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DENSITY PROPERTY FOR A PRODUCT OF TRANSLATION INVARIANT DENSITY DIFFERENTIATION BASES

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Abstract. We prove that if $B_1,\dots,B_k$ are translation invariant density differentiation bases in $\mathbb{R}^{n_1},\dots,\mathbb{R}^{n_k}$, respectively, then their product $B_1\times\cdots\times B_k$ (i.e., the basis in $\mathbb{R}^{n_1+\cdots+n_k}$ for which $B_1\times\cdots\times B_k$ $(x_1,\dots,x_k)$ consists of all sets $R_1\times\cdots\times R_k$ with $R_1\in B_1(x_1),\dots,R_k\in B_k(x_k)$) is a density basis, as well.

1. Definitions and Notation

A mapping B defined on $\mathbb{R}^n$ is called a *differentiation basis* (briefly, basis) if for each $x\in\mathbb{R}^n$ the value $B(x)$ is a collection of bounded measurable sets of positive measure which contain x and there exists a sequence (R_k) of sets from $B(x)$ with $\lim\limits_{k\to\infty}\operatorname{diam}R_k=0$.

Let B be a basis. For $f\in L(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$, the upper and lower limits of the integral means $\frac{1}{|R|}\int\limits_R f$, where R is an arbitrary set from $B(x)$ and $\operatorname{diam}R\to 0$, are called the *upper and lower derivatives with respect to B of the integral of f at the point x*, and denoted by $\overline{D}_B(\int f,x)$ and $\underline{D}_B(\int f,x)$, respectively. If the two derivatives coincide, then their common value is called the derivative of $\int f$ at x and denoted by $D_B(\int f,x)$. We say that B *differentiates* $\int f$ (or $\int f$ *is differentiable with respect to* B) if $\overline{D}_B(\int f,x)=\underline{D}_B(\int f,x)=f(x)$ for almost all $x\in\mathbb{R}^n$. If this is true for each f in a class $F\subset L(\mathbb{R}^n)$ of functions, we say that B *differentiates* F.

For a basis B, we denote by $\overline{B}$ the collection $\bigcup\limits_{x\in\mathbb{R}^n}B(x)$.

A basis B is called:

- *homothecy invariant* if for every $x\in\mathbb{R}^n$, every $R\in B(x)$ and every homothecy $H:\mathbb{R}^n\to\mathbb{R}^n$, we have $H(R)\in B(H(x))$;
- *translation invariant* if for every $x\in\mathbb{R}^n$, every $R\in B(x)$ and every translation $T:\mathbb{R}^n\to\mathbb{R}^n$, we have $T(R)\in B(T(x))$;
- *convex* if it is formed by convex sets, i.e., each set from $\overline{B}$ is convex;
- *density basis* if B differentiates $\int\chi_E$ for every bounded measurable set $E\subset\mathbb{R}^n$.

Note that each homothecy invariant basis is also translation invariant.

The maximal operator M_B and the truncated maximal operator M_B^r ($r\in(0,\infty]$) corresponding to a basis B are defined as follows:

$$M_B(f)(x)=\sup_{R\in B(x)}\frac{1}{|R|}\int\limits_R|f|,$$

$$M_B^r(f)(x)=\sup_{R\in B(x),\ \operatorname{diam}R<r}\frac{1}{|R|}\int\limits_R|f|,$$

where $f\in L_{\text{loc}}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$. Obviously, $M_B=M_B^\infty$ and $M_B^r=M_{B^r}$, where B^r denotes the truncation of the basis B at the level r, i.e., $B^r(x)=\{R\in B(x):\ \operatorname{diam}R<r\}$ $(x\in\mathbb{R}^n)$.

Let $B_1,\dots,B_k$ be the bases in $\mathbb{R}^{n_1},\dots,\mathbb{R}^{n_k}$, respectively. Denote by $B_1\times\cdots\times B_k$ the product of the basis $B_1,\dots,B_k$, i.e., the basis in $\mathbb{R}^{n_1+\cdots+n_k}$ for which $B_1\times\cdots\times B_k$ $(x_1,\dots,x_k)$ consists of all sets $R_1\times\cdots\times R_k$ with $R_1\in B_1(x_1),\dots,R_k\in B_k(x_k)$.

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It is easy to see that the product of translation invariant (homothecy invariant) bases is translation invariant (homothecy invariant) basis, as well.

2. Result

The following characterizations for homothecy invariant density bases and translation invariant density bases are known.

Theorem 2.1. *Let B be a homothecy invariant basis. Then the following two properties are equivalent:*

(a) *B is a density basis.*
(b) *For each $\lambda \in (0,1)$, there exists a positive constant $c(B,\lambda)$ such that*

$$\big|\big\{M_B(\chi_E) > \lambda\big\}\big| \le c(B,\lambda)|E|,$$

for each bounded measurable set E.

Theorem 2.2. *Let B be a translation invariant basis. Then the following two properties are equivalent:*

(a) *B is a density basis.*
(b) *For each $\lambda \in (0,1)$, there exist positive constants $r(B,\lambda)$ and $c(B,\lambda)$ such that*

$$\big|\big\{M_B^{r(B,\lambda)}(\chi_E) > \lambda\big\}\big| \le c(B,\lambda)|E|,$$

for each bounded measurable set E.

Theorem 2.1 belongs basically to Busemann and Feller (see, e.g., [1, Chapter II, Theorem 1.2]). Theorem 2.2 under some additional restrictions is proved in the works of Oniani [4] and Hagelstein and Parissis [2]. Although the method of proving given in these works allows to establish the statement of Theorem 2.2 in a full generality.

Some other fundamental properties of density bases can be found in [4, Chapter III] and [3, 5, 6].

The main result of the paper is the following

Theorem 2.3. *Let $B_1, \dots, B_k$ be translation invariant density bases. Then their product $B_1 \times \cdots \times B_k$ is a density basis, as well.*

Note that in the proof of Theorem 2.3 the key tool is the above given characterization of translation invariant density bases.

3. Auxiliary Propositions

Lemma 3.1. *Let B be a translation invariant basis. Then for every $f \in L_{\text{loc}}(\mathbb{R}^n)$ and $\lambda > 0$ the set $\{M_B(f) > \lambda\}$ is open.*

Proof. First note that for an arbitrary bounded measurable set $E \subset \mathbb{R}^n$,

$$|E \cap (t+E)| \longrightarrow |E| \ \ (t \to 0). \tag{3.1}$$

Expression (3.1) is clear if E is an elementary set (i.e., a union of a finite number of cubes). After taking into account the possibility of approximation of a bounded measurable set by elementary sets, we derive (3.1) for the general case.

Take an arbitrary point x from the set $\{M_B(f) > \lambda\}$. Then there is $R \in B(x)$ with $\frac{1}{|R|}\int\limits_R |f| > \lambda$. Using (3.1) and the property of an absolute continuity of the integral, we have $\frac{1}{|t+R|}\int\limits_{t+R} |f| > \lambda$ for every t with the norm small enough. Hence, by the translation invariance of B, some neighbourhood of x is included in the set $\{M_B(f) > \lambda\}$. The lemma is proved. □

For a set $E \subset \mathbb{R}^{n+m}$ and the points $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we denote

$$E_x = \big\{t \in \mathbb{R}^m : \ (x,t) \in E\big\}, \quad E^y = \big\{t \in \mathbb{R}^n : \ (t,y) \in E\big\}.$$

Lemma 3.2. *Let B and S be translation invariant bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then for every open set $E \subset \mathbb{R}^{n+m}$ and a number $\lambda > 0$ the sets*

$$\big\{(x,y) \in \mathbb{R}^{n+m} : \ M_B(\chi_{E^y})(x) > \lambda\big\}, \quad \big\{(x,y) \in \mathbb{R}^{n+m} : \ M_S(\chi_{E_x})(y) > \lambda\big\}$$

are open.

Proof. Denote

$$G = \big\{(x,y) \in \mathbb{R}^{n+m} : \ M_B(\chi_{E^y})(x) > \lambda\big\}.$$

Suppose $(x,y) \in G$. We have to prove that (x,y) is an interior point of G. Let $R \in B(x)$ be such that

$$\frac{|R \cap E^y|}{|R|} = \frac{1}{|R|} \int\limits_R \chi_{E^y} > \lambda.$$

Here, $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^n$. Taking into account the fact that E^y is open, by the approximation argument we can find a compact set $K \subset E^y$ for which

$$\frac{|R \cap K|}{|R|} = \frac{1}{|R|} \int\limits_R \chi_K > \lambda.$$

By the reasoning similar to the one given in the proof of Lemma 3.1, we can find an open ball V in $\mathbb{R}^n$ with center at the origin such that

$$\frac{|(t+R) \cap K|}{|t+R|} = \frac{1}{|t+R|} \int\limits_{t+R} \chi_K > \lambda, \tag{3.2}$$

for every $t \in V$.

On the other hand, there is an open ball U in $\mathbb{R}^m$ with center at the origin such that

$$K \subset E^{y+\tau}, \ \text{ for every } \ \tau \in U. \tag{3.3}$$

Indeed, assuming the opposite, we find the sequences $\tau_j \in \mathbb{R}^m$ $(j \in \mathbb{N})$ with $\tau_j \to 0$ $(j \to \infty)$ and $x_j \in K$ $(j \in \mathbb{N})$ such that $x_j \notin E^{y+\tau_j}$ for every $j \in \mathbb{N}$. Then by virtue of the compactness of K, we can choose a subsequence (x_{j_p}) which tends to some point x^* from K. Hence we have

$$(x_{j_p}, y + \tau_{j_p}) \to (x^*, y) \in E \ \ (p \to \infty)$$

and

$$(x_{j_p}, y + \tau_{j_p}) \notin E \ \ (p \in \mathbb{N}).$$

But this is impossible, since (x^*, y) is an interior point for E.

Now, using (3.2) and (3.3), we conclude that

$$\lambda < \frac{1}{|t+R|} \int\limits_{t+R} \chi_K \le \frac{1}{|t+R|} \int\limits_{t+R} \chi_{E^{y+\tau}},$$

for every $t \in V$ and $\tau \in U$. Clearly, this implies that $(x,y) + V \times U \subset G$. Hence (x,y) is an interior point for the set G.

The reasoning is analogous for the second set $\{(x,y) \in \mathbb{R}^{n+m} : \ M_s(\chi_{E_x})(y) > \lambda\}$. The lemma is complete. $\square$

Lemma 3.3. *Let B and S be translation invariant bases in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then for every open set $E \subset \mathbb{R}^{n+m}$ and a number $\lambda > 0$,*

$$\big\{M_{B\times S}(\chi_E) > \lambda\big\} \subset \Big\{(x,y) \in \mathbb{R}^{n+m} : \ M_S(\chi_{F_x})(y) > \frac{\lambda}{2}\Big\},$$

where F is the set

$$\Big\{(x,y) \in \mathbb{R}^{n+m} : \ M_B(\chi_{E^y})(x) > \frac{\lambda}{2}\Big\}$$

(*which is open by Lemma* 3.2).

Proof. Suppose a point $(x_0, y_0) \in \mathbb{R}^{n+m}$ is such that

$$M_S(\chi_{F_{x_0}})(y_0) \le \frac{\lambda}{2}\,. \tag{3.4}$$

Let us consider an arbitrary set $I \times J \in B \times S(x_0, y_0)$. By (**??**) we have

$$\frac{|J \cap F_{x_0}|_m}{|J|_m} = \frac{1}{|J|_m} \int\limits_J \chi_{F_{x_0}} \le \frac{\lambda}{2}\,. \tag{3.5}$$

Note that if $y \in J \setminus F_{x_0}$, then $M_B(\chi_{E^y})(x_0) \le \frac{\lambda}{2}$ and, consequently,

$$\frac{|I \cap E^y|_n}{|I|_n} = \frac{1}{|I|} \int\limits_I \chi_{E^y} \le \frac{\lambda}{2}\,. \tag{3.6}$$

By (3.5) and (3.6), we have

$$\begin{aligned} \big|E \cap (I \times J)\big| &= \int\limits_J |E^y \cap I|_n \, dy \\ &= \int\limits_{J \cap F_{x_0}} |E^y \cap I|_n \, dy + \int\limits_{J \setminus F_{x_0}} |E^y \cap I|_n \, dy \le |I|_n \cdot |J \cap F_{x_0}|_n + \Big(\frac{\lambda}{2}\,|I|_n\Big)|J \setminus F_{x_0}|_m \\ &\le |I|_n \cdot \Big(\frac{\lambda}{2}\,|J|_m\Big) + \frac{\lambda}{2}\,|I|_n|J|_m = \lambda \cdot |I|_n|J|_m = \lambda|I \times J|. \end{aligned}$$

Thus

$$\frac{1}{|I \times J|} \int\limits_{I \times J} \chi_E = \frac{|E \cap (I \times J)|}{|I \times J|} \le \lambda.$$

Hence, $M_{B\times S}(\chi_E)(x_0, y_0) \le \lambda$. The lemma is complete. □

4. Proof of Theorem 2.3

Without loss of generality, let us consider the case of two bases $B_1 = B$ and $B_2 = S$ in the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Let $\lambda \in (0, 1)$. Since B and S are translation invariant bases, by Theorem 2.2, there exist positive constants $c(B, \lambda/2)$, $c(S, \lambda/2)$, $r(B, \lambda/2)$ and $r(S, \lambda/2)$ such that

$$\Big|\Big\{M_B^{r(B,\lambda/2)}(\chi_P) > \frac{\lambda}{2}\Big\}\Big| \le c\Big(P, \frac{\lambda}{2}\Big)|P| \tag{4.1}$$

and

$$\Big|\Big\{M_S^{r(S,\lambda/2)}(\chi_Q) > \frac{\lambda}{2}\Big\}\Big| \le c\Big(Q, \frac{\lambda}{2}\Big)|Q|, \tag{4.2}$$

for every bounded measurable sets P and Q in the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

Set

$$r(B \times S, \lambda) = \min\Big(r\Big(B, \frac{\lambda}{2}\Big), r\Big(S, \frac{\lambda}{2}\Big)\Big)$$

and

$$c(B \times S, \lambda) = c\Big(B, \frac{\lambda}{2}\Big)c\Big(S, \frac{\lambda}{2}\Big).$$

Let us show that for any bounded open set E in $\mathbb{R}^{n+m}$,

$$\big|\big\{M_{B\times S}^{r(B\times S,\lambda)}(\chi_E) > \lambda\big\}\big| \le c(B \times S, \lambda)|E|. \tag{4.3}$$

It is clear that from (4.3), by Theorem 2.2, it follows that $B \times S$ is a density basis.

Note that

$$(B \times S)^{r(B\times S,\lambda)} \subset B^{r(B\times S,\lambda)} \times S^{r(B\times S,\lambda)}. \tag{4.4}$$

Denote

$$F = \Big\{(x, y) \in \mathbb{R}^{n+m} :\ M_B^{r(B\times S,\lambda)}(\chi_{E^y})(x) > \frac{\lambda}{2}\Big\}.$$

By Lemma 3.2, F is an open subset of $\mathbb{R}^{n+m}$.

By (4.1) and the estimation $r(B\times S,\lambda)\le r(B,\lambda/2)$, for every $y\in\mathbb{R}^m$, we have

$$\begin{aligned}|F^y|_n&=\Big|\Big\{x\in\mathbb{R}^n:\ M_B^{r(B\times S,\lambda)}(\chi_{E^y})(x)>\frac{\lambda}{2}\Big\}\Big|_n\\&\le\Big|\Big\{x\in\mathbb{R}^n:\ M_B^{r(B,\lambda/2)}(\chi_{E^y})(x)>\frac{\lambda}{2}\Big\}\Big|_n\\&\le c\Big(B,\frac{\lambda}{2}\Big)|E^y|_n.\end{aligned}$$

Hence

$$|F|=\int\limits_{\mathbb{R}^m}|F^y|_n\,dy\le\int\limits_{\mathbb{R}^m}c\Big(B,\frac{\lambda}{2}\Big)|E^y|_n\,dy=c\Big(B,\frac{\lambda}{2}\Big)|E|. \tag{4.5}$$

Now, let us consider the set

$$G=\Big\{(x,y)\in\mathbb{R}^{n+m}:\ M_S^{r(B\times S,\lambda)}(\chi_{E_x})(y)>\frac{\lambda}{2}\Big\},$$

which by Lemma 3.2 is open and by Lemma 3.3 and (4.4) contains the set $\{M_{B\times S}^{r(B\times S,\lambda)}(\chi_E)>\lambda\}$.

Using (4.2) and the estimation $r(B\times S,\lambda)\le r(S,\lambda/2)$ for every $x\in\mathbb{R}^n$, we write

$$\begin{aligned}|G_x|_m&=\Big|\Big\{y\in\mathbb{R}^m:\ M_S^{r(B\times S,\lambda)}(\chi_{F_x})(y)>\frac{\lambda}{2}\Big\}\Big|_m\\&\le\Big|\Big\{y\in\mathbb{R}^m:\ M_S^{r(S,\lambda/2)}(\chi_{F_x})(y)>\frac{\lambda}{2}\Big\}\Big|_m\\&\le c\Big(S,\frac{\lambda}{2}\Big)|F_x|_m.\end{aligned}$$

Hence by (4.5), we have

$$|G|=\int\limits_{\mathbb{R}^n}|G_x|_m\,dx\le\int\limits_{\mathbb{R}^n}c\Big(S,\frac{\lambda}{2}\Big)|F_x|_m\,dx=c\Big(S,\frac{\lambda}{2}\Big)|F|=c\Big(S,\frac{\lambda}{2}\Big)c\Big(B,\frac{\lambda}{2}\Big)|E|.$$

Thus from (4.3) and all written within it, we conclude that the theorem is complete.

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