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I_2-CONVERGENT SERIES THROUGH COMPACT OPERATOR AND ORLICZ FUNCTIONS IN IFNS

CARLOS GRANADOS[1], IVAN PADILLA[2] AND YESIKA ROJAS[3]

Abstract. Recently, Khan et al. [17] defined the notion of ideal convergence of single sequences and also some notions of a compact operator and Orlicz function in intuitionistic fuzzy normed space. The aim of this paper is to generalize this notion to the double sequences in such spaces, i.e., we define the notions of $\mathcal{M}^{I_2}_{(\mu,v)}(T)$, $\mathcal{M}^{I_2^0}_{(\mu,v)}(T)$, $\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$ and $\mathcal{M}^{I_2^0}_{(\mu,v)}(T,F)$ for double sequences in an intuitionistic fuzzy normed space. For the sake of generalizing, we contribute basically to outcomes that we came up with and study some basic topological properties.

1. Introduction

The notion of statistical convergence was derived from the convergence of real sequences by Fast [5] and Schoenberg [30]. After the studies of Salát [27], many studies in this area have been conducted by Fridy [7], and Connor [4]. Kostyrko et al. [19], introduced the concept of ideal convergence by expanding the concept of statistical convergence. After the basic properties of I-convergence given by Kostyrko et al. [20], some works [11, 22, 28, 31] have turned out to be the basis of other studies. Furthermore, following this idea, the notion of natural density of a subset K of $\mathbb{N}\times\mathbb{N}$ is defined as

$$\delta_2(K)=\lim_{n,m\to\infty}\frac{K(n,m)}{nm}\ (Limit\ taken\ in\ Pringsheim's\ sense),$$

and statistical convergence of a double sequence is as follows: A double sequence $x=(x_{ij})$ of real or complex terms is said to be statistically convergent to the number l if for each $\varepsilon>0$, the set

$$\{(j,k): j\leqslant n, k\leqslant m: |x_{ji}-l|\geqslant\varepsilon\}$$

has double natural density zero.

The notion of generalized statistical convergence of double sequence is also studied by Mursaleen et al. [23]. In 2007, Karakus et al. [22] studied statistical convergence of double sequence in probabilistic normed space and this work was further extended by Mohiuddin et al. [31] and Savas et al. [29], respectively. Important development of the statistical convergence of double sequence or more is being carried out by [9, 10, 12, 13].

On the other hand, in the last decade, the concept of fuzzy set has been as the most active field of research in many branches of mathematics, computer and engineering [1]. Taking into account the work proposed by Zadeh [32], a huge amount of researches have been done on a fuzzy set theory and its applications, as well as, on fuzzy analogues of the classical theories. Fuzzy set has a wide number of applications in various fields such as population dynamics [2], nonlinear dynamical system [14], chaos control [6], computer programming [8] and much more. In 2006, Saadati and Park [26] defined the concept of intuitionistic fuzzy normed spaces. After that, the study of intuitionistic fuzzy topological spaces [3], intuitionistic fuzzy 2-normed space [24] and intuitionistic fuzzy Zweier ideal convergent sequence spaces [16] are the latest developments in fuzzy topology.

In this paper, we introduce and study the concepts of $\mathcal{M}^{I_2}_{(\mu,v)}(T)$, $\mathcal{M}^{I_2^0}_{(\mu,v)}(T)$, $\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$ and $\mathcal{M}^{I_2^0}_{(\mu,v)}(T,F)$ for double sequences in intuitionistic fuzzy normed space taking into account the notions presented in [17]. Besides, we show some topological properties on these spaces.

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2. Preliminaries

In this section, we recall some well-known notions which are useful for the development of this paper.

Definition 2.1 ([26]). The five-tuple $(X,\mu,v,*,\diamond)$ is called an intuitionistic fuzzy normed space (simply, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and μ, v are fuzzy sets on $X\times(0,\infty)$ satisfying for every $r,u\in X$ and $s,t>0$ the following condition:

(1) $\mu(r,t)+v(r,t)\leqslant 1$,
(2) $\mu(r,t)>0$,
(3) $\mu(r,t)=1$ if and only if $r=0$,
(4) $\mu(\alpha r,t)=\mu\big(r,\frac{t}{|\alpha|}\big)$ for each $\alpha\neq 0$,
(5) $\mu(r,t)*\mu(u,s)\leqslant\mu(r+u,t+s)$,
(6) $\mu(r,.):(0,\infty)\to[0,1]$ is continuous,
(7) $\lim_{t\to\infty}\mu(r,t)=1$ and $\lim_{t\to 0}\mu(r,t)=0$,
(8) $v(r,t)<1$,
(9) $v(r,t)=0$ if and only if $r=0$,
(10) $v(\alpha r,t)=v\big(r,\frac{t}{|\alpha|}\big)$ for each $\alpha\neq 0$,
(11) $v(r,t)\diamond v(u,s)\geqslant v(r+u,t+s)$,
(12) $v(x,.):(0,\infty)\to[0,1]$ is continuous,
(13) $\lim_{t\to\infty}v(r,t)=1$ and $\lim_{t\to 0}v(r,t)=0$.

In this case, (μ,v) is said to be an intuitionistic fuzzy norm.

Example 2.1. Let $(X,\|.\|)$ be a normed space. Denote $a*b=ab$ and $a\diamond b=\min(a+b,1)$ for all $a,b\in[0,1]$ and let μ_0 and v_0 be fuzzy sets on $X\times(0,\infty)$ defined as follows:

$$\mu_0(x,t)=\frac{t}{t+\|x\|}\ \text{ and }\ v_0(x,t)=\frac{\|x\|}{t+\|x\|},$$

for all $t\in\mathbb{R}^+$. Then $(X,\mu,v,*,\diamond)$ is an intuitionistic fuzzy normed space.

Definition 2.2. An ideal I is a non-empty collection of subsets of X which satisfies conditions (1) and (2) of the following statements:

(1) If $A\subset B$ and $B\in I$, then $A\in I$,
(2) If $A,B\in I$, then $A\cup B\in I$.
(3) I is called as a non-trivial ideal if $X\notin I$ and $I\neq\emptyset$.
(4) A non-trivial ideal I on X is said to be admissible if $\{I\supseteq\{x\}\}$. Besides, it is called maximal if there cannot exist any non-trivial ideal $J\neq I$ containing I as a subset.

A non-empty family of subsets of $F\subset 2^X$ is a filter on X if it satisfies conditions (1), (2) and (3) of the following statements:

(1) $\emptyset\in F$,
(2) If $A,B\in F$, then $A\cap B\in F$,
(3) If $A\in F$ and $A\subset B$, then $B\in F$.
(4) $I\subset 2^X$ is a non-trivial ideal if and only if $F=F(I)=\{X-A:A\in I\}$ is a filter on X.

Throughout this paper, I_2 will be a strongly non-trivial ideal on $\mathbb{N}\times\mathbb{N}$.

Definition 2.3 ([25]). Let I_2 be a non-trivial ideal of $\mathbb{N}\times\mathbb{N}$ and $(X,\mu,v,*,\diamond)$ be an intuitionistic fuzzy normed space. A double sequence $x=(x_{kj})$ of elements of X is said to be I_2-convergent to $L\in X$ with respect to the intuitionistic fuzzy norm (μ,v), if for each $\epsilon>0$ and $t>0$,

$$\{(j,k)\in\mathbb{N}\times\mathbb{N}:\mu(x_{kj}-L,t)\leqslant 1-\epsilon\ \text{ or }\ v(x_{kj}-L,t)\geqslant\epsilon\}\in I_2.$$

In this case, we wrtie $I^2_{(\mu,v)}$-$\lim x=L$.

Definition 2.4 ([18]). Let X and Y be two normed linear spaces and $T:D(T)\to Y$ be a linear operator, where $D\subset X$. Then the operator T is said to be bounded if there exists a positive real n

such that $\|Tx\| \leqslant n\|x\|$, for all $x \in D(T)$. The set of all bounded linear operators x $B(X,Y)$ [21] are the normed linear spaces normed by $\|T\| = \sup_{x\in X, \|x\|=1} \|Tx\|$ and $B(X,Y)$ is a Banach space if Y is a Banach space.

Definition 2.5 ([18])**.** Let X and Y be two normed linear spaces. An operator $T : X \to Y$ is said to be a compact linear operator, (or completely continuous linear operator), if it satisfies the following conditions:

(1) T is linear.
(2) T maps every bounded sequence (x_n) in X onto a sequence $(T(x_n))$ in Y which has a convergent subsequence.

The set of all compact linear operators $C(X,Y)$ is a closed subspace of $B(X,Y)$.

Definition 2.6 ([17])**.** An Orlicz function is a function $F : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $F(0) = 0$, $F(x) > 0$ for $x > 0$ and $F(x) \to \infty$ as $x \to \infty$. If the convexity of Orlicz function F is replaced by $F(x+y) \leqslant F(x) + F(y)$, then this function is called modulus function. Besides, if F is an Orlicz function, then $F(\lambda x) \leqslant \lambda F(x)$.

3. I_2-convergent Sequences by Using Compact Operator in IFNS

In this section, we introduce the double ideal sequence spaces on compact operator in intuitionistic fuzzy normed spaces.

$$\begin{aligned} \mathcal{M}^{I_2}_{(\mu,v)}(T) = \{(x_{qw}) \in l_\infty : \{(q,w) \in \mathbb{N}\times\mathbb{N} : \mu(T(x_{qw}) - L, t) \leqslant 1-\epsilon \text{ or} \\ v(T(x_{qw}) - L, t) \geqslant \epsilon \in I_2\}\} \\ \mathcal{M}^{I_2^0}_{(\mu,v)}(T) = \{(x_{qw}) \in l_\infty : \{(q,w) \in \mathbb{N}\times\mathbb{N} : \mu(T(x_{qw}), t) \leqslant 1-\epsilon \text{ or} \\ v(T(x_{qw}), t) \geqslant \epsilon \in I_2\}\}. \end{aligned}$$

Besides, we define an open ball with center x and radius r with respect to t as follows:

$$\begin{aligned} B^2_x(r,t)(T) = \{(y_{qw}) \in l_\infty : \{(q,w) : \mu(T(x_{qw}) - T(y_{qw}), t) \leqslant 1-\epsilon \text{ or} \\ v(T(x_{qw}) - T(y_{qw}), t) \geqslant \epsilon \in I_2\}\}. \end{aligned}$$

Now, we will show and prove our main results.

Theorem 3.1. *The sequence spaces* $\mathcal{M}^{I_2}_{(\mu,v)}(T)$ *and* $\mathcal{M}^{I_2^0}_{(\mu,v)}(T)$ *are the linear spaces.*

Proof. Let $x = (x_{qw}), y = (y_{qw}) \in \mathcal{M}^{I_2}_{(\mu,v)}(T)$ and α, β be scalars. Then for a given $\epsilon > 0$, we have the sets:

$$\begin{aligned} P_1 = \Big\{(q,w) \in \mathbb{N}\times\mathbb{N} : \mu\Big(T(x_{qw}) - L_1, \frac{t}{2|\alpha|}\Big) \leqslant 1-\epsilon \text{ or} \\ v\Big(T(x_{qw}) - L_1, \frac{t}{2|\alpha|}\Big) \geqslant \epsilon\Big\} \in I_2, \\ P_1 = \Big\{(q,w) \in \mathbb{N}\times\mathbb{N} : \mu\Big(T(y_{qw}) - L_2, \frac{t}{2|\beta|}\Big) \leqslant 1-\epsilon \text{ or} \\ v\Big(T(y_{qw}) - L_2, \frac{t}{2|\beta|}\Big) \geqslant \epsilon\Big\} \in I_2. \end{aligned}$$

This implies that

$$P_1^c = \Big\{(q,w) \in \mathbb{N}\times\mathbb{N} : \mu\Big(T(x_{qw}) - L_1, \frac{t}{2|\alpha|}\Big) > 1-\epsilon \text{ or } v\Big(T(x_{qw}) - L_1, \frac{t}{2|\alpha|}\Big) < \epsilon\Big\} \in F(I_2);$$

$$P_2^c = \Big\{(q,w) \in \mathbb{N}\times\mathbb{N} : \mu\Big(T(y_{qw}) - L_2, \frac{t}{2|\beta|}\Big) > 1-\epsilon \text{ or } v\Big(T(y_{qw}) - L_2, \frac{t}{2|\beta|}\Big) < \epsilon\Big\} \in F(I_2).$$

Now, define the set $P_3 = P_1 \cup P_2$, thus $P_3 \in I_2$ and P_3^c is a non-empty set in $F(I_2)$. Then we prove that for each $(x_{qw}), (y_{qw}) \in \mathcal{M}^{I_2}_{(\mu,v)}(T)$,

$$P_3^c \subset \{(q,w) \in \mathbb{N}\times\mathbb{N} : \mu((\alpha T(x_{qw}) + \beta T(y_{qw})) - (\alpha L_1 + \beta L_2), t) > 1-\epsilon \text{ or } v((\alpha T(x_{qw}) + \beta T(y_{qw})) - (\alpha L_1 + \beta L_2), t) < \epsilon\}.$$

Let $(n,m) \in P_3^c$, in this case

$$\mu\Big(T\Big(x_{nm} - L_1, \frac{t}{2|\alpha|}\Big)\Big) > 1-\epsilon \text{ or } v\Big(T\Big(x_{nm} - L_1, \frac{t}{2|\alpha|}\Big)\Big) < \epsilon$$

and

$$\mu\Big(T\Big(y_{nm} - L_2, \frac{t}{2|\beta|}\Big)\Big) > 1-\epsilon \text{ or } v\Big(T\Big(y_{nm} - L_2, \frac{t}{2|\beta|}\Big)\Big) < \epsilon.$$

Thus we have

$$\begin{aligned}
&\mu((\alpha T(x_{qw}) + \beta T(y_{nm})) - (\alpha L_1 + \beta L_2), t)\\
&\geqslant \mu\Big(\alpha T(x_{qw}) - \alpha L_1, \frac{t}{2}\Big) * \mu\Big(\beta T(x_{nm}) - \beta L_2, \frac{t}{2}\Big)\\
&= \mu\Big(T(x_{nm}) - L_1, \frac{t}{2|\alpha|}\Big) * \mu\Big(T(x_{nm}) - L_2, \frac{t}{2|\beta|}\Big)\\
&> (1-\epsilon) * (1-\epsilon) = 1-\epsilon
\end{aligned}$$

and

$$\begin{aligned}
&v((\alpha T(x_{nm}) + \beta T(y_{nm})) - (\alpha L_1 + \beta L_2), t)\\
&\leqslant v\Big(\alpha T(x_{nm}) - \alpha L_1, \frac{t}{2}\Big) \diamond v\Big(\beta T(x_{nm}) - \beta L_2, \frac{t}{2}\Big)\\
&= v\Big(T(x_{nm}) - L_1, \frac{t}{2|\alpha|}\Big) \diamond v\Big(T(x_{nm}) - L_2, \frac{t}{2|\beta|}\Big)\\
&< \epsilon \diamond \epsilon = \epsilon.
\end{aligned}$$

This implies that

$$P_3^c \subset \{(q,w) \in \mathbb{N}\times\mathbb{N} : \mu((\alpha T(x_{qw}) + \beta T(y_{qw})) - (\alpha L_1 + \beta L_2), t) > 1-\epsilon \text{ or } v((\alpha T(x_{qw}) + \beta T(y_{qw})) - (\alpha L_1 + \beta L_2), t) < \epsilon\}.$$

Therefore the sequence space $\mathcal{M}^{I_2}_{(\mu,v)}(T)$ is a linear space. The proof of $\mathcal{M}^{I_2^0}_{(\mu,v)}(T)$ runs similarly. □

Remark 3.1. In the following theorems, we will discuss some problems on the convergence in double sequence spaces. Towards this end, first at all, we have to discuss about the topology of this space. Let

$$\tau^{I_2}_{(\mu,v)}(T) = \{H \subset \mathcal{M}^{I_2}_{(\mu,v)}(T) : \text{ for each } x \in H, \text{ there exist } t > 0 \text{ and } r \in (0,1) \text{ such that } B_x^2(r,t)(T) \subset A\}.$$

Then $\tau^{I_2}_{(\mu,v)}(T)$ is a topology on $\mathcal{M}^{I_2}_{(\mu,v)}(T)$.

Theorem 3.2. *Let $\mathcal{M}^{I_2}_{(\mu,v)}(T)$ be an IFNS and $\tau^{I_2}_{(\mu,v)}(T)$ be a topology on $\mathcal{M}^{I_2}_{(\mu,v)}(T)$. Then a double sequence $(x_{qw}) \in \mathcal{M}^{I_2}_{(\mu,v)}(T)$, $x_{qw} \to x$ if and only if $\mu(T(x_{qw}) - T(x), t) \to 1$ and $v(T(x_{qw}) - T(x), t) \to 0$ as $q, w \to \infty$.*

Proof. Fix $t_0 > 0$ and consider $x_{qw} \to x$. Then, for $r \in (0, 1)$, there exist $n_0, m_0 \in \mathbb{N}$ such that $(x_{qw}) \in B^2_x(r, t)(T)$ for all $q \geqslant n_0$ and $w \geqslant m_0$. Thus we have

$$B^2_x(r, t_0)(T) = \{(q, w) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{qw}) - T(x), t_0) \leqslant 1 - r \ \text{ or } \\ v(T(x_{qw}) - T(x), t_0) \geqslant r\} \in I_2$$

such that $(B^2_x)^c(T) \in F(I_2)$. Then $1 - \mu(T(x_{qw}) - T(x), t_0) < r$ and $v(T(x_{qw}) - T(x), t_0) < r$. Therefore $\mu(T(x_{qw}) - T(x), t_0) \to 1$ and $v(T(x_{qw}) - T(x), t_0) \to 0$ as $q, w \to \infty$.

Conversely, if for each $t > 0$, $\mu(T(x_{qw}) - T(x), t) \to 1$ and $v(T(x_{qw}) - T(x), t_0) \to 0$ as $q, w \to \infty$, then for $r \in (0, 1)$, there exist $n_0, m_0 \in \mathbb{N}$ such that $1 - \mu(T(x_{qw}) - T(x), t) < r$ and $v(T(x_{qw}) - T(x), t) < r$, for all $q \geqslant n_0$ and $w \geqslant m_0$. This shows that $\mu(T(x_{qw}) - T(x), t) > 1 - r$ and $v(T(x_{qw}) - T(x), t) < r$ for all $q \geqslant n_0$ and $w \geqslant m_0$. Therefore $(x_{qw}) \in (B^2_x)^c(r, t)(T)$ for all $q \geqslant n_0$ and $w \geqslant m_0$ and then $x_{qw} \to x$. □

Theorem 3.3. *A double sequence $x = (x_{qw}) \in \mathcal{M}^{I_2}_{(\mu,v)}(T)$ is I_2-convergent if and only if for every $\epsilon > 0$ and $t > 0$, there exist the numbers $N = N(x, \epsilon, t)$ and $M = M(x, \epsilon, t)$ such that*

$$\Big\{(N, M) : \mu\Big(T(X_{NM}) - L, \frac{t}{2}\Big) > 1 - \epsilon \ \text{ or } \ v(T(x_{NM} - L, \frac{t}{2}) < \epsilon\Big\} \in F(I_2).$$

Proof. Consider that $I^2_{(\mu,v)}$-$\lim x_{qw} = L$ and let $t > 0$. For a given $\epsilon > 0$, take $s > 0$ such that $(1 - \epsilon) * (1 - \epsilon) > 1 - s$ and $\epsilon \diamond \epsilon < s$. Then for each $x \in \mathcal{M}^{I_2}_{(\mu,v)}(T)$,

$$R_2 = \Big\{(q, w) \in \mathbb{N} \times \mathbb{N} : \mu\Big(T(x_{qw}) - L, \frac{t}{2}\Big) \leqslant 1 - \epsilon \ \text{ or } \ v\Big(T_{qw} - L, \frac{t}{2}\Big) \geqslant \epsilon\Big\} \in I_2,$$

which implies that

$$R^c_2 = \Big\{(q, w) \in \mathbb{N} \times \mathbb{N} : \mu\Big(T(x_{qw}) - L, \frac{t}{2}\Big) > 1 - \epsilon \ \text{ or } \ v\Big(T(x_{qw}) - L, \frac{t}{2}\Big) < \epsilon\Big\} \in F(I_2).$$

Conversely, let's choose $N, M \in R^c_2$. Then

$$\mu\Big(T(x_{NM}) - L, \frac{t}{2}\Big) > 1 - \epsilon \ \text{ or } \ v\Big(T(x_{NM}) - L, \frac{t}{2}\Big) < \epsilon.$$

Now, we have to prove that there exist the numbers $N = N(x, \epsilon, t)$ and $M = M(x, \epsilon, t)$ such that

$$\{(q, w) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{qw}) - T(x_{NM}), t) \leqslant 1 - s \ \text{ or } \\ v(T(x_{qw}) - T(x_{NM}), t) \geqslant s\} \in I_2.$$

To this end, we define that for each $x \in \mathcal{M}^{I_2}_{(\mu,v)}(T)$,

$$S_2 = \{(q, w) \in \mathbb{N} \times \mathbb{N} : \mu(T(x_{qw} - T(x_{NM}, t) \leqslant 1 - s \ \text{ or } \\ v(T(x_{qw}) - T(x_{NM}), t) \geqslant s\} \in I_2.$$

Thus we have to prove that $S_2 \subset R_2$. Let's suppose that $S \subsetneq R$, then there exist $n, m \in S_2$ such that $n, m \notin R_2$. Then we have

$$\mu(T(x_{nm} - T(x_{NM}, t) \leqslant 1 - s \ \text{ or } \ \mu\Big(T(x_{nm}) - L, \frac{t}{2}\Big) > 1 - \epsilon.$$

In particular, $\mu\big(T(x_{NM}) - L, \frac{t}{2}\big) > 1 - \epsilon$. Hence we have

$$1 - s \geqslant \mu(T(x_{nm} - T(x_{NM}), t) \geqslant \mu\Big(T(x_{nm}) - L, \frac{t}{2}\Big) * \mu\Big(T(x_{NM}) - L, \frac{t}{2}\Big) \\ \geqslant (1 - \epsilon) * (1 - \epsilon) > 1 - s,$$

but this is not possible. Otherwise,

$$v(T(x_{nm}) - T(x_{NM}), t) \geqslant s \ \text{ or } \ v\Big(T(x_{nm}) - L, \frac{t}{2}\Big) < \epsilon.$$

In particular, $v\big(T(x_{NM})-L,\frac{t}{2}\big)<\epsilon$. Thus we have

$$s\leqslant v(T(x_{nm})-T(x_{NM}),t)\leqslant v\Big(T(x_{nm})-L,\frac{t}{2}\Big)\diamond v\Big(T(x_{NM})-L,\frac{t}{2}\Big)\leqslant \epsilon\diamond\epsilon<s,$$

but this is not possible. Therefore $S_2\subset R_2$. Thus, $R_2\in I_2$ which implies that $S\in I_2$. □

4. I_2-convergent Sequences by Using Orlicz Function in IFNS

In this section, we use the notion of compact operator and Orlicz function for defining a new double ideal sequence space in intuitionistic fuzzy normed spaces.

$$\mathcal{M}^{I_2}_{(\mu,v)}(T,F)=\Big\{(x_{qw})\in l_\infty:\Big\{(q,w)\in\mathbb{N}\times\mathbb{N}:F\Big(\frac{\mu(T(x_{qw})-L,t)}{\rho}\Big)\leqslant 1-\epsilon\ \text{ or }\ F\Big(\frac{v(T(x_{qw})-L,t)}{\rho}\Big)\geqslant\epsilon\Big\}\in I_2\Big\},$$

$$\mathcal{M}^{I_2^0}_{(\mu,v)}(T,F)=\Big\{(x_{qw})\in l_\infty:\Big\{(q,w)\in\mathbb{N}\times\mathbb{N}:F\Big(\frac{\mu(T(x_{qw}),t)}{\rho}\Big)\leqslant 1-\epsilon\ \text{ or }\ F\Big(\frac{v(T(x_{qw}),t)}{\rho}\Big)\geqslant\epsilon\Big\}\in I_2\Big\}.$$

Moreover, we define an open ball with center x and radius r with respect to t as follows:

$$B^2_x(r,t)(T,F)=\Big\{(y_{qw})\in l_\infty:(q,w)\in\mathbb{N}\times\mathbb{N}:F\Big(\frac{\mu(T(x_{qw})-T(y_{qw}),t)}{\rho}\Big)\leqslant 1-\epsilon\ \text{ or }\ F\Big(\frac{v(T(x_{qw})-T(y_{qw}),t)}{\rho}\Big)\geqslant\epsilon\in I_2\Big\}.$$

Remark 4.1. The sequences $\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$ and $\mathcal{M}^{I_2^0}_{(\mu,v)}(T,F)$ are the linear spaces.

Theorem 4.1. *Every open ball $B^2_x(r,t)(T,F)$ is an open set in $\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$.*

Proof. Let $B^2_x(r,t)(T,F)$ be an open ball with center x and radius r with respect to t. This is

$$B^2_x(r,t)(T,F)=\Big\{y=(y_{qw})\in l_\infty:\Big\{(q,w)\in\mathbb{N}\times\mathbb{N}:F\Big(\frac{\mu(T(x_{qw})-T(y_{qw}),t)}{\rho}\Big)\leqslant 1-r\ \text{ or }\ F\Big(\frac{v(T(x_{qw})-T(y_{qw}),t)}{\rho}\Big)\geqslant r\Big\}\in I_2\Big\}.$$

Let $y\in B^2_x(r,t)(T,F)$, then $F\big(\frac{\mu(T(x_{qw})-T(y_{qw}),t)}{\rho}\big)>1-r$ and $F\big(\frac{v(T(x_{qw})-T(y_{qw}),t)}{\rho}\big)<r$. Since $F\big(\frac{\mu(T(x_{qw})-T(y_{qw}),t)}{\rho}\big)>1-r$, there exists $t_0\in(0,t)$ such that $F\big(\frac{\mu(T(x_{qw})-T(y_{qw}),t_0)}{\rho}\big)>1-r$ and $F\big(\frac{v(T(x_{qw})-T(y_{qw}),t_0)}{\rho}\big)<r$.

Taking $r_0=F\big(\frac{\mu(T(x_{qw})-T(y_{qw}),t_0)}{\rho}\big)$, we have $r_0>1-r$, and there exists $s\in(0,1)$ such that $r_0>1-s>1-r$. For $r_0>1-s$, we have $r_1,r_2\in(0,1)$ such that $r_0*r_1>1-s$ and $(1-r_0)\diamond(1-r_0)\leqslant s$. Put $r_3=\max\{r_1,r_2\}$. Now, we take a ball $B^c_y(1-r_e,t-t_0)(T,F)$ and prove that $B^2_y(1-r_3,t-t_0)(T,F)\subset B^2_x(r,t)(T,F)$. Let $z=(z_{qw})\in B^2_y(1-r_3,t-t_0)(T,F)$, then $F\big(\frac{\mu(T(y_{qw})-T(z_{qw}),t-t_0)}{\rho}\big)>r_3$ and $F\big(\frac{v(T(y_{qw})-T(z_{qw}),t-t_0)}{\rho}\big)<1-r_3$.

Hence we have

$$\begin{aligned}&F\Big(\frac{\mu(T(x_{qw})-T(z_{qw}),t)}{\rho}\Big)\\ &\geqslant F\Big(\frac{\mu(T(x_{qw})-T(y_{qw}),t_0)}{\rho}\Big)*F\Big(\frac{\mu(T(y_{qw})-T(z_{qw}),t-t_0)}{\rho}\Big)\\ &\geqslant(r_0*r_3)\geqslant(r_0*r_1)\geqslant(1-s)\geqslant(1-r)\end{aligned}$$

and

$$F\Big(\frac{v(T(x_{qw})-T(z_{qw}),t)}{\rho}\Big)$$
$$\leqslant F\Big(\frac{v(T(x_{qw})-T(y_{qw}),t_0)}{\rho}\Big)\diamond F\Big(\frac{v(T(y_{qw})-T(z_{qw}),t-t_0)}{\rho}\Big)$$
$$\leqslant (1-r_0)\diamond(1-r_3)\leqslant(1-r_0)\diamond(1-r_2)\leqslant s\leqslant r.$$

Therefore $z\in B_x(r,t)(T,F)$ and hence we have $B_y^2(1-r_3,t-t_0)(T,F)\subset B_x^2(r,t)(T,F)$. □

Remark 4.2. $\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$ is an IFNS.

Define $\tau^{I_2}_{(\mu,v)}(T,F)=\{A\subset\mathcal{M}^{I_2}_{(\mu,v)}(T,F):$ for each $x\in H$, there exists $t>0$ and $r\in(0,1)$ such that $B_x^2(r,t)(T,F)\subset H\}$. Then $\tau^{I_2}_{(\mu,v)}/(T,F)$ is a topology on $\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$.

Remark 4.3. The topology $\tau^{I_2}_{(\mu,v)}(T,F)$ on $\mathcal{M}^{I_2^0}_{(\mu,v)}(T,F)$ is first countable.

Theorem 4.2. *$\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$ and $\mathcal{M}^{I_2^0}_{(\mu,v)}(T,F)$ are Hausdorff spaces.*

Proof. Let $u,v\in\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$ such that $u\neq v$. Then $0<F\big(\frac{\mu(T(u)-T(v),t)}{\rho}\big)<1$ and $0<F\big(\frac{v(T(u)-T(v),t)}{\rho}\big)<1$. Take $r_1=F\big(\frac{\mu(T(u)-T(v),t)}{\rho}\big)$, $r_2=F\big(\frac{v(T(u)-T(v),t)}{\rho}\big)$ and $r=\max\{r_1,1-r_2\}$. For each $r_0\in(r,1)$, there exist r_3 and r_4 such that $r_3*r_4\geqslant r_0$ and $(1-r_3)\diamond(1-r_4)\leqslant 1-r_0$. Put $r_5=\max\{r_3,1-r_4\}$ and consider the open balls $B_u^2\big(1-r_5,\frac{t}{2}\big)$ and $B_v^2\big(1-r_5,\frac{t}{2}\big)$. Then it is clear that $\big(B_u^2\big)^c(1-r_5,\frac{t}{2})\cap\big(B_v^2\big)^c(1-r_5,\frac{t}{2})=\emptyset$, thus

$$r_1=F\Big(\frac{\mu(T(u)-T(v),t)}{\rho}\Big)\geqslant F\Big(\frac{\mu\big(T(u)-T(b),\frac{t}{2}\big)}{\rho}\Big)*F\Big(\frac{\mu\big(T(b)-T(v),\frac{t}{2}\big)}{\rho}\Big)$$
$$\geqslant r_5*r_5\geqslant r_3*r_3\geqslant r_0\geqslant r_1$$

and

$$r_2=F\Big(\frac{v(T(u)-T(v),t)}{\rho}\Big)\leqslant F\Big(\frac{v\big(T(u)-T(v),\frac{t}{2}\big)}{\rho}\Big)\diamond F\Big(\frac{v\big(T(b)-T(v),\frac{t}{2}\big)}{\rho}\Big)$$
$$\leqslant(1-r_5)\diamond(1-r_5)\leqslant(1-r_4)\diamond(1-r_4)\leqslant(1-r_0)\leqslant r_2,$$

and this is a contradiction. Therefore $\mathcal{M}^{I_2}_{(\mu,v)}(T,F)$ is Hausdorff.

The proof of $\mathcal{M}^{I_2^0}_{(\mu,v)}(T,F)$ runs similarly. □

5. Conclusion

In this paper, we have proved new notions on double sequence spaces by using the results presented in [17]. These new notions can be extended to three-dimensional or higher order spaces. On the other hand, applications problem can be obtained for such areas as artificial intelligence, decision making, computational simulation and even neutrosophic double sequence spaces [13].

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[1]Estudiante de Doctorado en Matemáticas, Magister en Ciencias Matemáticas, Universidad de Antioquia, Medellín, Colombia

[2]Magister en Educación Universidad del Atlántico, Barranquilla, Colombia

[3]Magister en Informática Educativa Universidad del Atlántico, Barranquilla, Colombia
Email address: carlosgranadosortiz@outlook.es
Email address: iapadilla@mail.uniatlantico.edu.co
Email address: yesikarojas@mail.uniatlantico.edu.co