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PRIMELY FILTERS IN BL-ALGEBRAS

AFSANEH BEHZADI[1], SOMAYEH MOTAMED[2,*] AND FARHAD KHAKSAR HAGHANI[1]

Abstract. In this paper, we introduce the concept of primely filters in BL-algebras. As for the concept, we present some related results in BL-algebras. In particular, we show some relations between primely filters and other types of filters in BL-algebras.

1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic (BL-logic) in order to investigate many-valued logic by algebraic means [7]. Filters theory plays an important role in studying these logical algebras and ordered semi groups. From logical point of view, various filters correspond to various sets of provable formulas. Hájek introduced the concepts of filters and prime filters in BL-algebras. Using prime filters in BL-algebras, Hájek proved the completeness of Basic Logic BL. Haveshki et al. extended the algebraic analysis of BL-algebras and introduced positive implicative filters of BL-algebras [8]. Kondo and Dudek [10] proved that positive implicative filters (introduced by Haveshki et al.) and the original implicative deductive systems (introduced by Turunen [17]) coincide and every positive implicative filter is a Boolean filter.

Today this many-valued logic has been developed into a fuzzy logic, which meets together with the motivations from the beginning of the twentieth century in one theory that connects quantum mechanics, mathematical logic, probability theory, computer science, algebra, soft computing and many other important aspects of our modern world. This was the motivation for the researchers of this study to introduce some new filters in BL-algebras. BL-algebras are important classes of algebras inspired by logic. In fact, the objective of this paper is to develop and define new concepts for investigating BL-algebras.

The structure of the paper is as follows:

Section 2 is a recall of some definitions and results about BL-algebras that are used in the paper. In Section 3, a new filter (primely filter) in BL-algebra is introduced and many properties of it are obtained. Also, the relationships between this new filter and other types of filters in BL-algebra are investigated. In Section 4, we define the notion of primely BL-algebras and the related algebra, called primely BL-algebra.

2. Preliminaries

Definition 2.1 ([7])**.** A BL-algebra is an algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:

(BL_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
(BL_2) $(A, *, 1)$ is a commutative monoid;
(BL_3) $*$ and $\rightarrow$ form an adjoint pair, i.e., $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in A$;
(BL_4) $a \wedge b = a * (a \rightarrow b)$, for all $a, b \in A$;
(BL_5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$, for all $a, b \in A$.

• Let F be a non-empty subset of a BL-algebra A and $a, b \in A$. Then the following conditions are equivalent:

(i) F is a filter of A.
(ii) If $a, b \in F$ imply $a * b \in F$ and if $a \in F$ and $a \leq b$, then $b \in F$.

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*Corresponding author.

(iii) $1 \in F$, and if $a, a \to b \in F$, then $b \in F$ [7].

Definition 2.2. A non-empty subset F of a BL-algebra A is called:

- a prime filter of A, if F is a proper filter and for all $x, y \in A$, if $x \vee y \in F$ implies $x \in F$ or $y \in F$ [7];
- a maximal filter of A, if F is a proper filter and it is not properly contained in any other proper filter of A [7];
- an integral filter of A, if F is a proper filter and for all $x, y \in A$, if $(x * y)^- \in F$, implies $x^- \in F$ or $y^- \in F$ [4];
- an obstinate filter of A, if F is a proper filter and for all $x, y \in A$, if $x, y \notin F$ implies that $x \to y \in F$ and $y \to x \in F$ [3];
- a semi-integral filter of A, if F is a proper filter and for all $x, y \in A$, if $(x * y)^- \in F$, implies $x^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N}$ [12].

Definition 2.3. Let F be a filter, M be a non-empty subset of BL-algebra A and $x, x_1, x_n, a, m \in A$. Then:

(i) the intersection of all maximal filters of A containing F is called the radical of F and denoted by $\mathrm{Rad}(F)$ [14].

(ii) $(F : x) = \{r \in A : r \vee x \in F\}$ [13].

(iii) $[M) = \{a \in A : x_1 * \cdots * x_n \leq a, \text{for some } x_1, \ldots, x_n \in M\}$ and $[M)$ is a filter of A generated by M [15].

(iv) $[m) = \{x \in A : x \geq m^n, \text{for some } n \in \mathbb{N}\}$, then $[m)$ is a filter of A generated by m [15].

(v) $D(F) = \{x \in A : x^{--} \in F\}$ is a filter of A and, clearly, $F \subseteq D(F)$ [3].

3. Primely Filters in BL-algebras

Here, we introduce the concept of a primely filter in a BL-algebra.

Definition 3.1. A proper filter F of a BL-algebra A is called a primely filter if for $x, y \in A$, $(x \wedge y)^- \in F$, implies $x^- \in F$ or $y^- \in F$.

In the following we present some examples of infinite, finite primely filters in BL-algebras.

Example 3.2. (i) Let $A = \{-\infty, \ldots, -3, -2, -1, 0, a, b, 1\}$, where $-\infty < \cdots < -3 < -2 < -1 < 0 < a$, $b < c < 1$. The operations $*$, $\to$ are defined as follows:

$*$	$-\infty$	$\ldots$	-3	-2	-1	0	a	b	1
$-\infty$	$-\infty$	$\cdots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	-6	-5	-4	-3	-3	-3	-3
-2	$-\infty$	$\cdots$	-5	-4	-3	-2	-2	-2	-2
-1	$-\infty$	$\cdots$	-4	-3	-2	-1	-1	-1	-1
0	$-\infty$	$\cdots$	-3	-2	-1	0	0	0	0
a	$-\infty$	$\cdots$	-3	-2	-1	0	a	0	a
b	$-\infty$	$\cdots$	-3	-2	-1	0	0	b	b
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1

and

$\to$	$-\infty$	$\dots$	-3	-2	-1	0	a	b	1
$-\infty$	1	$\cdots$	1	1	1	1	1	1	1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\cdots$	1	1	1	1	1	1	1
-2	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
-1	$-\infty$	$\cdots$	-1	1	1	1	1	1	1
0	$-\infty$	$\cdots$	-3	-2	-1	1	1	1	1
a	$-\infty$	$\cdots$	-3	-2	-1	b	1	b	1
b	$-\infty$	$\cdots$	-3	-2	-1	a	a	1	1
1	$-\infty$	$\cdots$	-3	-2	-1	0	a	b	1,

then $(A, \wedge, \vee, *, \to, -\infty, 1)$ is a BL-algebra [9]. All filters of A are $F_1 = \{1\}$, $F_2 = \{a, 1\}$, $F_3 = \{b, 1\}$, $F_4 = \{0, a, b, 1\}$ and $F_5 = \{\dots, -3, -2, -1, 0, a, b, 1\} - \{-\infty\}$. It is clear that all filters are primely.

(ii) Let $A = \{0, a, b, c, d, 1\}$, where $0 < d < c < a, b < 1$. If the operations $*$ and $\to$ are defined as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

and

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1,

then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra [11]. It is easy to see that $F_1 = \{1\}$, $F_2 = \{a, 1\}$, $F_3 = \{b, 1\}$ and $F_4 = \{a, b, c, 1\}$ are primely filters of A.

(iii) Let $A = \{0, a, b, c, d, 1\}$, where $0 < a < c < 1$ and $0 < b < c, d < 1$. If the operations $*$ and $\to$ are defined as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

and

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1,

then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra [9]. Clearly, $F_1 = \{1\}$ and $F_2 = \{c, 1\}$ are not primely filters of A. Since $(a \wedge b)^- = 1$, while $a^- = d$ and $b^- = a$.

Theorem 3.3. *Let F be a proper filter of a BL-algebra A. Then F is a primely filter if and only if $x^- \to y^- \in F$ or $y^- \to x^- \in F$, for all $x, y \in A$.*

Proof. Let F be a primely filter and $x, y \in A$. We known that $((x^- \to y^-) \vee (y^- \to x^-))^{--} = 1 \in F$, so, $((x^- \to y^-)^- \wedge (y^- \to x^-)^-)^- = 1 \in F$. Hence, according to the hypothesis $(x^- \to y^-)^{--} = x^- \to y^- \in F$ or $(y^- \to x^-)^{--} = y^- \to x^- \in F$, for all $x, y \in A$. Now, let $x^- \to y^- \in F$ or $y^- \to x^- \in F$, for all $x, y \in A$. We also assume that $x^- \vee y^- = (x \wedge y)^- \in F$, for $x, y \in A$. Since $x^- \vee y^- \leq (x^- \to y^-) \to y^-$ and $x^- \vee y^- \leq (y^- \to x^-) \to x^-$, so, $(x^- \to y^-) \to y^- \in F$ and $(y^- \to x^-) \to x^- \in F$. Now, if $x^- \to y^- \in F$, then $y^- \in F$. Hence F is a primely filter. Also, if $y^- \to x^- \in F$, then $x^- \in F$. Therefore F is a primely filter. $\square$

According to Theorem 3.3:

Theorem 3.4 (Extension property for primely filters)**.** *Let F and G be two proper filters of BL-algebra A such that $F \subseteq G$ and let F be a primely filter. Then G is a primely filter.*

Proposition 3.5. *Let A be a BL-algebra. Then the following conditions are equivalent:*
(i) $\{1\}$ *is a primely filter;*
(ii) *all filters of A are primely filters;*
(iii) $x^- \leq y^-$ *or* $y^- \leq x^-$*, for all* $x, y \in A$.

Proof. ((i) $\Rightarrow$ (ii)) Based on Theorem 3.4, the proof is clear.
((ii) $\Rightarrow$ (i)) is trivial.
((i) $\Rightarrow$ (iii)) Let $\{1\}$ be a primely filter. Then based on Theorem 3.3, $x^- \to y^- \in \{1\}$ or $y^- \to x^- \in \{1\}$, for all $x, y \in A$. Hence $x^- \leq y^-$ or $y^- \leq x^-$, for all $x, y \in A$.
((iii) $\Rightarrow$ (i)) Let $x^- \leq y^-$ or $y^- \leq x^-$, for all $x, y \in A$. Then $x^- \to y^- \in \{1\}$ or $y^- \to x^- \in \{1\}$, for all $x, y \in A$. Therefore $\{1\}$ is a primely filter of A. □

Proposition 3.6. *Let A be a BL-algebra. Then the following conditions are equivalent:*
(i) *F is a primely filter of A;*
(ii) $\{1/F\}$ *is a primely filter of A/F;*
(iii) *all filters of A/F are primely filters.*

Proof. ((i) $\Rightarrow$ (ii)) Let F be a primely filter and $(x/F \wedge y/F)^- \in \{1/F\}$, for all $x/F, y/F \in A/F$. Then $(x \wedge y)^- \in F$. According to the hypothesis, $x^- \in F$ or $y^- \in F$, so $x^-/F \in \{1/F\}$ or $y^-/F \in \{1/F\}$. Thus $\{1/F\}$ is a primely filter of A/F.
((ii) $\Rightarrow$ (i)) Let $\{1/F\}$ be a primely filter and $(x \wedge y)^- \in F$, for $x, y \in A$. Then $(x \wedge y)^-/F \in \{1/F\}$. So $x^-/F \in \{1/F\}$ or $y^-/F \in \{1/F\}$. Hence $x^- \in F$ or $y^- \in F$. Therefore F is a primely filter of A.
((ii) $\Rightarrow$ (iii)) According to Theorem 3.4, the proof is clear.
((iii) $\Rightarrow$ (ii)) The proof is easy. □

Based on Theorem 3.4:

Corollary 3.7. *Let F be a primely filter of a BL-algebra A. Then the following conditions are hold:*
(i) $(F : x)$ *is a primely filter, for* $x \in A - F$;
(ii) $\mathrm{Rad}(F)$ *is a primely filter of A;*
(iii) $D(F)$ *is a primely filter of A.*

Proposition 3.8. *Let F and G be two proper filters of a BL-algebra A such that $F \subseteq G$. Then G is a primely filter of A if and only if G/F is a primely filter of a BL-algebra A.*

Proposition 3.9. *Let F_1 and F_2 be two primely filters of a BL-algebra A such that $D(F_1) = D(F_2)$. Then $F_1 \cap F_2$ is a primely filter.*

Proof. Let F_1 and F_2 be two primely filters and $(x \wedge y)^- \in F_1 \cap F_2$, for $x, y \in A$. Then $(x \wedge y)^- \in F_1$. According to the hypothesis, $x^- \in F_1$ or $y^- \in F_1$. If $x^- \in F_1 \subseteq D(F_1) = D(F_2)$. Then $x^- \in F_2$, hence $x^- \in F_1 \cap F_2$. Similarly, if $y^- \in F_1$, then $y^- \in F_1 \cap F_2$. Therefore $F_1 \cap F_2$ is a primely filter. □

Recall that the union of two arbitrary filters is not a filter. In fact, if $\{F_i : i \in I\}$ is a non-empty totally ordered set of filters, then $\cup_{i \in I} F_i$ is a filter. So, clearly if $\{F_i : i \in I\}$ is a non-empty totally ordered set of primely filters, then $\cup_{i \in I} F_i$ is a primely filter.

Proposition 3.10. *Let F be a primely filter of an MV-algebra A. Then $F = (F : x)$, for all $x \in A - F$.*

Proof. Let F be a primely filter of an MV-algebra A and $x \in A - F$. It is clear that $F \subseteq (F : x)$. Now, let $y \in (F : x)$. Then $(x^- \wedge y^-)^- = x^{--} \vee y^{--} = x \vee y \in F$. According to the hypothesis, $x^{--} \in F$ or $y^{--} \in F$, and so, $x \in F$ or $y \in F$. As $x \in A - F$, then $y \in F$. Hence $(F : x) \subseteq F$. Therefore the proof is completed. □

In the following, we give the relations between primely filters and some other filters in BL-algebras.

Proposition 3.11. (i) *In any BL-algebra, every prime filter is a primely filter.*
(ii) *In any BL-algebra, every maximal filter is a primely filter.*

Proof. (i) Let F be a prime filter and $(x \wedge y)^- \in F$, for $x, y \in A$. Then $x^- \vee y^- \in F$. So, according to the hypotheses, $x^- \in F$ or $y^- \in F$. Thus F is a primely filter.
(ii) Based on part(i), the proof is clear. □

The following example shows that the converse of Proposition 3.11 is not generally true.

Example 3.12. Let $A = \{0, a, b, c, 1\}$, where $0 < c < a, b < 1$. If the operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

and

$\rightarrow$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1,

then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra [11]. It is easy to check that $F = \{1\}$ is a primely filter, while is not a prime filter, since $a \rightarrow b \notin F$ and $b \rightarrow a \notin F$.

The following corollary is a straightforward consequence of [16, Proposition 7] and Proposition 3.11.

Corollary 3.13. *Any proper filter of a BL-algebra can be extended to a primely filter.*

Based on [16, Proposition 7] and Proposition 3.11(i), we have the following

Corollary 3.14. *Let F and P be two proper filters of a BL-algebra A. Then for any $a \in A - F$, there exists a primely filter P such that $F \subseteq P$ and $a \notin P$.*

Recall that a non-unit element $m \in A$ is said to be a comolecule of A if whenever $m \leq x, y \leq 1$, then $x \vee y < 1$ [1].

The following corollary is a straightforward consequence of Proposition 3.11(i) and [1, Theorem 4.3].

Corollary 3.15. *Let A be a BL-algebra and $Comol(A) = A - \{1\}$. Then $\perp [m)$ is a primely filter.*

Proposition 3.16. *Let F be a primely filter of a BL-algebra A. Then the following conditions hold:*
(i) *$D(F)$ is a prime filter;*
(ii) *$D(F)$ is a primary filter;*
(iii) *$A/D(F)$ is a local BL-algebra.*

Proof. (i) Let F be a primely filter and $x \vee y \in D(F)$, for $x, y \in A$. Then according to the definition of $D(F)$, $(x \vee y)^{--} \in F$. As $(x \vee y)^{--} = (x^- \wedge y^-)^- \in F$ and F is a primely filter, so, $x^{--} \in F$ or $y^{--} \in F$. Hence $x \in D(F)$ or $y \in D(F)$. Thus $D(F)$ is a prime filter.

(ii) Let F be a primely filter. Then according to part(i) and [13, Theorem 3.1(3)], $D(F)$ is a primary filter of A.

(iii) Based on part(i), the proof is clear. □

Proposition 3.17. *Let F be a primely and normal filter of a BL-algebra A. Then F is a primary filter.*

Proof. Let F be a primely and normal filter. Then according to [2, Theorem 3.25], $F = D(F)$. Hence based on Proposition 3.16, F is a primary filter. □

Open Problem: If F is a normal and primary filter of a BL-algera, then is F a primely filter?

Recall that a BL-algebra A is an MV-algebra if for all $x \in A$, $x^{--} = x$ [7].

Proposition 3.18. *Let F be a primely filter of an MV-algebra A. Then the following conditions hold:*
(i) *F is a prime filter;*
(ii) *F is a primary filter.*

Proof. (i) Let F be a primely filter of an MV-algebra A and $x \vee y \in F$, for $x, y \in A$. Then $x \vee y = x^{--} \vee y^{--} = (x^- \wedge y^-)^- \in F$. As F is a primely filter, $(x^-)^- \in F$ or $(y^-)^- \in F$. According to the hypothesis, $x \in F$ or $y \in F$. Therefore F is a prime filter.

(ii) The proof based on [13, Theorem 3.1(3)] and part(i), is clear. □

Recall that a BL-algebra A is a semi-G-algebra if for all $x \in A$, $(x^2)^- = x^-$ [5].

Proposition 3.19. *Let F be a primely and fantastic filter of a semi-G-algebra A. Then the following conditions hold:*
(i) *F is an obstinate filter;*
(ii) *F is a prime filter;*
(iii) *F is a primary filter;*
(iv) *F is a maximal filter.*

Proof. (i) Let F be a fantastic and primely filter of a semi-G-algebrs A. Also, assume that $x, y \notin F$, for $x, y \in A$. Based on [5, Proposition 4.4], $x \wedge x^- = 0$, equivalenty $(x \wedge x^-)^- = 1 \in F$, for all $x \in A$. As F is a primely filter, $x^- \in F$ or $x^{--} \in F$. If $x^{--} \in F$, as F is a fantastic filter so, $x \in F$, which is a contradiction. Hence $x^- \in F$. It is known that $x^- = x \to 0 \leq x \to y$ and, so, $x \to y \in F$. Similarly, $y \to x \in F$. Therefore F is an obstinate filter.

(ii) Let F be a fantastic and primely filter of a semi-G-algebra A. Then based on [3, part (i) and Theorem 4.1], F is a prime filter.

(iii) Based on [13, part(ii) and Theorem 3.1], the proof is easy.

(iv) The proof based on [3, part (i) and Theorem 4.1], is clear. □

In the following example, it is shown that the condition "A is a semi-G-algebra" in Proposition 3.18 is necessary.

Example 3.20. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. If the operations $*$ and $\to$ are defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

and

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1,

then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra [8]. Clearly, $F = \{1\}$ is a primely filter and fantastic filter, while $a \wedge a^- \neq 0$. Also, F is not an obstinate filter, since $b \notin F$ and $(b^-)^n \notin F$.

Proposition 3.21. *Every integral filter in a BL-algebra is a primely filter.*

Proof. Let F be an integral filter of a BL-algebra A and $(x \wedge y)^- \in F$, for $x, y \in A$. As $(x \wedge y)^- \leq (x * y)^-$, then $(x * y)^- \in F$. Hence according to the hypotheses, $x^- \in F$ or $y^- \in F$. Therefore F is a primely filter of A. □

Example 3.22. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. If the operations $*$ and $\to$ are defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

and

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1,

then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra [8]. Clearly, $F_1 = \{1\}$ and $F_2 = \{b, 1\}$ are primely filters and not integral filters. Since $(a * a)^- \in F_1$, F_2, while $a^- \notin F_1$, F_2.

We known that every obstinate filter is an integral filter. So, according to Proposition 3.21, we have

Proposition 3.23. *Every obstinate filter in BL-algebras is a primely filter.*

Remark 3.24. Consider Example 3.22. Clearly, $F_2 = \{b, 1\}$ is a primely filter of A and not an obstinate filter, since $a \notin F_2$ and $(a^-)^n = a^n = 0 \notin F_2$.

Proposition 3.25. *Let F be a primely filter of a semi-G-algebra A. Then $D(F)$ is an obstinate filter.*

Proof. Let F be a primely filter. Then according to Corollary 3.7, $D(F)$ is a primely filter. Now let $x \notin D(F)$. Then according to [5, Proposition 4.4], $x \wedge x^- = 0$, for all $x \in A$. So, $(x \wedge x^-)^- = 1 \in D(F)$, for all $x \in A$. Hence $x^- \in D(F)$ or $x^{--} \in D(F)$, since $D(F)$ is a primely filter. If $x^{--} \in D(F)$, then $x^{--} \in F$. Hence $x \in D(F)$, which is a contradiction. Thus $x^- \in D(F)$ and so, $D(F)$ is an obstinate filter. □

Proposition 3.26. *Let $D(F)$ be an obstinate and normal filter. Then F is a primely filter.*

Proof. Let $D(F)$ be an obstinate and normal filter. Then according to Proposition 3.23, $D(F)$ is a primely filter. Hence based on [2, Theorem 3.25], F is a primely filter. □

Proposition 3.27. *Let F be a primely filter of a semi-G-algebra A. Then the following conditions hold:*
(i) *$D(F)$ is a Boolean (positive implicative) filter;*
(ii) *$D(F)$ is an implicative filter.*

Proof. (i) Let F be a primely filter and $x \wedge x^- = 0$, for all $x \in A$. As $(x \wedge x^-)^- = 1 \in F$, based on the hypotheses, $x^- \in F$ or $x^{--} \in F$. So, according to the definition, $D(F)$, $x^- \in D(F)$ or $x \in D(F)$. Hence $x \vee x^- \in D(F)$, for all $x \in A$. Thus $D(F)$ is a Boolean (positive implicative) filter.
(ii) Based on part (i), the proof is clear. □

The following example shows that the converse of Proposition 3.27 is not true, in general.

Example 3.28. Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. If the operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

and

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1,

then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra [11]. Clearly, $F = \{1\} = D(F)$ is a Boolean filter of A, while F is not a primely filter, since $a^- \rightarrow b^- \notin F$ and $b^- \rightarrow a^- \notin F$.

The following remark shows that the condition $x \wedge x^- = 0$ is necessary in Proposition 3.27.

Remark 3.29. In Example 3.20, $F = \{1\}$ is a primely filter, while $a \wedge a^- \neq 0$ and $D(F) = \{1\}$ is not a Boolean filter, since $a \vee a^- \notin D(F)$.

Lemma 3.30. *Let F be a filter of a BL-algebra A. Then $D(F) \subseteq \mathrm{Rad}(F)$.*

Proof. Let $x \in D(F)$. Then $x^{--} \in F$. As $x^{--} \leq x^- \rightarrow x$, therefore $x \in \mathrm{Rad}(F)$. Hence $D(F) \subseteq \mathrm{Rad}(F)$. □

The following proposition is a straightforward consequence of Proposition 3.16, Lemma 3.30 and Theorem 3.4.

Proposition 3.31. *Let F be a primely filter of a BL-algebra A. Then the following conditions hold:*
(i) *$\mathrm{Rad}(F)$ is a prime filter;*
(ii) *$\mathrm{Rad}(F)$ is a primary filter.*

Corollary 3.32. *Let F be a primely and semi-maximal filter of a BL-algebra A. Then the following conditions hold:*
(i) *F is a prime filter;*
(ii) *F is a primary filter.*

Proof. According to Proposition 3.31 and the definition of a semi-maximal filter, the proof is easy. □

Proposition 3.33. *Let F be a primely filter of a BL-algebra A. Then* $\mathrm{Rad}(F : x) = \mathrm{Rad}(F)$*, for* $x \in A - F$.

Proof. Let F be a primely filter. Then according to Proposition 3.31, $\mathrm{Rad}(F)$ is a primary filter. So, $\mathrm{Rad}(\mathrm{Rad}(F))$ is a maximal filter, hence $\mathrm{Rad}(F)$ is a maximal filter. As $\mathrm{Rad}(F) \subseteq \mathrm{Rad}(F : x)$, for all $x \in A - F$, so, $\mathrm{Rad}(F) = \mathrm{Rad}(F : x)$, for all $x \in A - F$. □

Proposition 3.34. *Let F be a filter of a BL-algebra A. Then*
(i) $D(F)$ *is a fantastic filter;*
(ii) $\mathrm{Rad}(F)$ *is a fantastic filter.*

Proof. (i) We known that $(x^{--} \to x)^{--} = x^{--} \to x^{--} = 1$, for all $x \in A$. Hence $(x^{--} \to x)^{--} \in F$, for all $x \in A$. So $x^{--} \to x \in D(F)$, for all $x \in A$. Therefore $D(F)$ is a fantastic filter.
(ii) Based on part (i) and Lemma 3.30, the proof is clear. □

Proposition 3.35. *Let F be a filter of a BL-algebra A. Then the following conditions hold:*
(i) *F is a primary filter if and only if* $\mathrm{Rad}(F)$ *is a primely filter.*
(ii) *If F is a semi-integral filter, then* $\mathrm{Rad}(F)$ *is a primely filter.*

Proof. (i) Let F be a primary filter. Then according to [13, Theorem 3.3], $\mathrm{Rad}(F)$ is a maximal filter. Based on Proposition 3.11(ii), $\mathrm{Rad}(F)$ is a primely filter. Conversely, let $\mathrm{Rad}(F)$ be a primely filter. So according to Proposition 3.31(ii), $\mathrm{Rad}(\mathrm{Rad}(F))$ is a primary filter and based on [14, Theorem 3.7(8)], $\mathrm{Rad}(F)$ is a primary filter. Then according to [13, Theorem 3.3], $\mathrm{Rad}(\mathrm{Rad}(F)) = \mathrm{Rad}(F)$ is a maximal filter and so $\mathrm{Rad}(F)$ is a maximal filter. Therefore F is a primary filter.
(ii) Let F be a semi-integral filter. Then according to [12, Lemma 2], F is a primary filter. Based on part(i), $\mathrm{Rad}(F)$ is a primely filter. □

Open Problem [12]: Is every primary filter in BL-algebras, semi-integral?

Proposition 3.36. *Let F be a primely filter of a BL-algebra A. Then the following conditions hold:*
(i) *If $N(A/F) = \{[1]\}$, then F is a primary filter.*
(ii) *If any almost top element of A/F is a trivial element, then F is a primary filter.*

Proof. (i) Let F be a primely filter. Then based on [2, Proposition 3.16 and Theorem 4.8], F is a primary filter.
(ii) Let F be a primely filter. Then based on [2, Proposition 3.16 and Theorem 4.7], F is a primary filter. □

Proposition 3.37. *Let F be a primary filter of a semi-G-algebra A. Then F is a primely filter.*

Proof. Let F be a primary filter of a semi-G-algebra A. Also, assume that $(x \wedge y)^- \in F$, for $x, y \in A$. As $(x \wedge y)^- \leq (x * y)^-$ so, $(x * y)^- \in F$. According to the hypothesis, $(x^n)^- = x^- \in F$ or $(y^n)^- = y^- \in F$, for some $n \in \mathbb{N}$. Hence $x^- \in F$ or $y^- \in F$. Then F is a primely filter. □

Recall that a BL-algebra A is a SBL-algebra if for all $x, y \in A$, $(x * y)^- = (x^- \vee y^-)$ [6].

Proposition 3.38. *Let F be a primely filter of a SBL-algebra A. Then F is a primary filter.*

Proof. Let F be a primely filter of a SBL-algebra and $(x * y)^- \in F$, for $x, y \in A$. Then since A is a SBL-algebra, $(x * y)^- = (x^- \vee y^-) = (x \wedge y)^- \in F$. So, according to the hypothesis, $x^- \in F$ or $y^- \in F$. Thus F is a primary filter. □

Corollary 3.39. *Let F be a primely filter of a BL-algebra A and $x \vee y = 1$, for all $x, y \neq 1$. Then F is a primary filter.*

Proof. Let F be a primely filter and $(x * y)^- \in F$, for $x, y \in A$. As $x \vee y = 1$, then $x * y = x \wedge y$. Hence $(x * y)^- = (x \wedge y)^- = (x^- \vee y^-) \in F$. So, according to the hypothesis, $x^- \in F$ or $y^- \in F$. Thus F is a primary filter. □

Proposition 3.40. *Let A be a Gödel algebra and F be a filter of A. Then the following conditions are equivalent:*
(i) *F is an integral filter;*
(ii) *F is a primely filter;*
(iii) *F is a semi-integral filter;*
(iv) *F is a primary filter.*

Proof. ((i) $\Rightarrow$ (ii)) The proof, according to Proposition 3.21, is clear.

((ii) $\Rightarrow$ (i)) Let F be a primely filter and $(x * y)^- \in F$, for $x, y \in A$. Then, according to the hypothesis, $(x * y)^- = (x \wedge y)^- \in F$ and so, $x^- \in F$ or $y^- \in F$. Thus F is an integral filter, for $x, y \in A$.

((i) $\Leftrightarrow$ (iii)) Based on [12, Lemma 3], the proof is clear.

((i) $\Leftrightarrow$ (iv)) According to [4, Theorem 4.15], the proof is clear. $\square$

4. Primely BL-algebras

Definition 4.1. A BL-algebra A is called primely, if for $x, y \in A$, $x \wedge y = 0$ implies $x = 0$ or $y = 0$.

Further,we present some examples of finite and infinite primely BL-algebras.

Example 4.2. (i) Let $A = \{0, a, b, c, 1\}$, where $0 < c < a, b < 1$. If the operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

and

$\rightarrow$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1,

then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra [13]. Clearly, A is a primely BL-algebra.

(ii) Let $A = \{-\infty, \ldots, -1, 0, 1\}$. If the operations $*$ and $\rightarrow$ are defined as follows:

$\rightarrow$	$-\infty$	$\ldots$	-3	-2	-1	0	1
$-\infty$	1	$\ldots$	1	1	1	1	1
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\ldots$	1	1	1	1	1
-2	$-\infty$	$\ldots$	-2	-1	1	1	1
0	$-\infty$	$\ldots$	-3	-2	-1	1	1
1	$-\infty$	$\ldots$	-3	-2	-1	0	1

and

$*$	$-\infty$	$\ldots$	-3	-2	-1	0	1
$-\infty$	$-\infty$	$\ldots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\ldots$	-6	-5	-4	-3	-3
-2	$-\infty$	$\ldots$	-5	-4	-3	-2	-2
-1	$-\infty$	$\ldots$	-4	-3	-2	-1	-1
0	$-\infty$	$\ldots$	-3	-2	-1	0	0
1	$-\infty$	$\ldots$	-3	-2	-1	0	1,

then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra [9]. Clearly, A is a primely BL-algebra.

(iii) Consider Example 3.2(iii). Clearly, A is not a primely BL-algebra, since $a \wedge b = 0$, but $a \neq 0$ and $b \neq 0$.

Theorem 4.3. *Let F be a proper filter of a BL-algebra A. Then A/F is a primely BL-algebra if and only if F is a primely filter of A.*

Proof. Let A/F be a primely BL-algebra and $(x \wedge y)^- \in F$, for $x, y \in A$. Then $[x] \wedge [y] = [0]$. Based on the hypothesis, $[x] = [0]$ or $[y] = [0]$. Hence $x^- \in F$ or $y^- \in F$. Therefore F is a primely filter of A. Now, assume that F is a primely filter of A and $[x] \wedge [y] = [0]$, for $[x], [y] \in A/F$. Then $(x \wedge y)^- \in F$. As F is a primely filter of A, $x^- \in F$ or $y^- \in F$. Hence $[x] = [0]$ or $[y] = [0]$. Therefore A/F is a primely BL-algebra. □

Proposition 4.4. *A BL-algebra A is primely if and only if every filter of A is a primely filter.*

Proof. Let A be a primely BL-algebra. Then $A/\{1\}$ is a primely BL-algebra. So, according to Theorem 4.3, $\{1\}$ is a primely filter of A. Therefore every filter of A is a primely filter. Now, let every filter of A be a primely filter. Then $\{1\}$ is a primely filter of A. According to Theorem 4.3, $A/\{1\} \cong A$ is a primely BL-algebra. □

Proposition 4.5. *Every linearly ordered BL-algebra is a primely BL-algebra.*

Proof. Let A be a linearly ordered BL-algebra and $x \wedge y = 0$ for $x, y \in A$. Then, according to the hypothesis, $x \leq y$ or $y \leq x$. Therefore $0 = x \wedge y = x$ or $0 = x \wedge y = y$. Thus A is a primely BL-algebra. □

Remark 4.6. In Example 4.2(i), A is a primely BL-algebra, while it is not a linearly ordered BL-algebra.

Proposition 4.7. *Any integral BL-algebra is a primely BL-algebra.*

Proof. Let $x \wedge y = 0$, for $x, y \in A$. Then as $x * y \leq x \wedge y = 0$, so $x * y = 0$. According to the hypothesis, $x = 0$ or $y = 0$. Therefore A is a primely BL-algebra. □

Remark 4.8. In Example 3.22, A is a primely BL-algebra, while A is not an integral BL-algebra.

Proposition 4.9. *Every BL-algebra with Gödel negation is a primely BL-algebra.*

Proof. Let A be a BL-algebra with Gödel negation. Then according to [4, Lemma 3.11], $D_s(A) = \{x \in A : x^- = 0\} = A - \{0\}$. Now, let $x \wedge y = 0$, for x, $y \in A$. Assume that $x \neq 0$ and $y \neq 0$. Hence x, $y \in D_s(A)$ and so $x * y \in D_s(A)$, since $D_s(A)$ is a proper filter of A. As $x * y \leq x \wedge y = 0$, then $x * y = 0$, which is a contradiction, since $D_s(A)$ is a proper filter of A. Therefore the proof is completed. □

Remark 4.10. In Example 3.20, A is a primely BL-algebra, while A is not with a Gödel negation.

Proposition 4.11. *Every special BL-algebra is a primely BL-algebra.*

Proof. According to [11, Proposition 3.1] and Proposition 4.9, the proof is clear. □

Remark 4.12. In Example 3.22, A is a primely BL-algebra, while not a special BL-algebra.

Proposition 4.13. *Let A be a Gödel algebra and F be a proper filter of A. Then the following conditions are equivalent:*

(i) *A is an integral BL-algebra;*
(ii) *A is a primely BL-algebra;*
(iii) *A is a semi-integral BL-algebra;*
(iv) *A is a local BL-algebra.*

Proposition 4.14. *Let A be a BL-algebra and A have no any zero divisor element. Then A is a primely BL-algebra.*

Proof. Let A have no any zero divisor element and $x \wedge y = 0$, for $x, y \in A$. Then as $x * y \leq x \wedge y = 0$, so, $x * y = 0$. Therefore $x = 0$ or $y = 0$, i.e., A is a primely BL-algebra. □

Remark 4.15. Consider Example 3.20. It is clear that A is a primely BL-algebra and a is a zero divisor element of A.

5. Conclusions

Various logical-algebras have been proposed semantical systems of non-classical logic systems. Among these logical algebras, BL-algebras are basic and very important algebraic structures. BL-algebras have interesting algebraic properties and includes some significant classes of algebras such as MV-algebras. In this paper, primely filters in BL-algebras and a new class of BL-algebras (primely BL-algebras) are defined and characterized. The researchers have presented several different characterizations and their important properties. Also, the connections between primely filters, maximal filters, prime filters and primary filters in a BL-algebra are described.

The researchers intend to focus on applying the results of this paper to other algebraic structures. Since this idea may be used in other algebraic structures associated with logical systems, we hope that this article will pave the ground for further study on algebraic structures associated with logical systems.

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[1]Department of Mathematics, Shahrekord Branch, Islamic Azad University, Shahrekord, Iran

[2]Department of Mathematics, Bandar Abbas Branch, Islamic Azad University, Bandar Abbas, Iran
Email address: yasaman3391@gmail.com
Email address: s.motamed63@yahoo.com
Email address: haghani1351@yahoo.com