Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 1, 27–34

THE TOTAL OUTER CONNECTED MONOPHONIC NUMBER OF A GRAPH

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Abstract. For a connected graph $G = (V, E)$ of order at least two, a *total outer connected monophonic set* S of G is an outer connected monophonic set such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total outer connected monophonic set of G is the *total outer connected monophonic number* of G and is denoted by $cm_{to}(G)$. In this paper, we present several properties of this parameter. Also, some realization results of the total outer connected monophonic number of a graph are studied. This concept can be mainly used in fault tolerance of communication network.

1. Introduction

By a graph $G = (V, E)$ we mean a finite simple undirected connected graph, where V is the vertex set and E is the edge set of G. The order and size of G are denoted by p and q, respectively. For basic graph theoretical terminology we refer to Harary [11, 22]. The degree of a vertex v in a graph G denoted by $\deg(v)$, is the number of edges incident with v. The *distance* $d(x, y)$ between two vertices x and y in a connected graph G is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called $x - y$ *geodesic* [1]. A vertex v in a graph G is said to be an *end-vertex* if its degree is 1. A vertex v of G is called a *support vertex* of G if it is adjacent to an end-vertex of G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent to v. A vertex v of G is called an *extreme vertex* if the subgraph induced by its neighbors is complete. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $V_1 \cap V_2 = \phi$, then the join $G_1 + G_2$ is a graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ together with all the edges joining vertices of V_1 to vertices of V_2 and m_iK_j denotes m_i-copies of the complete graph K_j, where $m_i \geq 2$ and $j \geq 2$.

A *chord* of a cycle C is an edge not in C whose endpoints lie on C. A *chordless cycle* is a cycle of length at least four in G that has no chord. A graph G is *chordal* if it is simple and has no chordless cycle. A relationship between pairwise compatibility graphs and chordal graphs are studied in [21]. A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an $x - y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$. The monophonic number of a graph, and algorithmic aspects of monophonic concepts have been studied in [2–5, 12–16, 19]. A *total monophonic set* of a graph G is a monophonic set S such that the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total monophonic set of G is the *total monophonic number* of G and is denoted by $m_t(G)$. Several results on a total monophonic number can be found in [9, 10, 20]. A set S of vertices in a graph G is said to be an *outer connected monophonic set* if S is a monophonic set of G and either $S = V$, or the subgraph induced by $V - S$ is connected. The minimum cardinality of an outer connected monophonic set of G is the *outer connected monophonic number* of G and is denoted by $m_{oc}(G)$. The outer connected monophonic number of a graph was introduced in [6] and further studied in [7, 8].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u - v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The monophonic radius $\mathrm{rad}_m(G)$ and monophonic diameter $\mathrm{diam}_m(G)$ of G are defined, respectively, as the minimum and the maximum

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2020 *Mathematics Subject Classification.* 05C12.

Key words and phrases. Monophonic set; Outer connected monophonic set; Total outer connected monophonic set, Total outer connected monophonic number.

monophonic eccentricity among all the vertices in G. The monophonic distance was introduced in [17] and further studied in [18]. There are several interesting applications of these concepts to facility location in real life situations, routing of transport problems and communication network designs. As the paths involved in the discussion of this paper are monophonic, no intervention by hackers or enemies is possible to the respective facilities provided. Further, as monophonic paths are secured and longer than geodesic paths, it is advantageous to more customers in getting the service with protection.

The following theorems will be used in the sequel.

Theorem 1.1 ([6]). *Each extreme vertex of a connected graph G belongs to every outer connected monophonic set of G.*

Theorem 1.2 ([6]). *For the complete graph K_p $(p \geq 2)$, $m_{oc}(K_p) = p$.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Main Results

Definition 2.1. A total outer connected monophonic set S of G is an outer connected monophonic set such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total outer connected monophonic set of G is the total outer connected monophonic number of G and is denoted by $cm_{to}(G)$.

Example 2.2. For the graph G given in Figure 1, it is clear that $S = \{v_1, v_4, v_5\}$ is the unique minimum outer connected monophonic set of G and so, $m_{oc}(G) = 3$. Since the subgraph induced by S has an isolated vertex v_1, S is not a total outer connected monophonic set of G. It is easily verified that $S_1 = S \cup \{v_2\}$ is a minimum total outer connected monophonic set of G and so, $cm_{to}(G) = 4$. Thus the outer connected monophonic number and the total outer connected monophonic number of a graph are different for such an example.

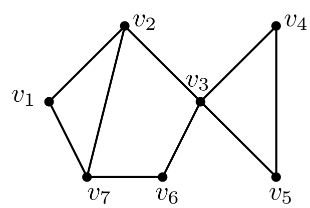

Figure 1

In an application point of view, a vertex represents a router of the communication network and an edge represents the information that travels along the routers. Communication among the vertices are restricted to the chordless (monophonic) path only. Vertices which are lying along a monophonic path are grouped together. Leading vertices (LVs) manage a group of vertices which lie on a monophonic path between the two LVs in which they are not isolated. Even though this set of LVs fails, in order to make the network fault tolerant, the rest of the vertices are able to communicate with each other. The problem is to identify the minimum number of LVs in such a way that each vertex of the communication network lies on some monophonic path between two LVs. Then for the model of the above communication network, we are interested in finding a minimum total outer connected monophonic set of the graph representing a communication network.

The following results are clear from the fact that every total outer connected monophonic set S of G is an outer connected monophonic set of G and the subgraph induced by S has no isolated vertices. Also, each extreme vertex of a connected graph G belongs to every outer connected monophonic set of G.

Theorem 2.3. *Each extreme vertex and each support vertex of a connected graph G belong to every total outer connected monophonic set of G. If the set S of all extreme vertices and support vertices* *form a total outer connected monophonic set, then it is the unique minimum total outer connected monophonic set of G.*

Corollary 2.4. *For the complete graph K_p $(p \geq 2)$, $cm_{to}(G) = p$.*

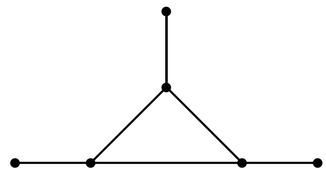

FIGURE 2

Remark 2.5. The converse of Corollary 2.4 need not be true. For the graph G given in Figure 2, every vertex of G is either an extreme vertex or a support vertex, by Theorem 2.3, $S = V(G)$ is the unique minimum total outer connected monophonic set of G and so, $cm_{to}(G) = p$, it is not a complete graph.

Theorem 2.6. *For a connected graph G of order p, $2 \leq m_{oc}(G) \leq cm_{to}(G) \leq p$.*

Proof. Any outer connected monophonic set of G needs at least two vertices and so $m_{oc}(G) \geq 2$. Since every total outer connected monophonic set of G is also an outer connected monophonic set of G, it follows that $m_{oc}(G) \leq cm_{to}(G)$. Since $V(G)$ is a total outer connected monophonic set of G, it is clear that $cm_{to}(G) \leq p$. Hence $2 \leq m_{oc}(G) \leq cm_{to}(G) \leq p$. □

Corollary 2.7. *Let G be a connected graph. If $cm_{to}(G) = 2$, then $m_{oc}(G) = 2$.*

For any path P_n of order $n \geq 4$, the outer connected monophonic number of P_n is 2 and the total outer connected monophonic number of P_n is 4. Thus, this shows that the converse of Corollary 2.7 is not necessarily true.

Remark 2.8. The bounds in Theorem 2.6 are sharp. For any path P_n of order $n \geq 3$, $m_{oc}(P_n) = 2$ and for the complete graph K_p $(p \geq 2)$, $cm_{to}(K_p) = p$. Also, all the inequalities in Theorem 2.6 can be strict. For the graph G given in Figure 1, $m_{oc}(G) = 3$, $cm_{to}(G) = 4$ and $p = 7$ so, $2 < m_{oc}(G) < cm_{to}(G) < p$.

Theorem 2.9. *For any non-trivial tree T, the set of all end-vertices and support vertices of T is the unique minimum total outer connected monophonic set of G.*

Proof. Since the set of all end-vertices and support vertices of T forms a total outer connected monophonic set, the result follows from Theorem 2.3. □

Now, we proceed to characterize the graph G for which the bounds in Theorem 2.6 are attained.

Theorem 2.10. *For any connected graph G, $cm_{to}(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $cm_{to}(G) = 2$. Conversely, let $cm_{to}(G) = 2$. Let $S = \{u, v\}$ be a minimum total outer connected monophonic set of G. Then uv is an edge. It is clear that a vertex, different from u and v, cannot lie on a $u - v$ monophonic path and so, $G = K_2$. □

Theorem 2.11. *Let G be a connected graph of order $p \geq 2$. Then every vertex of G is either an extreme vertex, or a support vertex if and only if $cm_{to}(G) = p$.*

Proof. Let G, a connected graph with every vertex of G, be either an extreme vertex, or a support vertex. Then the result follows from Theorem 2.3.

Conversely, let $cm_{to}(G) = p$. Suppose that there is a vertex x in G which is neither an extreme vertex nor a support vertex. Since x is not an extreme vertex, the subgraph induced by $N(x)$ is not complete. Then there exist vertices $u, v \in N(x)$ such that $d(u, v) = 2$. Clearly, x lies on a $u - v$ monophonic path in G. Also, since x is not a support vertex, $\deg(u) \geq 2$ and $\deg(v) \geq 2$.

It is clear that the subgraph induced by $V - \{x\}$ has no isolated vertices and $V - \{x\}$ is an outer connected monophonic set. Hence $V - \{x\}$ is a total outer connected monophonic set of G and so, $cm_{to}(G) \leq |V - \{x\}| = p - 1$, which is a contradiction. □

Theorem 2.12. *For the complete bipartite graph $G = K_{m,n}$ $(2 \leq m \leq n)$,*

$$cm_{to}(G) = \begin{cases} n+1, & \text{if } 2 = m \leq n, \\ 4, & \text{if } 3 \leq m \leq n. \end{cases}$$

Proof. Let $U = \{u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the bipartition of G, where $m \leq n$. We prove this theorem by considering two cases.

Case (i) $m = 2$. It is easy to observe that any subset $S \subseteq V(G)$ with cardinality $|S| \leq n$ is not an outer connected monophonic set of G. Clearly, $S = V(G) - \{v_1\}$ is a minimum outer connected monophonic set of G and the subgraph induced by S has no isolated vertices so, $cm_{to}(G) = n + 1$.

Case (ii) $3 \leq m \leq n$. Let $S = \{u_1, u_2, v_1, v_2\}$. Clearly, S is a total outer connected monophonic set of G, it follows that $cm_{to}(G) \leq 4$. It suffices to show that no 3-element subset of $V(G)$ forms a minimum total outer connected monophonic set of G. Let X be a 3-element subset of $V(G)$. If $|U| = 3$ and $X \subseteq U$, then it is clear that X is a monophonic set of G and the subgraph induced by $V - X$ is not connected. Hence X is not an outer connected monophonic set of G. If $|U| \geq 4$ and $X \subseteq U$, then there exists an element $u \in U$ and $u \notin X$, is not an internal vertex of any $x - y$ monophonic path in G, for some $x, y \in X$. Hence X is not an outer connected monophonic set of G. Therefore we may take that $X \cap U = \{u_i, u_j\}$ and $X \cap V = \{v_k\}$. It is clear that X is not an outer connected monophonic set of G. If $X \subseteq V$, then the argument is similar to the above. □

Theorem 2.13. *Let G be a connected graph of order p.*
(i) *If $G = K_1 + \bigcup m_i K_j$, where $j \geq 2$, $\sum m_i \geq 2$, then $cm_{to}(G) = p - 1$;*
(ii) *If $G = K_2 + \bigcup m_i K_j$, where $j \geq 2$, $\sum m_i \geq 2$, then $cm_{to}(G) = p - 2$.*

Proof. These results follow from Theorem 2.3. □

Remark 2.14. The converse of Theorem 2.13 (i) and (ii) need not be true. For the cycle C_4, $cm_{to}(C_4) = 3 = p-1$, it is not in the form $G = K_1 + \bigcup m_i K_j$ and for the cycle C_5, $cm_{to}(C_5) = 3 = p-2$, it is not in the form $G = K_2 + \bigcup m_i K_j$.

3. Some Realization Results on the Total Outer Connected Monophonic Number

For any connected graph G, $\text{rad}_m(G) \leq \text{diam}_m(G)$. It is shown in [17] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can also be extended so that the total outer connected monophonic number can be prescribed when $\text{rad}_m(G) < \text{diam}_m(G)$.

Theorem 3.1. *For positive integers r, d and $k \geq 5$ with $r < d$, there exists a connected graph G such that $\text{rad}_m(G) = r$, $\text{diam}_m(G) = d$ and $cm_{to}(G) = k$.*

Proof. Let $r = 1$ and $d \geq 2$. Let $P_{d+1} : v_1, v_2, \ldots, v_{d+1}$ be a path of length d. The graph G is obtained from the path P_{d+1} and the star $K_{1,k-3}$ having the vertex set $\{x, u_1, u_2, \ldots, u_{k-3}\}$ with x as the cut-vertex, by connecting the vertex x to the vertices v_i $(1 \leq i \leq d + 1)$ of P_{d+1}. The graph G is shown in Figure 3. It is clear that $e_m(x) = 1$, $e_m(v_1) = e_m(v_{d+1}) = d$ and $1 < e_m(v) \leq d$ for all other vertices v of G. Then $\text{rad}_m(G) = r$ and $\text{diam}_m(G) = d$. Let $S = \{u_1, u_2, \ldots, u_{k-3}, v_1, v_{d+1}, x\}$ be the set of all extreme vertices and support vertex of G. Since S is a total outer connected monophonic set of G, it follows from Theorem 2.3 that $cm_{to}(G) = k$.

Now, let $r \geq 2$ and $r < d$. Let $C : v_1, v_2, \ldots, v_{r+2}, v_1$ be the cycle of order $r + 2$ and let $W = K_1 + C_{d+1}$ be the wheel with $V(C_{d+1}) = \{w_1, w_2, \ldots, w_{d+1}\}$ and z as the central vertex. Let H be the graph obtained from C and W by identifying v_1 of C and the vertex w_1 of W; and also, joining each vertex $x \in \{v_3, v_4, \ldots, v_{r+1}\}$ to the vertex v_1 of C. Add $k - 5$ new vertices $u_1, u_2, \ldots, u_{k-5}$ to the graph H and join each u_i $(1 \leq i \leq k - 5)$ to the vertex v_1 of H and also join the vertex v_3 of H to the vertex z of W, thereby producing the graph G as shown in Figure 4. It is easily verified

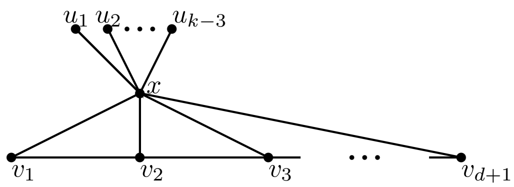

FIGURE 3

that $r \leq e_m(u) \leq d$ for any vertex u in G, $e_m(z) = r$ and $e_m(v_2) = d$. Then $\text{rad}_m(G) = r$ and $\text{diam}_m(G) = d$.

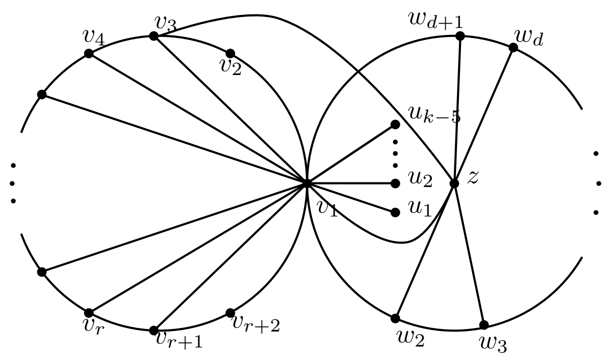

FIGURE 4

Note that $S = \{u_1, u_2, \ldots, u_{k-5}, v_2, v_{r+2}, v_1\}$ is the set of all extreme vertices and support vertex of G. By Theorem 2.3, every total outer connected monophonic set of G contains S. It is clear that S is not a total outer connected monophonic set of G. Also, for any $x \in V - S$, $S \cup \{x\}$ is not a total outer connected monophonic set of G. It is easy to verify that $S_1 = S \cup \{w_2, w_3\}$ is a minimum total outer connected monophonic set of G and so, $cm_{to}(G) = k$. □

We leave the following problem as an open question.

Problem 3.2. For any three positive integers r, d and $k \geq 5$ with $r = d$, does there exist a connected graph G with $\text{rad}_m(G) = r$, $\text{diam}_m(G) = d$ and $cm_{to}(G) = k$?

Theorem 3.3. *If p, d and k are positive integers such that $2 \leq d \leq p-2$, $k \geq 3$ and $p-d-k+2 \geq 0$, then there exists a connected graph G of order p with monophonic diameter d and $cm_{to}(G) = k$.*

Proof. We prove this theorem by considering two cases.

Case 1. Let $d = 2$. Add $k-1$ new vertices $w_1, w_2, \ldots, w_{k-1}$ to the complete graph K_{p-k+1} and join each vertex w_i $(1 \leq i \leq k-1)$ to all the vertices of K_{p-k+1}, thereby producing the graph G of order p as shown in Figure 5. It is easily verified that $1 \leq e_m(u) \leq 2$ for any vertex u in G, $e_m(w_i) = 2$ $(1 \leq i \leq k-1)$ and hence the monophonic diameter of G is 2.

Let $S = \{w_1, w_2, \ldots, w_{k-1}\}$ be the set of all extreme vertices of G. By Theorem 2.3, every total outer connected monophonic set of G contains S. It is clear that S is not a total outer connected monophonic set of G. It is easy to observe that for any $x \in V(K_{p-k+1})$, $S \cup \{x\}$ is a minimum total outer connected monophonic set of G and so, $cm_{to}(G) = k$.

Case 2. $d \geq 3$. First, let $k = 3$. Let $C_{d+2} : v_1, v_2, \ldots, v_{d+2}, v_1$ be the cycle of order $d+2$. Add $p-d-2$ new vertices $w_1, w_2, \ldots, w_{p-d-2}$ to C_{d+2} and join each vertex w_i $(1 \leq i \leq p-d-2)$ to both v_1 and v_3, thereby producing the graph G. Then G has order p and monophonic diameter d. It is clear that $S = \{v_3, v_4, v_5\}$ is a minimum total outer connected monophonic set of G and so

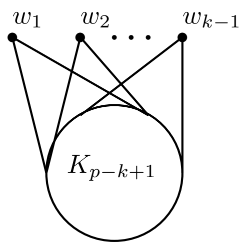

FIGURE 5

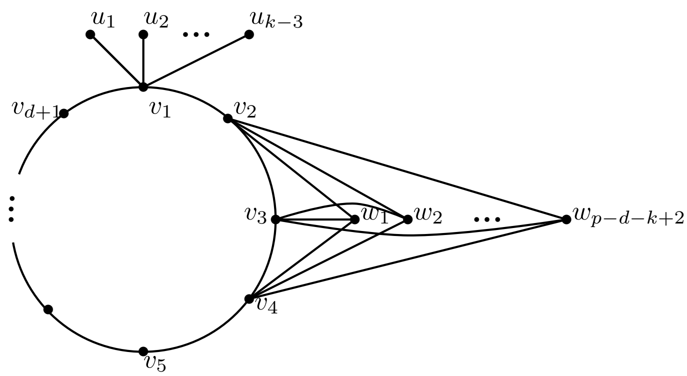

FIGURE 6

$cm_{to}(G) = 3 = k$. Now, let $k \geq 4$. The graph G_1 is obtained from the cycle $C_{d+1} : v_1, v_2, \ldots, v_{d+1}, v_1$ of order $d+1$ by adding $p-d-1$ new vertices $u_1, u_2, \ldots, u_{k-3}, w_1, w_2, \ldots, w_{p-d-k+2}$ and joining each vertex $u_i(1 \leq i \leq k-3)$ to the vertex v_1 of C_{d+1}; and joining each vertex $w_j(1 \leq j \leq p-d-k+2)$ to the vertices v_2, v_3 and v_4 of C_{d+1}. The graph G_1 of order p is shown in Figure 6. It is easily verified that $3 \leq e_m(u) \leq d$ for any vertex u in G_1, $e_m(u_i) = d(1 \leq i \leq k-3)$ and hence the monophonic diameter of G_1 is d. Let $S = \{u_1, u_2, \ldots, u_{k-3}, v_1\}$ be the set of all extreme vertices and support vertex of G_1. By Theorem 2.3, every total outer connected monophonic set of G_1 contains S. It is clear that S is not a total outer connected monophonic set of G_1. Also, for any vertex $x \notin S$, $S \cup \{x\}$ is not a total outer connected monophonic set of G_1. It is easily seen that $S \cup \{v_2, v_{d+1}\}$ is a minimum total outer connected monophonic set of G_1 and so $cm_{to}(G_1) = k$. □

Theorem 3.4. *For positive integers a, b such that $3 \leq a \leq b$ with $b \leq 2a$, there exists a connected graph G such that $m_{oc}(G) = a$ and $cm_{to}(G) = b$.*

Proof. We prove this theorem by considering two cases.

Case 1. $3 \leq a = b$. By Theorem 1.2 and Corollary 2.4, the complete graph K_a has the desired properties.

Case 2. $3 \leq a < b$. Let $b = a + k$, where $1 \leq k \leq a$. For $k = 1$, the star $K_{1,a}$ has the desired properties.

Now, let $k \geq 2$. The graph G is obtained from the star $K_{1,a-k+1}$ having the vertex set $\{x, x_1, x_2, \ldots, x_{a-k+1}\}$ with x as the cut-vertex and "$k-1$" copies of path $P_i : u_i, v_i$ $(1 \leq i \leq k-1)$ of order 2 by joining each vertex u_i $(1 \leq i \leq k-1)$ of P_i to the vertex x in $K_{1,a-k+1}$. The graph G is shown in Figure 7. Let $S = \{x_1, x_2, \ldots, x_{a-k+1}, v_1, v_2, \ldots, v_{k-1}\}$ be the set of all extreme vertices of G. By Theorem 1.1, every outer connected monophonic set of G contains S. It is clear that S is the unique minimum outer connected monophonic set of G and so, $m_{oc}(G) = a$. Since the subgraph induced by S contains only isolated vertices, S is not a total outer connected monophonic set of G. Since every vertex of G is either an extreme vertex, or a support vertex, by Theorem 2.11, $V(G)$ is the unique minimum total outer connected monophonic set of G and so, $cm_{to}(G) = a + k = b$. □

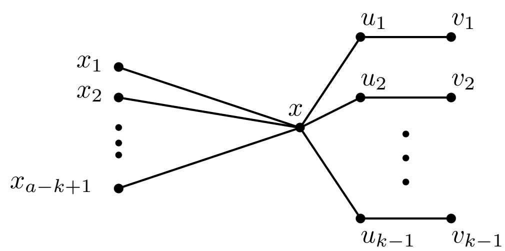

FIGURE 7

ACKNOWLEDGEMENT

Research is supported by NBHM Project No. NBHM/R.P.29/2015/Fresh/157.

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(Received 05.06.2021)

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