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THE COMPLEX LINE (p,q)-INTEGRAL AND (p,q)-GREEN'S FORMULA

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Abstract. In this study, we present the complex line (p,q)-integral and multiple (p,q)-integral by using the concept of (p,q)-calculus. The (p,q)-Green's formula and the (p,q)-Gauss formulas are obtained with appropriate conditions in a complex plane.

1. Introduction

The q-calculus, where q stands for quantum, plays an important role in understanding various fields of mathematics, such as fractional calculus, discrete geometric function theory, analytic function theory, etc. There is a vast list of publications [1–4, 9–11, 15, 17] and references therein. Roughly speaking, q-calculus substitutes the classical derivative by a difference operator. For basic definitions and more details, see [7, 13].

The (p,q)-calculus, which is more general than q-calculus, has been studied due to its important applications in various subfields of mathematics and quantum physics [5, 6, 8, 12, 18–20]. In [16], the author investigated some properties of the (p,q)-calculus. Furthermore, in the same paper, the fundamental theorem and the formula of integration by parts were obtained.

It is well-known that Green's formula provides the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. The aim of our paper is to get the complex (p,q)-Green's Formula. The paper is organised as follows. In Section 2, we recall some basic definitions and properties of the (p,q)-calculus. In Section 3, the (p,q)-complex integral is defined. In the next and final section, we give the complex line (p,q)-integral and the multiple (p,q)-integral, respectively.

2. Basic Definitions and Preliminaries

In this work, unless otherwise stated, for fixed (x_0, y_0) $(x_0 \geq 0,\ y_0 \geq 0)$, we consider the discrete set

$$K = \big\{p^m x_0 + i\, q^n y_0 \in \mathbb{C},\ \ m,n = 0,1,2,\dots;\ \ 0<p<1,\ \ 0<q<1\big\}.$$

Definition 2.1. For $\alpha \in \mathbb{R}$, the (p,q)-analogue of α is defined as

$$[\alpha]_{p,q} = \frac{p^\alpha - q^\alpha}{p-q}.$$

Also, for $n \in \mathbb{N}$, the (p,q)-factorial of n is defined as

$$[n]_{p,q}! = [1]_{p,q}\,[2]_{p,q}\dots[n]_{p,q}, \quad n \geq 1; \quad [0]_{p,q}! = 1.$$

Moreover, let us introduce the p,q-binomial coefficients

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}!\,[n-k]_{p,q}!}, \quad 0 \leq k \leq n; \quad n,k \in \mathbb{N}.$$

Definition 2.2. The (p,q)-differential of a complex discrete function $f(z)$, defined at the points of a discrete domain K, is written as

$$D_{p,q}\, f(z) = \frac{f(pz) - f(qz)}{(p-q)z}, \quad z \neq 0,$$

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and

$$D_{p,q}\, f(0) = \lim_{z\to 0} \frac{f(p^n z) - f(q^n z)}{(p^n - q^n)z}.$$

For example, if $f(z) = z^n$, $n \in \mathbb{N}$, one can see that

$$D_{p,q}\, z^n = [n]_{p,q}\, z^{n-1}.$$

Definition 2.3. For $z = x + iy := (x, y) \in K$,

$$(z) = \left\{(x, y), (px, y), (x, py), (qx, y), (x, qy), (qx, py), (px, qx), (px, py), (qx, qy)\right\} \subset K$$

is called a basic discrete set with respect to $z \in K$.

Definition 2.4. For $z = x + iy \in S(z)$, the (p, q)-differential of a complex discrete function $f(z)$ is defined as $d_{p,q}\, f(z) = f(pz) - f(qz) = f(px, py) - f(qx, qy)$.

Now, for $z = x + iy \in S(z)$, let us introduce the partial (p, q)-derivatives in real variables x and y.

$$D_{(p,q),x}\, f(x, y) = \frac{f(px, y) - f(qx, y)}{(p - q)x}, \quad D_{(p,q),y}\, f(x, y) = \frac{f(x, py) - f(x, qy)}{(p - q)y}. \tag{2.1}$$

Also, we consider the dilatation operators

$$M_q^x\, f(x, y) = f(qx, y), \quad M_q^y\, f(x, y) = f(x, qy) \tag{2.2}$$

in variables x and y, respectively.

Hence, the (p, q)-differential of $f(z)$ can be written as

$$d_{p,q}\, f(x, y) = M_p^y\, D_{(p,q),x}\, f(x, y)\, d_{p,q}x + M_q^x\, D_{(p,q),y}\, f(x, y)\, d_{p,q}y, \tag{2.3}$$

where $d_{p,q}x = (p - q)x$, $d_{p,q}y = (p - q)y$.

For any point $z \in S(z)$, it can be seen that $d_{p,q}z = d_{p,q}x + i\, d_{p,q}y = (p-q)z$, $d_{p,q}\bar{z} = d_{p,q}x - i\, d_{p,q}y = (p - q)\bar{z}$.

Definition 2.5. Let $D_{(p,q),x}$, $D_{(p,q),y}$ and M_p, M_q be as in (2.1) and (2.2), respectively. Then the complex (p, q)-differential operators $D_{(p,q),z}$ and $D_{(p,q),\bar{z}}$ are defined as follows:

$$D_{(p,q),z} := \frac{1}{2}\left[M_p^y\, D_{(p,q),x} - i\, M_q^x\, D_{(p,q),y}\right], \tag{2.4}$$

$$D_{(p,q),\bar{z}} := \frac{1}{2}\left[M_p^y\, D_{(p,q),x} + i\, M_q^x\, D_{(p,q),y}\right]. \tag{2.5}$$

Theorem 2.6. *For any point $z \in S(z)$, the (p, q)-differential of discrete function $f(z)$ defined at the points in K can be given by*

$$d_{p,q}\, f(z) = D_{(p,q),z}\, f(z)\, d_{p,q}z + D_{(p,q),\bar{z}}\, f(z)\, d_{p,q}\bar{z}. \tag{2.6}$$

Proof. The validity of (2.6) can be seen by using (2.1), (2.2), (2.4) and (2.5) through simple calculations. Also, we note that (2.3) is equivalent to (2.6). □

Definition 2.7. Let $B \subset K$ be a subdiscrete domain. For $S(z) \subset B$, if for all $z \in B$, $f(z)$ satisfies $D_{(p,q),\bar{z}}\, f(z) \equiv 0$, then $f(z)$ is called (p, q)-analytic on B.

Example 2.8. The function

$$f(z) = f(x, y) = qx^2 + (p + q)ixy - py^2 \tag{2.7}$$

is (p, q)-analytic. In fact, after some calculations, it is seen that

$$\frac{f(px, py) - f(qx, py)}{x} = (p^2 - q^2)(qx + ipy) = \frac{f(qx, py) - f(qx, qy)}{iy}.$$

So, this implies that $D_{(p,q),\bar{z}} f(z) \equiv 0$.

Remark 2.9. In [14], the criterion of (p,q)-analyticity is given as $D_{p,x}\, f(z) = D_{q,y}\, f(z)$, where

$$D_{p,x}\, f(z) = \frac{f(z) - f(px, y)}{(1-p)x}; \quad D_{q,y}\, f(z) = \frac{f(z) - f(x, qy)}{(1-q)iy}.$$

The function (2.7) is not (p,q)-analytic in the sense of M. A. Khan.

On the other hand, in [17], the criterion of q-analyticity was given as

$$D_{q,\bar{z}}\, f(z) = \frac{1}{2}\left[D_{q,x} + i\, M^{y}_{\frac{1}{q}}\, D_{q,y}\right] f(z) = 0.$$

The function (2.7) is not q-analytic in the sense of Pashaev. However, by interchanging p and q, and then choosing $p = 1$, the new function $g(z) = g(x,y) = x^2 + (1+q)ixy - qy^2$ is q-analytic in the sense of Pashaev.

Let us define the operator $L_{p,q}$ by

$$L_{p,q}\, f(z) := z\, f(qx, py) - x\, f(qx, qy) - i\, y\, f(px, py).$$

Theorem 2.10. *For $z = x + iy \in S(z)$,*

$$f(z) \quad \text{is} \quad (p,q)\text{-analytic on} \quad S(z) \Leftrightarrow L_{p,q} f(z) \equiv 0.$$

Corollary 2.11. *If $f(z)$ is (p,q)-analytic, it is easily seen that $d_{p,q}\, f(z) = D_{(p,q),z}\, f(z)\, d_{p,q} z$.*

3. The (p,q)-complex Integral

In this section, the (p,q)-Jackson integral is defined for an arbitrary function $f(z)$.

Definition 3.1. Let $f(z)$ be an arbitrary function and $F(z)$ be a function such that $D_{(p,q),z}\, F(z) = f(z)$, then

$$\frac{F(pz) - F(qz)}{(p-q)z} = f(z).$$

The function $F(z)$ is a (p,q)-antiderivative of $f(z)$. $F(z)$ is denoted by

$$\int f(z) d_{p,q} z.$$

Definition 3.2 ([16])**.** Let f be an arbitrary function and a be a real number. The (p,q)-integral of f is defined as follows:

$$\int_0^a f(x)\, d_{p,q} x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}}\, f\left(\frac{q^k}{p^{k+1}} a\right), \quad \left|\frac{q}{p}\right| < 1. \tag{3.1}$$

Definition 3.3. A function f is called (p,q)-integrable on $[0,\infty)$ if the series in (3.1) converges absolutely.

Remark 3.4. If we choose $p = 1$, then formula (3.1) for the well-known Jackson integral (see [13, p. 67]) gives

$$\int f(x)\, d_q x = (1-q)x \sum_{k=0}^{\infty} q^k f\left(q^k x\right).$$

For $p = r^{1/2}$, $q = s^{-1/2}$,

$$\left|\frac{p}{q}\right| < 1 \Leftrightarrow |rs| < 1,$$

and formula (3.1) gives

$$\int_0^a f(x) d_{p,q} x = \left(s^{-1/2} - r^{1/2}\right) a \sum_{k=0}^{\infty} r^{k/2}\, s^{(k+1)/2}\, f\left(r^{k/2}\, s^{(k+1)/2} a\right),$$

which is formula (11) given in [5].

The detailed information related to the (p,q)-integrals can be found in [16].

4. Line Integrals

On the complex plane $\mathbb{C}$, let

$$\gamma : z(t) = x(t) + i\,y(t), \quad t \in [0, \quad a], \quad (a > 0)$$

be a curve which is piecewise smooth and rectifiable. Now, let us consider a function $f(z(t)) = u(z(t)) + iv(z(t))$, $t \in [0, a]$, on the curve γ.

Definition 4.1. If for $z = x + iy$,

$$\lim_{n,k\to\infty} h\left(\frac{q^n}{p^n}x, \frac{q^k}{p^k}y\right) := \lim_{n\to\infty}\left[\lim_{k\to\infty} h\left(\frac{q^n}{p^n}x, \frac{q^k}{p^k}y\right)\right] = \lim_{k\to\infty}\left[\lim_{n\to\infty} h\left(\frac{q^n}{p^n}x, \frac{q^k}{p^k}y\right)\right] = h(0,0),$$

then $h(z)$ is called (p,q)-regular at the point $(0,0)$.

Definition 4.2. Let $f(z)$ be given on a piecewise smooth, rectifiable curve $\gamma \subset \mathbb{C}$, or indirectly, as a composite function $f(z(t))$ on $I = [0,a]$. It is said to be (p,q)-uniformly continuous on the curve γ if for any given positive number ε, we can find a number $\delta = \delta(\varepsilon)$ such that $\left|f\left(z\left(a\frac{q^k}{p^k}\right)\right) - f\left(z\left(a\frac{q^{k+1}}{p^{k+1}}\right)\right)\right| < \varepsilon$ for sufficiently large numbers k, provided $\left|z\left(a\frac{q^k}{p^k}\right) - z\left(a\frac{q^{k+1}}{p^{k+1}}\right)\right| < \delta$.

Lemma 4.3. *Let $\gamma : z(t) = x(t) + iy(t)$ be a continuous curve on $\mathbb{C}$ with $t \in [0,a]$, and let $f(z) = u(z) + iv(z) = u(z(t)) + iv(z(t))$ be a complex-valued function which is (p,q)-uniformly continuous on the given curve γ. If $x(t), y(t)$ have (p,q)-derivative and u, v are (p,q)-integrable, then the Jackson integral of $f(z)$ on γ is given by*

$$\begin{aligned}\int_\gamma f(z)d_{p,q}z &= \int_0^a f(z(t))\, D_{p,q}z(t)\, d_{p,q}t \\ &= (p-q)a\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(z\left(a\frac{q^n}{p^{n+1}}\right)\right) D_{p,q}z\left(a\frac{q^n}{p^{n+1}}\right). \end{aligned} \tag{4.1}$$

Here, the (p,q)-antiderivative of $f(z)$ is (p,q)-regular at $z = 0$.

Proof. Let $\gamma : z(t) = x(t) + i\,y(t)$, for $0 \le t \le a$, and $f(z) = f(z(t)) = u(z(t)) + i\,v(z(t))$. Starting with

$$I = \int_\gamma f(z)d_{p,q}z = \int_\gamma u(z)d_{p,q}z + i\int_\gamma v(z)d_{p,q}z,$$

and making the substitution $z = z(t)$ in each of integrals on the right-hand side, we have

$$\begin{aligned} I &= \int_0^a u(z(t))D_{p,q}z(t)\, d_{p,q}t + i\int_0^a v(z(t))D_{p,q}z(t)\, d_{p,q}t = \int_0^a f(z(t))D_{p,q}z(t)\, d_{p,q}t \\ &= \int_0^a [u(z(t)) + iv(z(t))][D_{p,q}x(t) + iD_{p,q}y(t)]\, d_{p,q}t \\ &= \int_0^a u(z(t))D_{p,q}x(t)\, d_{p,q}t + i\int_0^a v(z(t))D_{p,q}x(t)\, d_{p,q}t \\ &\quad + i\int_0^a u(z(t))D_{p,q}y(t)\, d_{p,q}t - \int_0^a v(z(t))D_{p,q}y(t)\, d_{p,q}t. \end{aligned}$$

By calculating the real integrals and using (3.1), we get

$$\begin{aligned} I &= \sum_{n=0}^{\infty}\left[u\left(z\left(a\frac{q^n}{p^{n+1}}\right)\right)+iv\left(z\left(a\frac{q^n}{p^{n+1}}\right)\right)\right] \\ &\quad\times\left[x\left(a\frac{q^n}{p^{n+1}}\right)-x\left(a\frac{q^n}{p^{n+1}}\right)+i\left(y\left(a\frac{q^n}{p^{n+1}}\right)-y\left(a\frac{q^n}{p^{n+1}}\right)\right)\right] \\ &= \sum_{n=0}^{\infty} f\left(z\left(a\frac{q^n}{p^{n+1}}\right)\right)\left(z\left(a\frac{q^n}{p^n}\right)-z\left(a\frac{q^{n+1}}{p^{n+1}}\right)\right) \\ &= (p-q)a\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(z\left(a\frac{q^n}{p^{n+1}}\right)\right)D_{p,q}z\left(a\frac{q^n}{p^{n+1}}\right). \end{aligned}$$

The proof is complete. □

Definition 4.4. If the series in (4.1) is convergent, then the complex function $f(z)$ is called (p,q)-integrable on the curve γ.

Remark 4.5. If $f(z)$ is (p,q)-integrable on the continuous curve γ, then

$$\lim_{p\to 1}\int_{\gamma} f(z)\,d_{p,q}z = \int_{\gamma} f(z)\,d_q z.$$

Example 4.6. If we use (4.1) for $\gamma : z(t) = t^2 + it$, $0 \le t \le 2$, and the function $f(z) = \overline{z}$ for $0 < q < p < 1$, we get

$$\begin{aligned} \int_{\gamma}\overline{z}\,d_{p,q}z &= \int_{t=0}^{2}(t^2-it)D_{p,q}z(t)\,d_{p,q}t = \int_{0}^{2}(t^2-it)[t(p+q)+i]\,d_{p,q}t \\ &= 4\Big(\frac{4}{p^2+q^2}-\frac{i(1-(p+q))}{p^2+q+q^2}+\frac{1}{p+q}\Big). \end{aligned}$$

Note that the result above is the same as the following (p,q)-integral result as $p \to 1$:

$$\lim_{p\to 1}\int_{\gamma}\overline{z}\,d_{p,q}z = \int_{\gamma}\overline{z}\,d_q z = 4\Big(\frac{4}{1+q^2}-\frac{2i}{1+q+q^2}+\frac{1}{1+q}\Big).$$

Let us assume that the rectifiable piecewise smooth curve

$$\gamma : z(t) = x(t) + iy(t), \quad 0 \le t \le a$$

is positively oriented. In that case, the initial point of γ is $z(0) = z_0 = x(0) + i\,y(0)$ and the final point of γ is $z(a) = z_a = x(a) + i\,(a)$.

Let $f(z)$ be a (p,q)-integrable function on γ. Let $\widetilde{\gamma}$ be the curve having the opposite orientation of γ, in the classical sense. Then the line (p,q) -integral of $f(z)$ on $\widetilde{\gamma}$ is defined by

$$\int_{\widetilde{\gamma}} f(z)\,d_{p,q}z := -\int_{\gamma} f(z)\,d_{p,q}z.$$

5. Multiple (p,q)-Integrals

Let $h(x,y)$ be a function given in $D = \big\{(x,y) \in \mathbb{R}^2 : 0 \le x \le a,\ 0 \le y \le a\big\}$. For $(x,y) \in D$, the multiple (p,q)-integral of $h(x,y)$ with $0 < q < p < 1$ can be given by,

$$I = \int_0^x\int_0^y h(s,t)\,d_{p,q}t\,d_{p,q}s = (p-q)^2 xy\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{n+k}}{q^{n+k+2}}h\left(\frac{q^n}{p^{n+1}}x, \frac{q^k}{p^{k+1}}y\right). \tag{5.1}$$

Now, let us investigate under which conditions the series (5.1) will be convergent. For all $(x,y) \in D$, if $M \in (0,+\infty)$ exists such that $|h(x,y)| \le M$, integral (5.1) always exists. However, even if $h(x,y)$ is unbounded, integral (5.1) may exist, as well.

The following theorem is related to a function in this situation:

Theorem 5.1. *Let $h(x,y)$ be a function given in $D=\{(x,y)\in\mathbb{R}^2 : 0\le x\le a,\, 0\le y\le a\}$. Assume that for all $(x,y)\in(0,a]\times(0,a]$, $0\le\alpha<1$, $0\le\beta<1$, there is a positive M such that*

$$x^\alpha y^\beta |h(x,y)| < M. \tag{5.2}$$

Then the series (5.1) is convergent for all $(x,y)\in(0,a]\times(0,a]$.

Proof. From (5.2), for $x>0$, $y>0$, we have

$$|h(x,y)| < Mx^{-\alpha}y^{-\beta}. \tag{5.3}$$

In (5.3), if we replace x with $\frac{q^n}{p^{n+1}}x$ and y with $\frac{q^k}{p^{k+1}}y$, we get

$$\left| h\left(\frac{q^n}{p^{n+1}}x, \frac{q^k}{p^{k+1}}y\right)\right| < M\left(\frac{q^n}{p^{n+1}}x\right)^{-\alpha}\left(\frac{q^k}{p^{k+1}}y\right)^{-\beta}.$$

Thus

$$\frac{q^n}{p^{n+1}}\frac{q^k}{p^{k+1}}\left|h(q^n x, q^k y)\right| < Mp^{\alpha-1}p^{\beta-1}\left(\frac{q}{p}\right)^{(1-\alpha)n}\left(\frac{q}{p}\right)^{(1-\beta)k} x^{-\alpha}y^{-\beta}.$$

For $0<q<p<1$, we have $0<\left(\frac{q}{p}\right)^{1-\alpha}<1$, $0<\left(\frac{q}{p}\right)^{1-\beta}<1$. The geometric series

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} Mx^{-\alpha}y^{-\beta}p^{\alpha-1}p^{\beta-1}\left(\frac{q}{p}\right)^{(1-\alpha)n}\left(\frac{q}{p}\right)^{(1-\beta)k} = \frac{Mx^{-\alpha}y^{-\beta}}{(p^{1-\alpha}-q^{1-\alpha})(p^{1-\beta}-q^{1-\beta})}$$

is pointwise convergent. Hence, for all $(x,y)\in(0,a]\times(0,a]$, we have

$$(p-q)^2 xy\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{q^{n+k}}{p^{n+k+2}}|h(x,y)| < \frac{Mx^{1-\alpha}y^{1-\beta}(p-q)^2}{(p^{1-\alpha}-q^{1-\alpha})(p^{1-\beta}-q^{1-\beta})}.$$

Finally, the series in (5.1) is absolutely convergent, therefore it is convergent. □

Remark 5.2. The condition (5.2) is sufficient but not necessary for the convergence of the series in (5.1).

Remark 5.3. For all $(x,y)\in D$, if we take $h(x,y)=1$, then

$$\int_0^a\int_0^a d_{p,q}x\, d_{p,q}y = a^2 = Area(D).$$

Remark 5.4. We note that

$$\lim_{p\to 1}\int_0^a\int_0^a h(x,y)\, d_{p,q}x\, d_{p,q}y = \int_0^a\int_0^a h(x,y)\, d_q x\, d_q y$$

(see [15]).

Definition 5.5. Let $h(x,y)$ be a function on a (p,q)-geometrical set $D=\{(x,y)\in\mathbb{R}^2 : 0\le x\le a,$ $0\le y\le a\}$ with $0<q<p<1$. For all $(x,y)\in D$ and $x\ne 0$, $y\ne 0$, if

$$\lim_{n\to\infty} h\left(\frac{q^n}{p^n}x, y\right) = h(0,y) \quad\text{and}\quad \lim_{k\to\infty} h\left(x, \frac{q^k}{p^k}y\right) = h(x,0)$$

hold, then $h(x,y)$ is called (p,q)-regular on line segments $D_1=\{(0,y)\in\mathbb{R}^2 : 0<y\le a\}$ and $D_2=\{(x,0)\in\mathbb{R}^2 : 0<x\le a\}$, respectively.

Remark 5.6. Since $\dfrac{q}{p}<1$, the definition of the (p,q)-regularity is the same as that of the q-regularity in [15]. Also, we note that (5.1) is always true if $h(x,y)$ is (p,q)-regular on D_1, D_2 and at $(0,0)$.

Lemma 5.7. *Let $h(x,y)$ be (p,q)-regular on $D_1 = \{(0,y) \in \mathbb{R}^2 : 0 < y \le a\}$, $D_2 = \{(x,0) \in \mathbb{R}^2 : 0 < x \le a\}$ and at the point $(0,0)$. If $h(0,y) = h(x,0) = h(0,0) = 0$, then*

$$D_{(p,q),x}\, D_{(p,q),y} \int_0^x \int_0^y h(s,t)\, d_{p,q}t\, d_{p,q}s = \int_0^x \int_0^y D_{(p,q),s}\ D_{(p,q),t} h(s,t)\, d_{p,q}t\, d_{p,q}s. \tag{5.4}$$

Proof. For $x \neq 0$ and $y \neq 0$, using the definition of the multiple (p,q)-integral, we get

$$\begin{aligned} D_{(p,q),x}\ D_{(p,q),y} F(x,y) &= D_{(p,q),x}\ D_{(p,q),y} \int_0^x \int_0^y h(s,t)\, d_{p,q}t\, d_{p,q}s \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{q^{n+k}}{p^{n+k+2}} \left[p^2 h\left(\frac{q^n}{p^n}x, \frac{q^k}{p^k}y\right) - pqh\left(\frac{q^n}{p^n}x, \frac{q^{k+1}}{p^{k+1}}y\right) \right. \\ &\qquad \left. - pqh\left(\frac{q^{n+1}}{p^{n+1}}x, \frac{q^k}{p^k}y\right) + q^2 h\left(\frac{q^{n+1}}{p^{n+1}}x, \frac{q^{k+1}}{p^{k+1}}y\right)\right] = h(x,y). \end{aligned}$$

On the other hand, for the right-hand side of (5.4), we get

$$\begin{aligned} \int_0^x \int_0^y D_{(p,q),t} h(s,t)\, d_{p,q}t\, d_{p,q}s &= \int_0^x \int_0^y \frac{h(s,pt) - h(s,qt)}{(p-q)t}\, d_{p,q}t\, d_{p,q}s \\ &= \int_0^x (p-q)y \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} \left[\frac{h\left(s, \frac{q^k}{p^k}y\right) - h\left(s, \frac{q^{k+1}}{p^{k+1}}y\right)}{(p-q)\frac{q^k}{p^{k+1}}y} \right] d_{p,q}s \\ &= \int_0^x \sum_{k=0}^{\infty} \left[h\left(s, \frac{q^k}{p^k}y\right) - h\left(s, \frac{q^{k+1}}{p^{k+1}}y\right)\right] d_q s. \end{aligned}$$

It can be seen that

$$\sum_{k=0}^{N} \left[h\left(s, \frac{q^k}{p^k}y\right) - h\left(s, \frac{q^{k+1}}{p^{k+1}}y\right)\right] = h(s,y) - h\left(s, \frac{q^{N+1}}{p^{N+1}}y\right),$$

$$\lim_{N\to\infty} \sum_{k=0}^{N} \left[h\left(s, \frac{q^k}{p^k}y\right) - h\left(s, \frac{q^{k+1}}{p^{k+1}}y\right)\right] = h(s,y) - h(s,0).$$

Hence we have

$$\int_0^x \int_0^y D_{(p,q),t} h(s,t)\, d_{p,q}t\, d_{p,q}s = \int_0^x [h(s,y) - h(s,0)] d_{p,q}s.$$

Finally, we get

$$\begin{aligned} \int_0^x \int_0^y D_{(p,q),s}\, D_{(p,q),t} h(s,t) d_{p,q}t\, d_{p,q}s &= \int_0^x D^s_{p,q} \left[h(s,y) - h(s,0)\right] d_{p,q}s \\ &= h(x,y) - h(x,0) - h(0,y) + h(0,0). \end{aligned}$$

With our assumptions $h(0,y) = h(x,0) = h(0,0) = 0$, the lemma is proved. $\square$

Theorem 5.8. *Let the real-valued functions $u(x,y)$, $v(x,y)$ be given on the set $D = [0,a] \times [0,a]$, and let the functions u, v, $D_{(p,q),x}v$, $D_{(p,q),y}u$ be (p,q)-integrable on D with respect to both x and y. Then the (p,q)-Green's formula*

$$\int_{\partial D} u(x,y) d_{p,q}x + v(x,y) d_{p,q}y = \iint_D \left[D_{(p,q),x} v(x,y) - D_{(p,q),y} u(x,y)\right] d_{p,q}x\, d_{p,q}y \tag{5.5}$$

is valid. Here, ∂D is oriented positively.

Proof. Let $\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$. γ_1 is given by $\gamma_1 : x = t, 0 \le t \le a,\ y = 0$, so we have $d_{p,q}x(t) = d_{p,q}t,\ d_{p,q}y(t) = 0$. Then, calculating the left-hand side of (5.5) on γ_1, we arrive at

$$\int\limits_{\gamma_1} u(x,y)d_{p,q}x + v(x,y)d_{p,q}y = \int\limits_0^a u(t,0)d_{p,q}t.$$

For the curve γ_2, $\gamma_2 : x = a, y = t,\ 0 \le t \le a$, so we have $d_{p,q}x(t) = 0$, $d_{p,q}y(t) = d_{p,q}t$, and

$$\int\limits_{\gamma_2} u(x,y)d_{p,q}x + v(x,y)d_{p,q}y = \int\limits_0^a v(a,t)d_{p,q}t.$$

On the other hand, the curve γ_3 is the curve $\widetilde{\gamma_3} : x = t,\ y = a,\ 0 \le t \le a$ oriented oppositely. Hence on $\widetilde{\gamma_3}$, $d_{p,q}y = 0$ and $d_{p,q}x = d_{p,q}t$, we have

$$\int\limits_{\gamma_3} u(x,y)d_{p,q}x + v(x,y)d_{p,q}y = -\int\limits_{\widetilde{\gamma_3}} u(x,y)d_{p,q}x + v(x,y)d_{p,q}y = -\int\limits_0^a u(t,a)d_{p,q}t.$$

Similarly, the curve γ_4 is the curve $\widetilde{\gamma_4} : x = 0,\ y = t,\ 0 \le t \le a$ oriented oppositely. Thus

$$\int\limits_{\gamma_4} u(x,y)d_{p,q}x + v(x,y)d_{p,q}y = -\int\limits_{\widetilde{\gamma_4}} u(x,y)d_{p,q}x + v(x,y)d_{p,q}y = -\int\limits_0^a v(0,t)d_{p,q}t.$$

Hence we have

$$\int\limits_{\partial D} u(x,y)d_{p,q}x + v(x,y)d_{p,q}y = \int\limits_0^a \big[u(t,0) + v(a,t) - u(t,a) - v(0,t)\big]d_{p,q}t. \tag{5.6}$$

On the other hand, by (5.1), we can write

$$\iint\limits_D D_{(p,q),x}v(x,y)d_{p,q}x\, d_{p,q}y = \int\limits_0^a \int\limits_0^a D_{(p,q),x}v(x,y)d_{p,q}x\, d_{p,q}y = \int\limits_0^a \big[v(a,t) - v(0,t)\big]d_{p,q}t, \tag{5.7}$$

and

$$\iint\limits_D D_{(p,q),y}u(x,y)d_{p,q}x\, d_{p,q}y = \int\limits_0^a \int\limits_0^a D_{(p,q),y}u(x,y)d_{p,q}x\, d_{p,q}y = \int\limits_0^a \big[u(t,a) - u(t,0)\big]d_{p,q}t. \tag{5.8}$$

Using (5.7) and (5.8), we have

$$\iint\limits_D \big[D_{(p,q),x}v(x,y) - D_{(p,q),y}u(x,y)\big]d_{p,q}x\, d_{p,q}y = \int\limits_0^a \big[v(a,t) - v(0,t) - u(t,a) + u(t,0)\big]d_{p,q}t. \tag{5.9}$$

If we compare (5.6) and (5.9), it is seen that the (p,q)-Green's formula (5.5) holds. □

Remark 5.9. If the complex-valued function $f(z) = f(x,y)$ is q-periodic in x, i.e., $f(x,y) = f(qx,y)$, and p-periodic in y, i.e., $f(x,y) = f(x,py)$, then using (p,q)-Green's formula (5.5), we get the following (p,q)-Gauss formulas

$$\int\limits_{\partial D} f(z)\, d_{p,q}z = 2i \iint\limits_D D_{(p,q),\overline{z}}f(z)d_{p,q}x\, d_{p,q}y, \tag{5.10}$$

$$\int\limits_{\partial D} f(z)d_{p,q}\overline{z} = -2i \iint\limits_D D_{(p,q),z}f(z)d_{p,q}x\, d_{p,q}y. \tag{5.11}$$

Remark 5.10. It is not difficult to seen that the function

$$f(z) = f(x, y) = e^{2\pi i\left(m\,\frac{\ln|x|}{\ln q} + k\,\frac{\ln|y|}{\ln p}\right)}, \quad m, k \in \mathbb{Z}, \tag{5.12}$$

is q-periodic in x and p-periodic in y. It can be easily verified that (5.12) satisfies (5.10) and (5.11).

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