}}{h_r^{\alpha}} \leq \frac{1}{\delta^{\alpha}}.$$

From here, $\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} B(x_k - \zeta, \lambda) < \varepsilon$ and $\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} Y(x_k - \zeta, \lambda) < \varepsilon$ are obtained. Thus $(G,B,Y)^{\alpha}_{\theta} - \lim x = \zeta$. □

Theorem 4. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, and let α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. If*

$$\lim_{r \to \infty} \inf \frac{(h'_r)^{\beta}}{(h_r)^{\alpha}} > 0 \quad \textit{and} \quad \lim_{r \to \infty} \frac{h'_r}{(h_r)^{\beta}} = 1 \tag{1}$$

hold and $A \subset F$ is neutrosophic-bounded (NB) in NNS V, then $(G,B,Y)^{\alpha}_{\theta} \subset (G,B,Y)^{\beta}_{\theta'}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$, $h'_r = s_r - s_{r-1}$.

Proof. Let $x \in (G,B,Y)^{\alpha}_{\theta}$ and assume that (1) holds. Since $A \subset F$ is neutrosophic-bounded (NB) in $NNSV$, there exists some $\lambda > 0$ such that $\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} G(x_k - \zeta, \lambda) > 1 - \varepsilon$ and $\frac{1}{h_r^{\alpha}} \sum_{k \in I_r} B(x_k -$

$\zeta,\lambda) < \ \varepsilon, \frac{1}{h_r^\alpha}\sum_{k\in I_r} Y(x_k-\zeta,\lambda) < \varepsilon$ for each $(x_k-\zeta)\in A$. Now, since $I_r \subseteq J_r$ and $h_r \leq h'_r$ for all $r\in\mathbb{N}$, we can write

$$\begin{aligned}\frac{1}{(h_r)^\alpha}\sum_{k\in I_r} G(x_k-\zeta,\lambda) \leq& \frac{1}{(h_r)^\alpha}\sum_{k\in J_r} G(x_k-\zeta,\lambda)\\ =& \frac{(h'_r)^\beta}{(h_r)^\alpha}\frac{1}{(h'_r)^\beta}\sum_{k\in J_r} G(x_k-\zeta,\lambda),\end{aligned}$$

for all $r\in\mathbb{N}$. Therefore we obtain

$$\begin{aligned}\frac{1}{(h'_r)^\beta}\sum_{k\in J_r} B(x_{p_k}-\zeta,\lambda) =& \frac{1}{(h'_r)^\beta}\sum_{k\in J_r-I_r} B(x_k-\zeta,\lambda)+\frac{1}{(h'_r)^\beta}\sum_{k\in I_r} B(x_k-\zeta,\lambda)\\ \leq& \frac{h'_r-h_r}{(h'_r)^\beta}\varepsilon+\frac{1}{(h'_r)^\beta}\sum_{k\in I_r} B(x_k-\zeta,\lambda)\\ \leq& \frac{h'_r-(h_r)^\beta}{(h_r)^\beta}\varepsilon+\frac{1}{(h_r)^\alpha}\sum_{k\in I_r} B(x_k-\zeta,\lambda)\\ \leq& \Big(\frac{h'_r}{(h_r)^\beta}-1\Big)\varepsilon+\frac{1}{(h_r)^\alpha}\sum_{k\in I_r} B(x_k-\zeta,\lambda),\end{aligned}$$

for every $r\in\mathbb{N}$. Therefore $\dfrac{1}{(h'_r)^\beta}\sum_{k\in J_r} G(x_k-\zeta,\lambda) > 1-\varepsilon$ and $\dfrac{1}{(h'_r)^\beta}\sum_{k\in J_r} B(x_k-\zeta,\lambda) < \varepsilon$. It can be shown to be $\dfrac{1}{(h'_r)^\beta}\sum_{k\in J_r} Y(x_k-\zeta,\lambda) < \varepsilon$ by similar operations. $(G,B,Y)_\theta^\alpha \subset (G,B,Y)_{\theta'}^\beta$ is obtained as the result. $\square$

Thus in the light of Theorem 4, we have the following result.

Corollary. *Let $\theta=(k_r)$ and $\theta'=(s_r)$ be two lacunary sequences such that $I_r\subset J_r$ for all $r\in\mathbb{N}$.*
If (1) *holds and is NB then:*
(i) $(G,B,Y)_\theta^\alpha \subset (G,B,Y)_{\theta'}$ *for* $0<\alpha\leq 1$,
(ii) $(G,B,Y)_\theta \subset (G,B,Y)_{\theta'}$.

Definition 8. Take an NNS V. A sequence $x=(x_k)$ is said to be strongly lacunary Cauchy of order α with respect to the NN N ($LCa-NN$) if for every $\varepsilon\in(0,1)$ and $\lambda>0$, there are $r_0,p\in\mathbb{N}$ satisfying

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} G(x_k-x_p,\lambda) > 1-\varepsilon \quad\text{and}\quad \frac{1}{h_r^\alpha}\sum_{k\in I_r} B(x_k-x_p,\lambda)<\varepsilon,\ \ \frac{1}{h_r^\alpha}\sum_{k\in I_r} Y(x_k-x_p,\lambda)<\varepsilon,$$

for all $r\geq r_0$.

Theorem 5. *If a sequence $x=(x_k)$ in an NNS is strongly lacunary convergent of order α with respect to NN N, then it is strongly Cauchy of order α with respect to NN N.*

Proof. Let $(G,B,Y)_\theta^\alpha-\lim x=\zeta$. Select $\varepsilon>0$. Then for a given $\rho\in(0,1)$, $(1-\rho)*(1-\rho)>1-\varepsilon$ and $\rho\Diamond\rho<\varepsilon$, we have

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} G\Big(x_k-\zeta,\frac{\lambda}{2}\Big) > 1-\rho \quad\text{and}\quad \frac{1}{h_r^\alpha}\sum_{k\in I_r} B\Big(x_k-\zeta,\frac{\lambda}{2}\Big)<\rho,\ \ \frac{1}{h_r^\alpha}\sum_{k\in I_r} Y\Big(x_k-\zeta,\frac{\lambda}{2}\Big)<\rho.$$

We have to show that

$$\frac{1}{h_r^\alpha}\sum_{k\in I_r} G(x_k-x_m,\lambda) > 1-\varepsilon \quad\text{and}\quad \frac{1}{h_r^\alpha}\sum_{k\in I_r} B(x_k-x_m,\lambda)<\varepsilon,\ \ \frac{1}{h_r^\alpha}\sum_{k\in I_r} Y(x_k-x_m,\lambda)<\varepsilon.$$

There are three possible cases:

Case (i). For $\lambda > 0$, we get

$$G(x_k - x_m, \lambda) \geq G\Big(x_k - \zeta, \frac{\lambda}{2}\Big) * G\Big(x_m - \zeta, \frac{\lambda}{2}\Big) > (1-\rho) * (1-\rho) > 1 - \varepsilon.$$

Case (ii). We obtain

$$B(x_k - x_m, \lambda) \leq B\Big(x_k - \zeta, \frac{\lambda}{2}\Big) \lozenge B\Big(x_m - \zeta, \frac{\lambda}{2}\Big) < \rho \lozenge \rho < \varepsilon.$$

Case (iii). We have

$$Y(x_k - x_m, \lambda) \leq Y\Big(x_k - \zeta, \frac{\lambda}{2}\Big) \lozenge Y\Big(x_m - \zeta, \frac{\lambda}{2}\Big) < \rho \lozenge \rho < \varepsilon.$$

This shows that (x_k) is strongly Cauchy of order α with respect to NN N. □

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