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ON SOME EMBEDDING THEOREMS IN GRAND NIKOLSKII–MORREY SPACES WITH DOMINANT MIXED DERIVATIVES

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Abstract. In this paper, we introduce grand Nikolskii–Morrey spaces with dominant mixed derivatives and, using the method of integral representation, we study some differential properties of functions from these spaces.

1. Introduction and Preliminary Notes

In connection with the study of different types of differential equations, for example, the equations of the type

$$u^{(3)}_{x^1y^2} + u^{(2)}_{xy} + u^{(1)}_x + u^{(1)}_y + u = f,$$

in which the mixed derivatives dominate, it is possible to study function spaces with the dominant mixed derivatives. Such spaces were first introduced and studied by S. M. Nikolskii [16] and later well discussed in the works of A. D. Dzabrailov [4], T. I. Amanov [2], A. M. Najafov [12] etc.

The grand Lebesgue spaces $L_{p)}(G)$ $(|G| < \infty)$ introduced in the paper by T. Ivaniec and C. Sbordone [7], and their generalizations such as grand Lebesgue–Morrey, grand grand Lebesgue–Morrey, grand grand Sobolev–Morrey, grand grand Sobolev–Morrey with dominant mixed derivatives, grand Nikolskii–Morrey spaces have been studied by many mathematicians (see, e.g., [1, 5–15, 17, 18]).

In this paper, we construct a grand Nikolskii–Morrey space with dominant mixed derivatives $S^l_{p),\varphi,\beta}H(G_\varphi)$ and, using the integral representation method, we study some properties of the functions from the constructed spaces from the point of view of embedding theory.

Definition 1.1. Denote by $S^l_{p),\varphi,\beta}H(G_\varphi)$ the Banach space of locally summable functions on G with the finite norm

$$\|f\|_{S^l_{p),\varphi,\beta}H(G_\varphi)} = \sum_{e\subseteq e_n} \sup_{\substack{0<h_j<h_{o,j}\\ j\in e_n}} \frac{\left\|\Delta^{m^e}(\varphi(h)G_{\varphi(h)})D^{k^e}f\right\|_{p),\varphi,\beta}}{\prod\limits_{j\in e}(\varphi_j(h_j))^{l_j-k_j}}, \tag{1.1}$$

where

$$\|f\|_{p),\varphi,\beta;G} = \|f\|_{L_{p),\varphi,\beta;G}} = \sup_{\substack{x\in G,\\ 0<t_j\le d_j,\\ j\in e_n\\ 0<\varepsilon<p-1}} \left(\frac{1}{|\varphi(t)|^\beta}\frac{\varepsilon}{|G_{\varphi(t)}(x)|}\int\limits_{G_{\varphi(t)}(x)} |f(y)|^{p-\varepsilon}\,dy\right)^{\frac{1}{p-\varepsilon}}, \tag{1.2}$$

and $l\in(0,\infty)^n$, $e_n = (1,2,\dots,n)$; $m \in N^n_0$, $m^e = (m^e_1, m^e_2, \dots, m^e_n)$, $m^e_j = m_j (j\in e)$, $m^e_j = 0 (j \in e' = e_n\backslash e)$; $p\in(1,\infty)$ $|\varphi(t)|^\beta = \prod\limits_{j\in e_n}(\varphi_j(t_j)^{\beta_j}$ $\beta_\gamma\in[0,1]$, $j\in e_n$ and $\varphi(t) = (\varphi_1(t_1), \varphi_2(t_2), \dots, \varphi_n(t_n))$, $\varphi_j(t_j)$, $j\in e_n$ are continuous functions on $[0, T_{0j}]$, $\varphi_j(t_j) > (t_j > 0)$, $\lim\limits_{t_j\to+0}\varphi_j(t_j) = 0$, $j\in e_n$; $T_0 = (T_{01},\dots,T_{0,n})$ and $h_0 = (h_{01},\dots,h_{0n})$ are positive fixed vectors, and suppose that $0 < d_j \le T_{0j}$, $d_j (j\in e_n)$ are diagonals of an n-dimensional parallelepiped

$$I_{\varphi(t)}(x) = \left\{y : |y_j - x_j| < \frac{1}{2}\varphi_j(t_j), j\in e_n\right\},\ \ G_{\varphi(t)}(x) = G\cap I_{\varphi(t)}(x).$$

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We denote the set of such vector functions φ by N.

Note that the Nikolskii–Morrey spaces $H^l_{p,\varphi,\beta}(G_\varphi)$ and grand Nikolskii–Morrey space $H^l_{p),\varkappa,a}(G_\lambda)$ are studied in [1] and [14], respectively. The spaces $S^l_{p),\varkappa,a}H(G_\varphi)$ in the case $\varphi_j(t_j) = t_j^{\varkappa}$, $\beta_j = \frac{a_2}{p}$ coincide with the spaces $S^l_{p),\varphi,\beta}H(G_\varphi)$, and in the case $\beta_j = 0$, $j \in e_n$ coincides with the grand Nikolskii spaces with the dominant mixed derivatives $S^l_{p)}H(G_\varphi)$.

In the case when for any $t_j > 0$ ($j \in e_n$), there exists a constant $C > 0$ such that $|\varphi(t)| \le C$, we have the embeddings:

$$L_{p),\varphi,\beta}(G_\varphi) \hookrightarrow L_{p)}(G_\varphi) \quad \text{and} \quad S^l_{p),\varphi,\beta}H(G_\varphi) \hookrightarrow \; S^l_{p)}H(G_\varphi) \text{ i.e.,}$$
$$\|f\|_{p),G} \le C\,\|f\|_{p),\varphi,\beta;G} \quad \text{and} \quad \|f\|_{S^l_{p)}H(G)} \le C\,\|f\|_{S^l_{p),\varphi,\beta}H(G_\varphi)}. \tag{1.3}$$

Definition 1.2. The open set $G \subset R^n$ is said to be an open set with the condition of a flexible φ-horn, if for some $\theta \in [0,1]^n, T \in (0,\infty)$, for any $x \in G$, there exists the vector-function

$$\rho(\varphi(t),x) = (\rho_1(\varphi_1(t_1),x), \rho_2(\varphi_2(t_2),x), \dots, \rho_n(\varphi_n(t_n),x)), \;\; 0 < t_j \le T_j, \;\; j \in e_n,$$

with the following properties:

1) for all $j \in e_n$, $\rho_j(\varphi_j(t_j),x)$ are absolutely continuous on $[0,T_j]$; $\left|\rho'_j(\varphi_j(t_j),x)\right| \le 1$ for almost all $t_j \in [0,T_j]$ $(j \in e_n)$.

2) $\rho_j(0,x) = 0$, $j \in e_n$, and

$$x + V(x,\theta) = x + \bigcup_{\substack{0 \le t_j \le T_j \\ j \in e_n}} [\rho(\varphi(t),x) + \varphi(t)\theta I] \subset G.$$

Assuming that $\varphi_j(t)$, $j \in e_n$ are also differentiable on $[0,T_j]$ $(j \in e_n)$, then (see [14]) for all $f \in H_{p-\varepsilon}(G)$ defined in n-dimensional domains, satisfying the condition of flexible φ-horn, the following integral representation $(x \in U \subset G)$ holds:

$$D^\nu f(x) = \sum_{e \subseteq e_n} (-1)^{|\nu|} \prod_{j \in e'} (\varphi_j(T_j))^{-2-\nu_j} \int_{0^e}^{T^e} \int_{r^n} \int_{-\infty^e}^{+\infty^e} K_e^{(\nu)}\Big(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho(\varphi(t^e + T^{e'})x)}{\varphi(t^e + T^{e'})}\Big)$$
$$\times \zeta_e\Big(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t)x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t)x)\Big)$$
$$\times \Delta^{m^e}(\varphi(\delta u)f(x + y + u^e) \prod_{j \in e} (\varphi_j(T_j))^{-\nu_j - 2} \prod_{j \in e} \frac{\varphi'_j(t_j)}{\varphi_j(t_j)} dt^e du^e dy, \tag{1.4}$$

where $K_e(\cdot,y) \in C_0^\infty(R^n)$, $(t^e + T^{e'}) = t_j (j \in e)$, $(t^e + T^{e'}) = T_j (j \in e')$,

$$\int_{a^e}^{b^e} f(x) dx^e = \Big(\prod_{j \in e} \int_{a_j}^{b_j} dx_j\Big) f(x), \quad \Delta^{m^e}(\varphi(t))\, f(x) = \prod_{j \in e} \Delta_j^{m^e}(\varphi_j(t_j))\, f(x).$$

Let $M(\cdot,y) \in C_0^\infty(R^n)$ be such that

$$S(M) = \sup pM \subset I_{\varphi(T_0)} = \Big\{ y : |y_j| < \frac{1}{2}\varphi_j(T_{0j}), j \in e_n \Big\}.$$

Assume that for any $0 < T_j \le T_{0j}, j \in e_n$,

$$V = \bigcup_{\substack{0 < t_j \le T_j, \\ j \in e_n}} \Big\{ y : \Big(\frac{y}{t^e + T^e}\Big) \in S(M) \Big\},$$

$U \subset G, V \subset I_{\varphi(T)}$ and suppose that $U + V \subset G$.

Lemma 1.1. *Let $1 < p < q \le r \le \infty$, $0 < \eta_j$, $t_j \le T_j \le T_{0j}$, $\nu = (\nu_1, \nu_2, \dots, \nu_n)$, $\nu_j \ge 0$ $(j \in e_n)$ be entire $\Delta^{m^e} f \in L_{p),\varphi,\beta}(G)$, and let*

$$\gamma_j = l_j - \nu_j + \beta_j - \Big(\frac{1}{p - \varepsilon} - \frac{1}{q - \varepsilon}\Big),$$

$$A_{\eta}^{e}(x)=\prod_{j\in e'}(\varphi_j(T_j))^{-2-\nu_j}\int_{0^e}^{\eta^e}L_e(x,t^e+T^{e'}),\prod_{j\in e}(\varphi_j(t_j))^{-\nu_j-2}\prod_{j\in e}\frac{\varphi_j'(t_j)}{\varphi_j(t_j)}dt^e, \tag{1.5}$$

$$A_{\eta T}^{e}(x)=\prod_{j\in e'}(\varphi_j(T_j))^{-2-\nu_j}\int_{\eta^e}^{T^e}L_e(x,t^e+T^{e'}),\prod_{j\in e}(\varphi_j(t_j))^{-\nu_j-2}\prod_{j\in e}\frac{\varphi_j'(t_j)}{\varphi_j(t_j)}dt^e, \tag{1.6}$$

where

$$\begin{aligned}&L_e(x,t^e+T^{e'})\\&=\int_{R^n}\int_{-\infty^e}^{+\infty^e}M\Big(\frac{y}{\varphi(t^e+T^{e'})}\frac{\rho(\varphi(t^e+T^{e'}),x)}{\varphi(t^e+T^{e'})}\Big)\zeta_e\left(\frac{u}{\varphi(t)},\frac{\rho(\varphi(t),x)}{\varphi(t)},\frac{1}{2}\rho'(\varphi(t),x)\right)\\&\times\Delta^{m^e}(\varphi(\delta)u)f(x+yu^e)du^e dy.\end{aligned} \tag{1.7}$$

Then for any $\overline{x}\in U$, $0<\varepsilon<p-1$, the following inequalities:

$$\begin{aligned}&\sup_{\overline{x}\in U}\left\|A_{\eta}^{e}(x)\right\|_{q-\varepsilon,U_{\psi(\xi)}(\overline{x})}\\&\le C_1\left\|\prod_{j\in e}(\varphi_j(t_j))^{-l_j}\Delta^{m^e}(\varphi(t))f\right\|_{p),\varphi,\beta;(G)}\prod_{j\in e'}(\varphi_j(t_j))^{-\nu_j+\beta_j-\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)}\\&\times\varepsilon^{-\frac{1}{p-\varepsilon}}\prod_{j\in e_n}(\varphi_j(\xi_j))^{-\beta_j\frac{p-\varepsilon}{q-\varepsilon}}\prod_{j\in e}(\varphi_j(\eta_j))^{\gamma_j}\ \ (\gamma_j>0),\end{aligned} \tag{1.8}$$

$$\begin{aligned}&\sup_{\overline{x}\in U}\left\|A_{\eta T}^{e}(x)\right\|_{q-\varepsilon,U_{\psi(\xi)}(\overline{x})}\\&\le C_1\left\|\prod_{j\in e}(\varphi_j(t_j))^{-l_j}\Delta^{m^e}(\varphi(t))f\right\|_{p),\varphi,\beta;(G)}\prod_{j\in e'}(\varphi_j(T_j))^{-\nu_j+\beta_j-\left(\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}\right)}\\&\times\varepsilon^{-\frac{1}{p-\varepsilon}}\prod_{j\in e_n}(\varphi_j(\xi_j))^{-\beta_j\frac{p-\varepsilon}{q-\varepsilon}}\begin{cases}\prod_{j\in e}(\varphi_j(T_j))^{\gamma_j}, & \gamma_j>0,\\ \prod_{j\in e}\ln\frac{\varphi_j(T_j)}{\varphi_j(\eta_j)}, & \gamma_j=0,\\ \prod_{j\in e}(\varphi_j(\eta_j))^{\gamma_j}, & \gamma_j<0\end{cases}\end{aligned} \tag{1.9}$$

are valid.

Here, $\bigcup_{\psi(\xi)}(\overline{x})=\left\{x:|x_j-\overline{x}_j|<\frac{1}{2}\psi_j(\xi_j),j\in e_n\right\}$, $\psi\in N$ and C_1,C_2 are constants independent of f, ξ, η an T.

Proof. Applying sequentially the generalized Minkowskii inequality for any $\overline{x}\in U$,

$$\begin{aligned}&\left\|A_{\eta}^{e}\right\|_{q-\varepsilon,U_{\psi(\xi)}(\overline{x})}\\&\le\prod_{j\in e'}(\varphi_j(T_j))^{-2-\nu_j}\int_{0^e}^{\eta^e}\left\|L_e(x,t^e+T^{e'})\right\|_{q-\varepsilon,U_{\psi(\xi)}(\overline{x})}\prod_{j\in e}(\varphi_j(t_j))^{-\nu_j-2}\prod_{j\in e}\frac{\varphi_j'(t_j)}{\varphi_j(t_j)}dt^e.\end{aligned} \tag{1.10}$$

From the Hölder inequality ($q\le r$), we have

$$\left\|L_e(\cdot,t^e+T^{e'})\right\|_{q-\varepsilon,U_{\psi(\xi)}(\overline{x})}\le\left\|L_e(\cdot,t^e+T^{e'})\right\|_{r-\varepsilon,U_{\psi(\xi)}(\overline{x})}\prod_{j\in e_n}(\varphi_j(t_j))^{\frac{1}{p-\varepsilon}-\frac{1}{q-\varepsilon}}. \tag{1.11}$$

Further, we assume that there exists a function $M_1(x)$ such that $|M(x,y)|\le C|M_1(x)|$ for all $y\in R^n$. Let χ be a characteristic function of the set $S(M)$. Applying again the Hölder inequality for representing the function in the form (1.7) in the case $1 < p < r \leq \infty$, $s \leq r \left(\frac{1}{S} = 1 - \frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon} \right)$, we have

$$\left| M \int_{-\infty^e}^{+\infty^e} \zeta_e \Delta^{m^e} f du^e \right|$$

$$= \left(\left| \int_{-\infty^e}^{+\infty^e} \zeta_e \Delta^{m^e} f du^e \right| |M|^s \right)^{\frac{1}{r-\varepsilon}} \left(\left| \int_{-\infty^e}^{+\infty^e} \zeta_e \Delta^{m^e} f du^e \right| \chi \right)^{\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}} \left(|M|^s \right)^{\frac{1}{s} - \frac{1}{r-\varepsilon}}.$$

Applying the Hölder inequality and the inequality $\left(\frac{1}{r-\varepsilon} + \left(\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon} \right) + \left(\frac{1}{s} - \frac{1}{r-\varepsilon} \right) = 1 \right)$ to L_e, we get

$$\left\| L_e(\cdot, t^e + T^{e'}) \right\|_{r-\varepsilon, U_{\psi(\xi)}(x)}$$

$$\leq C^1 \sup_{\overline{x} \in U_{\psi(\xi)}(\overline{x})} \left(\int_{R^n} \left| \int_{-\infty^e}^{+\infty^e} \zeta_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2} \rho'(\varphi(t), x) \right) \Delta^{m^e} (\varphi(\delta u) f(x + y + u^e) du^e \right|^{p-\varepsilon} \right.$$

$$\left. \times \left(\frac{y}{\varphi(t^e + T^{e'})} \right) dy \right)^{\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}} \sup_{y \in V} \left(\int_{U_{\psi(\xi)}(\overline{x})} \left| \int_{-\infty^e}^{+\infty^e} \zeta_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2} \rho'(\varphi(t), x) \right) \right. \right.$$

$$\left. \left. \times \Delta^{m^e} (\varphi(\delta u) f(x + y + u^e) du^e \right|^{\frac{1}{p-\varepsilon} - \frac{1}{r-\varepsilon}} \right) \left(\int_{R^n} \left| M_1 \left(\frac{y}{\varphi(t^e + T^{e'})} \right) \right|^s dy \right)^{\frac{1}{s}}. \tag{1.12}$$

For $x \in U$, we have

$$\int_{R^n} \left| \int_{-\infty^e}^{+\infty^e} \zeta_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2} \rho'(\varphi(t), x) \right) \right.$$

$$\left. \times \Delta^{m^e} (\varphi(\delta u) f(x + y + u^e) du^e \right|^{p-\varepsilon} \chi \left(\frac{y}{\varphi(t^e + T^{e'})} \right) dy$$

$$\int_{(U+V)} \left| \int_{-\infty^e}^{+\infty^e} \zeta_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2} \rho'(\varphi(t), x) \Delta^{m^e} (\varphi(\delta u) \right) f(x + y + u^e) du^e \right|^{p-\varepsilon} dy$$

$$\leq \prod_{j \in e} (\varphi_j(t_j))^{(p-\varepsilon) l_j} \left\| \int_{-\infty^e}^{+\infty^e} \zeta_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2} \rho'(\varphi(t), x) \right) \right.$$

$$\left. \times \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \Delta^{m^e} (\varphi(\delta u) f(x + y + u^e) du^e \right\|^{p-\varepsilon}_{p-\varepsilon, G_{\varphi(t^e + T^{e'})}(x)}$$

$$\leq \prod_{j \in e'} (\varphi_j(T_j))^{p-\varepsilon} \prod_{j \in e} (\varphi_j(t_j))^{p-\varepsilon + (p-\varepsilon) l_j} \varepsilon^{-1} \prod_{j \in e'} (\varphi_j(t_j))^{\beta_j (p-\varepsilon)} \prod_{j \in e'} (\varphi_j(t_j))^{\beta_j (p-\varepsilon)}$$

$$\times \left\| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \Delta^{m^e} (\varphi(t) f \right\|^{p-\varepsilon}_{p), \varphi, \beta; G}$$

$$= \prod_{j \in e'} (\varphi_j(T_j))^{(1+\beta_j)(p-\varepsilon)} \prod_{j \in e} (\varphi_j(t_j))^{(1+\beta_j + l_j)(p-\varepsilon)} \varepsilon^{-1}$$

$$\times \left\| \prod_{j \in e} (\varphi_j(t_j))^{-l_j} \Delta^{m^e} (\varphi(t) f \right\|^{p-\varepsilon}_{p), \varphi, \beta; G}. \tag{1.13}$$

For $y \in V$, we have $((U+V)_\psi \subset G_\varphi)$

$$\int\limits_{U_{\psi(x)}(\overline{x})} \left| \int\limits_{-\infty^e}^{+\infty^e} \zeta_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t),x)\Delta^{m^e}(\varphi(\delta u)\right) f(x+y+u^e)du^e \right|^{p-\varepsilon} dx$$

$$\leq \int\limits_{(U+V)_{\psi(x)}(\overline{x}+y)} \left| \int\limits_{-\infty^e}^{+\infty^e} \zeta_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t),x)\right) \Delta^{m^e}(\varphi(\delta u)) f(x+u^e)du^e \right|^{p-\varepsilon} dx$$

$$\leq \prod_{j\in e} (\varphi_j(t_j))^{(p-\varepsilon)l_j} \left\| \int\limits_{-\infty^e}^{+\infty^e} \zeta_e \left(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t),x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t),x)\right) \right.$$

$$\left. \times \prod_{j\in e} (\varphi_j(t_j))^{-l_j} \Delta^{m^e}(\varphi(\delta u)) f(x+u^e)du^e \right\|^{p-\varepsilon}_{p-\varepsilon, U_{\psi(x)}(\overline{x}+y)}$$

$$\leq \prod_{j\in e'} (\varphi_j(T_j))^{1+\beta_j(p-\varepsilon)} \prod_{j\in e} (\varphi_j(t_j))^{(1+\beta_j+l_j)(p-\varepsilon)} \varepsilon^{-1} \prod_{j\in e'} (\varphi_j(t_j))^{\beta_j(p-\varepsilon)}$$

$$\times \left\| \prod_{j\in e} (\varphi_j(t_j))^{-l_j} \Delta^{m^e}(\varphi(t)f \right\|^{p-\varepsilon}_{p),\varphi,\beta;G}, \tag{1.14}$$

$$\int\limits_{R^n} \left| M_1 \left(\frac{y}{\varphi(t^e+T^{e'})} \right) \right|^s dy = \left\| M_1 \right\|_s^s \prod_{j\in e'} \varphi_j(T_j) \prod_{j\in e} \varphi_j(t_j). \tag{1.15}$$

From inequalities (1.11)–(1.15), for $r=q$, we get

$$\left\| L_e(\cdot, t^e+T^{e'}) \right\|_{r-\varepsilon, U_{\psi(\xi)}(x)} \leq C^2 \left\| \prod_{j\in e} (\varphi_j(t_j))^{-l_j} \Delta^{m^e}(\varphi(\delta u)) f \right\|_{p),\varphi,\beta;G}$$

$$\times \varepsilon^{-\frac{1}{p-\varepsilon}} \prod_{j\in e'} \varphi_j(T_j)^{2+\beta_j - \left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)}$$

$$\times \prod_{j\in e_n} (\varphi_j(\xi_j))^{\beta_j \frac{p-\varepsilon}{q-\varepsilon}} \prod_{j\in e} \varphi_j(t_j)^{2+\beta_j+l_j - \left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)}. \tag{1.16}$$

Using (1.10), for any $\overline{x} \in U$, we have

$$\left\| A^e_\eta \right\|_{q-\varepsilon, U_{\psi(\xi)}(\overline{x})} \leq C^3 \left\| \prod_{j\in e} (\varphi_j(t_j))^{-l_j} \right.$$

$$\left. \times \Delta^{m^e}(\varphi(t)) f \right\|_{p),\varphi,\beta;(G)} \varepsilon^{-\frac{1}{p-\varepsilon}} \prod_{j\in e'} (\varphi_j(T_j))^{-\nu_j+\beta_j-\left(\frac{1}{p-\varepsilon} - \frac{1}{q-\varepsilon}\right)}$$

$$\times \prod_{j\in e} \psi_j(t_j)^{\beta_j \frac{p-\varepsilon}{q-\varepsilon}} \prod_{j\in e} \psi_j(\eta_j)^{\gamma_j} \quad (\gamma_j > 0).$$

In a similar manner, (1.9) holds. □

2. Main Results

We prove two theorems for functions from the spaces $S^l_{p),\varphi,\beta} H(G_\varphi)$.

Theorem 2.1. *Let the open set $G \subset R^n$ satisfy the condition of flexible $\varphi-$ horn, $1 < p < \infty$; $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \geq 0$ be entire, $j \in e_n$, $\gamma_j > 0$ $(j \in e_n)$, and $f \in S^l_{p),\varphi,\beta} H(G_\varphi)$.*

Then

$$D^\nu : S^l_{p),\varphi,\beta} H(G_\varphi) \hookrightarrow L_{q-\varepsilon}(G),$$

for any $\varepsilon\,(0,p-1)$, *i.e., for* $f\in S^{l}_{p),\varphi,\beta}H\left(G_{\varphi}\right)$ *there exist the generalized derivatives* $D^{\nu}f$ *such that the following inequality:*

$$\|D^{\nu}f\|_{q-\varepsilon,G}\le c\,(\varepsilon)\sum_{e=e_n}\prod_{j\in e_n}\left(\varphi_j\left(T_j\right)\right)^{s_{e,j}}\sup_{\substack{0<t_j<T_{0,j}\\ j\in e_n}}\left\|\frac{\Delta^{m^e}(\varphi(t)G_{\varphi(t)})f}{\prod\limits_{j\in e_n}\left(\varphi_j\left(t_j\right)\right)^{l_j}}\right\|_{p),\varphi,\beta}\tag{2.1}$$

is valid.

In particular, if

$$\gamma_{j,0}=l_j-\nu_j+\beta_j-\frac{1}{p-\varepsilon}>0\ \ (j\in e_n),$$

then $D^{\nu}f(x)$ is continuous in the domain G, and

$$\sup_{x\in G}\left|D^{\nu}f\left(x\right)\right|\le c\,(\varepsilon)\sum_{e=e_n}\prod_{j\in e_n}\left(\varphi_j\left(T_j\right)\right)^{s^0_{e,j}}\sup_{\substack{0<t_j<T_{0,j}\\ j\in e_n}}\left\|\frac{\Delta^{m^e}(\varphi(t)G_{\varphi(t)})f}{\prod\limits_{j\in e_n}\left(\varphi_j\left(t_j\right)\right)^{l_j}}\right\|_{p),\varphi,\beta},\tag{2.2}$$

where

$$s_{e,j}=\begin{cases}\gamma_j, & j\in e\\ \gamma_j-l_j, & j\in e'\end{cases}\quad\left(s^0_{e,j}=\begin{cases}\gamma_{j,0}, & j\in e_n\\ \gamma_{j,0}-l_j, & j\in e'\end{cases}\right)\tag{2.3}$$

$0<T_j\le T_{j,0}$, $j\in e_n$; T_0 is a fixed positive vector, $c\,(\varepsilon)=c\varepsilon^{-\frac{1}{p-\varepsilon}}$ and c is a constant, independent of f,T and ε.

Proof. Initially note that under the conditions of our theorem, there exist generalized derivatives $D^{\nu}f$ on G. Indeed, if $\gamma_j>0(j\in e_n)$, then for $f\in S^{l}_{p),\varphi,\beta}H\left(G_{\varphi}\right)\to S^{l}_{p)}H\left(G_{\varphi}\right)\to S^{l}_{p-\varepsilon}H\left(G_{\varphi}\right)$ $(p-\varepsilon>1)$. Further, $D^{\nu}f$ exists on G, and $D^{\nu}f\in L_{p-\varepsilon}(G)$. Moreover, for almost each point $x\in G$, the integral representation (1.4) holds.

Based on the Minkowskii inequality, from identity (1.4), we get

$$\|D^{\nu}f\|_{q-\varepsilon,G}\le\sum_{e=e_n}\|A^{e}_{T}\|_{q-\varepsilon,G}\,.\tag{2.4}$$

By means of inequality (1.8), $U=G, L_e\equiv K^{(\nu)}_{e}, \eta_j=T_j$, $j\in e_n$, we get inequality (2.1)

Now, let conditions $\gamma_{j,0}=\gamma_j$ $(q=\infty)$, $j\in e_n$, hold. Then based on identity (1.4), from inequality (2.4), we get

$$\left\|D^{\nu}f-f^{\nu}_{\varphi(T)}\right\|_{\infty,G}\le c\,(\varepsilon)\sum_{\substack{e=e_n\\ e\neq\emptyset}}\prod_{j\in e_n}\left(\varphi_j\left(t_j\right)\right)^{\gamma_{j,0}}\sup_{\substack{0<t_j<T_{0,j}\\ j\in e_n}}\left\|\frac{\Delta^{m^e}(\varphi(t)G_{\varphi(t)})f}{\prod\limits_{j\in e_n}\left(\varphi_j\left(t_j\right)\right)^{l_j}}\right\|_{p),\varphi,\beta}.$$

As $T_j\to 0$ $(j\in e_n)$, the left-hand side of the inequality tends to zero, and since $f^{(\nu)}_{\varphi(T)}$ is continuous on G, and the convergence on $L_{\infty}\left(G\right)$ coincides with the uniform convergence, thus the limit function $D^{\nu}f$ is continuous on G. □

Let ξ be an n-dimensional vector. We have the following

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *be satisfied. Then for* $\gamma_j>0$, $j\in e_n$ *the generalized derivatives* $D^{\nu}f$ *satisfy on* G *the Hölder condition in the metric* $L_{q-\varepsilon}$, *with the exponent* $\sigma_j(j\in e_n)$, *i.e., the following inequality:*

$$\|\Delta\left(\xi,G\right)D^{\nu}f\|_{q-\varepsilon,G}\le C\left(\varepsilon\right)\|f\|_{S^{l}_{p),\varphi,\beta}H(G_{\varphi})}\prod_{j\in e_n}\sigma_j(|\xi_j|)\tag{2.5}$$

is valid, where $C(\varepsilon)$ *is defined in Theorem* 2.1, *and* σ_j *are the numbers satisfying the inequalities*

$$\sigma_j(|\xi_j|)=\begin{cases}\max\{\varphi_j\left(|\xi_j|\right))^{\gamma_j},\varphi_j\left(|\xi_j|\right))^{\gamma_j-1}\}, & \textit{for}\ \ j\in e\\ \varphi_j\left(|\xi_j|\right), & \textit{for}\ \ j\in e'.\end{cases}\tag{2.6}$$

If $\gamma_{j,0} > 0$, $(j \in e_n)$, *then*

$$\sup_{x\in G} |\Delta(\xi, G)\, D^{\nu} f(x)| \le c(\varepsilon)\, \|f\|_{S^{l}_{p),\varphi,\beta} H(G_\varphi)} \prod_{j\in e_n} \sigma_{j,0}(|\xi_j|), \tag{2.7}$$

where $\sigma_{j,0}$ *satisfies the same conditions as* $\sigma_{j,0}$, *but with* γ_j *replaced by* $\gamma_{j,0}$.

Proof. By Lemma 8.6 of [3], there is a domain

$$G_\omega \subset G\ (\omega = \omega_1, \dots, \omega_n), \omega_j = \lambda_j \theta(x),\ \ \lambda_j > 0, (j \in e_n),\ \ \theta(x) = dist(\varkappa, \partial G), x \in G).$$

Suppose that $|\xi_j| < \omega_j (j \in e_n)$; then for any $x \in G_\omega$, the segment connecting the points x, $x + \xi$ is contained in G.

Consequently, for all the points of this segment, identity (1.4) with the same kernels is valid.

By the same arguments, from (1.4) we get

$$\begin{aligned}
|\Delta(\xi, G)\, D^{\nu} f(x)| \le C_1 \sum_{e=e_n} \Bigg\{ &\prod_{j\in e'} (\varphi_j(T_j))^{-\nu_j-2} \int_0^{|\xi_1^e|} \cdots \int_0^{|\xi_n^e|} \prod_{j\in e} (\varphi_j(t_j))^{-\nu_j-2} \prod_{j\in e} \frac{\varphi_j'(t_j)}{\varphi_j(t_j)} \Bigg\} \\
&\times \int_{-\infty^e}^{\infty^e} \int_{R^n} \Big| K_i^{(\nu)}\Big(\frac{y}{\varphi(t^e + T^e)}, \frac{\rho(\varphi(t^e + T^{e'}), x)}{\varphi(t^e + T^{e'})}\Big) \zeta_e\Big(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t), x)\Big)\Big| \\
&\times \Big| \Delta(\xi, G)(\Delta^{m^e}(\varphi(\delta u) f(x + y + u^e) \Big| dt^e du^e dy \\
&+ \prod_{j\in e'} (\varphi_j(T_j))^{-\nu_j-3} \prod_{j\in e'} |\xi_j| \int_{|\xi_1^e|}^{T_1} \cdots \int_{|\xi_n^e|}^{T_n} \prod_{j\in e} (\varphi_j(t_j))^{-\nu_j-3} \prod_{j\in e} \frac{\varphi_j'(t_j)}{\varphi_j(t_j)} \\
&\times \int_{-\infty^e}^{\infty^e} \int_{R^n} \Big| K_i^{(\nu+1)}\Big(\frac{y}{\varphi(t^e + T^{e'})}, \frac{\rho(\varphi(t^e + T^{e'}), x)}{\varphi(t^e + T^{e'})}\Big) \zeta_e\Big(\frac{u}{\varphi(t)}, \frac{\rho(\varphi(t), x)}{\varphi(t)}, \frac{1}{2}\rho'(\varphi(t), x)\Big)\Big| \\
&\times \int_0^1 \cdots \int_0^1 \Big| \Delta^{m^e}(\varphi(\delta)u) f(x + y + u^e + \xi v) \Big| dv dy du^e dt^e
\end{aligned} \tag{2.8}$$

$$= C_1 \sum_{e=e_n} (E(x,\xi) + F(x,\xi)), \tag{2.9}$$

where $\left|\xi_j^e\right| = |\xi_j|$, $\left|\xi_j^e\right| = 0$, $(j \in e')$, $0 < T_j \le T_{0,j} (j \in e_n)$. We also assume that $|\xi_j| < T_j (j \in e_n)$. Consequently, $|\xi_j| < \min(\omega_j, T_j)\ (j \in e_n)$. If $x \in G \backslash G_\omega$, then

$$\Delta(\xi, G)\, D^{\nu} f(x) = 0.$$

Based on (2.8) we have

$$\|\Delta(\xi, G)\, D^{\nu} f\|_{q-\varepsilon, G} \le C^1 \sum_{e=e_n} \Big(\|E(\cdot, \xi)\|_{q-\varepsilon, G_\omega} + \|F(\cdot, \xi)\|_{q-\varepsilon, G_\omega} \Big). \tag{2.10}$$

By means of inequality (1.8) for $U = G$, $\eta_j = f$, $\eta_j = |\xi_j|$, $j \in e_n$, we get

$$\|E(\cdot, \xi)\|_{q-\varepsilon, G_\omega} \le C_1(\varepsilon) \Big\| \prod_{j\in e} (\varphi_j(t_j))^{-l_j} \varphi(t) f \Big\|_{p),\varphi,\beta;G} \prod_{j\in e} \varphi_j(|\xi_j|)^{\gamma_j}, \tag{2.11}$$

and by taking into account (1.9) for $U = G$, $\eta = |\xi_j|$, $j \in e_n$, we arrive at

$$\begin{aligned}
\|F(\cdot, \xi)\|\, \|_{q-\varepsilon, G_\omega} \le C_2(\varepsilon) &\Big\| \prod_{j\in e} (\varphi_j(t_j))^{-l_j} \Delta^{m^e} \varphi(t) f \Big\|_{p),\varphi,\beta;G} \\
&\times \prod_{j\in e_n} (\varphi_j(|\xi_j|)) \prod_{j\in e_n} (\varphi_j(|\xi_j|))^{\gamma_j - 1}.
\end{aligned} \tag{2.12}$$

From inequalities (2.9)–(2.11) we find that

$$\|\Delta\left(\xi,G\right)D^{\nu}f\|_{q-\varepsilon,G}\leq C(\varepsilon)\,\|f\|_{S^{l}_{p),\varphi,\beta}H(G_{\varphi})}\prod_{j\in e_n}\sigma_j(|\xi_j|).$$

Now, suppose that $|\xi_j|\geq\min\{\omega_j,T_j\}$, $j\in e_n$; then

$$\|\Delta\left(\xi,G\right)D^{\nu}f\|_{q-\varepsilon,G}\leq 2\,\|D^{\nu}f\|_{q-\varepsilon,G}\leq C(\omega,T)\,\|D^{\nu}f\|_{q-\varepsilon,G}\prod_{j\in e_n}\sigma_j(|\xi_j|).$$

Estimating $\|D^{\nu}f\|_{q-\varepsilon,G}$ and using inequality (2.1), we again get the required inequality. □

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