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ON THE UNIQUENESS OF THE CAUCHY PROBLEM FOR SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The Cauchy problem for singular functional differential equations is considered. The sufficient conditions of the unique solvability are established.

In the present work, we consider a vector functional differential equation

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1}$$

with the weighted initial codition

$$\lim_{t\to a} \frac{\|x(t) - x_0\|}{h(t)} = 0, \tag{2}$$

where $f : C([a,b];\mathbb{R}^n) \to L_{\rm loc}(]a,b];\mathbb{R}^n)$ is a continuous Volterra operator, $c_0 \in \mathbb{R}^n$, and $h : [a,b] \to [0,+\infty[$ is a continuous function such that $h(t) > 0$ for $0 < t \le b$.

Equation (1) is said to be regular, if the operator f has a summable in $[a,b]$ majorant in every ball of the space $C([a,b];\mathbb{R}^n)$, and is singular, otherwise.

For regular equations of type (1), the Cauchy problem is investigated thoroughly (see [1–6]), but for singular equations this problem remains still little studied.

Throughout the paper, we use the following notation. $\mathbb{R}$ is a set of real numbers; $\mathbb{R}_+ = [0,+\infty[$. $\mathbb{R}^n$ is a space of n-dimensional vector columns $x = (x_i)_{i=1}^n$ with elements $x_i \in \mathbb{R}$ $(i = 1,2,\dots,n)$ and with the norm

$$\|x\| = \sum_{i=1}^n |x_i|,$$

$$\mathbb{R}^n_+ = \Big\{x = (x_i)_{i=1}^n \in \mathbb{R}^n : \ \ x_i \ge 0 \ \ (i = 1,2,\dots,n)\Big\},$$

$$\mathbb{R}^n_\rho = \big\{x \in \mathbb{R}^n : \ \ \|x\| \le \rho\big\}.$$

If $x = (x_i)_{i=1}^n$, then

$$\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^n.$$

$C([a,b];\mathbb{R}^n)$ is the space of vector functions with the norm

$$\|x\|_C = \max\big\{\|x(t)\| : \ \ a \le t \le b\big\},$$

$$C_\rho([a,b];\mathbb{R}^n) = \Big\{x \in C([a,b];\mathbb{R}^n) : \ \ \|x\|_C \le \rho\Big\},$$

$$C([a,b];\mathbb{R}_+) = \Big\{x \in C([a,b];\mathbb{R}) : \ \ x(t) \ge 0 \ \text{ for } \ a \le t \le b\Big\}.$$

If $x \in C([a,b];\mathbb{R}^n)$ and $a \le s \le t \le b$, then

$$\overline{\nu}(x)(s,t) = \max\big\{\|x(\xi)\| : \ \ s \le \xi \le t\big\}.$$

$L_{\rm loc}(]a,b];\mathbb{R}^n)$ is the space of vector functions $x :]a,b] \to \mathbb{R}^n$, summable in every segment contained in $]a,b]$ in which under the convergence is understood a mean convergence on every segment contained in $]a,b]$.

$L_{\rm loc}(]a,b];\mathbb{R}_+) = \big\{x \in L_{\rm loc}(]a,b];\mathbb{R}) : \ x(t) \ge 0 \text{ for almost all } t \in [a,b]\big\}$.

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The solvability and continuity of problem (1), (2) are studied in papers [7] and [8]. To formulate the theorems on the uniqueness, we will need the following

Definition 1. The operator $f : C([a,b];\mathbb{R}^n) \to L_{\rm loc}(]a,b];\mathbb{R}^n)$ is said **to be Volterra** if for any $t_0 \in]a,b]$ and arbitrary vector functions x and $y \in C([a,b];\mathbb{R}^n)$ satisfying the condition

$$x(t) = y(t) \ \text{ for } \ a < t \leq t_0,$$

almost everywhere on $]a,t_0[$, the equality

$$f(x)(t) = f(y)(t)$$

is fulfilled. System (1) with the Volterra right-hand side is called evolutionary.

Definition 2. If $f : C([a,b];\mathbb{R}^n) \to L_{\rm loc}(]a,b];\mathbb{R}^n)$ is the Volterra operator and $b_0 \in]a,b]$, then:

(a) for any $x \in f : C([a,b];\mathbb{R}^n)$, under $f(x)$ is understood a vector function given by the equality

$$f(x)(t) = f(\overline{x})(t) \ \text{ for } \ z`t \leq b,$$

where

$$\overline{x}(t) = \begin{cases} x(t) & \text{for } \ a \leq t \leq b_0 \\ x(b_0) & \text{for } \ b_0 < t \leq b; \end{cases}$$

(b) the continuous vector function $x : [a,b_0] \to \mathbb{R}^n$ is said **to be a solution of system** (1) in the segment $[a,b_0]$ if x is absolutely continuous in every segment contained in $]a,b_0]$ and almost everywhere in $]a,b_0[$ satisfies (1).

Definition 3. We say that the operator $f : C([a,b];\mathbb{R}^n) \to L_{\rm loc}(]a,b];\mathbb{R}^n)$ satisfies **the local Carathéodory conditions** if it is continuous and there is a nondecreasing in the second argument function $\gamma :]a,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\gamma(\,\cdot\,,p) \in L_{\rm loc}(]a,b];\mathbb{R}) \ \text{ for } \ \rho \in \mathbb{R}_+,$$

and for any $x \in C([a,b];\mathbb{R}^n)$, almost everywhere in $]a,b[$, the inequality

$$\|f(x)(t)\| \leq \gamma(t,\|x\|_C)$$

is fulfilled.

Along the whole work, it is assumed that $f : C([a,];\mathbb{R}^n) \to L_{\rm loc}(]a,b];\mathbb{R}^n)$ is the Volterra operator satisfying the local Carathéodory conditions.

Definition 4. The solution x of system (1) defined in the segment $[a,b_0] \subset [a,b[$ **is said to be continuable** if for some $b_1 \in [b_0,b]$ system (1) in the segment $[a,b_1]$ has a solution y satisfying the condition

$$x(t) = y(t) \ \text{ for } \ a \leq t \leq b.$$

A solution x **is called non-continuable**, otherwise.

Definition 5. Problem (1), (2) is said to be locally solvable (globally solvable) if system (1) in some segment $[a,b_0] \subset [a,b[$ has a solution x satisfying the initial condition (2).

Definition 6. Problem (1), (2) is said to be locally uniquely solvable (globally uniquely solvable) if it is locally solvable (globally solvable) and in an arbitrary segment $[a,b_0] \subset [a,b]$ it has no more than one solution.

Definition 7. We say that problem (1), (2) has no more than one solution if for an arbitrary $t_0 \in]a,b[$ it either does not have a solution $[a,t_0]$, or has one and only one solution.

Definition 8. We say that the operator $\omega : C([a,b];\mathbb{R}_+) \to L_{\rm loc}(]a,b[\,;\mathbb{R}_+)$ belongs to the set $U_h([a,b])$ if:

(a) ω is the Volterra one, continuous, does not decrease and $\omega(0)(t) \equiv 0$;

(b) there is a positive number μ_0 and summable functions p and $q : [a,b] \to \mathbb{R}_+$ such that

$$\limsup_{t\to a} \left(\frac{1}{h(t)} \int_a^t p(s)\,ds \right) < 1, \quad \lim_{t\to a} \left(\frac{1}{h(t)} \int_a^t q(s)\,ds \right) = 0 \tag{3}$$

and for any $\mu \in [0,\mu_0]$, almost everywhere in $]a,b[$, the inequality

$$\omega(h\mu)(t) \le p(t)\mu + q(t)$$

is fulfilled;

(c) for any $t_0 \in]a,b]$, the problem

$$\frac{du(t)}{dt} = \omega([a]_+)(t), \quad \lim_{t\to a} \frac{u(t)}{h(t)} = 0$$

has only a trivial solution $[a,t_0]$, where

$$[u]_+ = \Big(\frac{|u_i| + u_i}{2} \Big)_{i=1}^n .$$

Lemma 1. *Let $\omega : C([a,b];\mathbb{R}_+) \to L_{\rm loc}(]a,b];\mathbb{R}_+)$ be the operator given by the equality*

$$\omega(u)(t) = p(t)\nu\Big(\frac{u}{h}\Big)(a,t),$$

where $p : [a,b] \to \mathbb{R}_+$ is a summable function such that

$$\limsup_{t\to a} \left(\frac{1}{h(t)} \int_a^t p(s)\,ds \right) < 1.$$

Then $\omega \in U_h([a,b])$.

Lemma 2. *Let $b_0 \in]a,a+1[$, $h : [a,b_0] \to \mathbb{R}_+$ be a nondecreasing function, $\ell > 0$, $\varepsilon > 0$, $\lambda_i \ge 1$ $(i = 1,\dots,m)$, and let $\omega : C([a,b_0];\mathbb{R}_+) \to L_{\rm loc}([a,b];\mathbb{R}_+)$ be the operator given by the equality*

$$\omega(u)(t) = e(t-a)^{\varepsilon-1}h(t) \sum_{i=1}^m \Big[\nu\Big(\frac{u}{h}\Big)\big(a, a + (t-a)^{\lambda_i}\big) \Big]^{\frac{1}{\lambda_i}} .$$

Let, moreover,

$$\ell(b_0 - a)^{\frac{\varepsilon}{2}} < \frac{\varepsilon}{m} .$$

Then $\omega \in U_h([a,b_0])$.

Lemma 3. *Let m be a natural number, $b_0 \in]a,a+1[$, $\ell > 0$, $h : [a,b_0] \to \mathbb{R}_+$ be a nondecreasing function and let $\omega : C([a,b_0];\mathbb{R}_+) \to L_{\rm loc}(]a,b_0];\mathbb{R}_+)$ be the operator given by the equality*

$$\omega(u)(t) = \ell(t-a)^{\varepsilon-1}h(t) \sum_{i=1}^m \Big[\ln_i \Big(\frac{1}{\nu(\frac{u}{h})(a, a + e_{in}^{-1}(\frac{1}{t-a}))} \Big) \Big]^{-1} .$$

Then $\omega \in U_h([a,b_0])$.

Theorem 1. *Let the function h be nondecreasing and for every positive number ρ let there exist the operator*

$$\omega_\rho \in U_h([a,b])$$

such that for arbitrary y and $z \in C_\rho([a,b];\mathbb{R}^n)$, almost everywhere in $]a,b[$, the inequality

$$\big[f(c_0 + hy)(t) - f(c_0 + hz)(t) \big] \operatorname{sgn}(y(t) - z(t)) \le \omega_\rho(h\|y - z\|)(t) \tag{4}$$

is fulfilled. Then problem (1), (2) has no more than one solution.

Theorem 2. *Let the function h be nondecreasing and for every positive number ρ let there exist the operator $\omega_\rho \in U_h([a,b])$ such that for arbitrary y and $z \in C_\rho([a,b];\mathbb{R}^n)$, almost everywhere in $]a,b[$, inequality* (4) *is fulfilled. Let, moreover,*

$$\lim_{t\to a}\left(\frac{1}{h(t)}\int\limits_a^t \|f(c_0)(s)\|\,ds\right) = 0. \tag{5}$$

Then problem (1), (2) *is locally uniquely solvable.*

Corollary 1. *Let the function h be nondecreasing and for every positive number ρ let there be a summable function $p_\rho : [a,b] \to \mathbb{R}_+$ such that*

$$\limsup_{t\to a}\left(\frac{1}{h(t)}\int\limits_a^t p_\rho(s)\,ds\right) < 1 \tag{6}$$

and for arbitrary y and $z \in C_\rho([a,b];\mathbb{R}^n)$, almost everywhere in $]a,b[$, the inequality

$$\big[f(c_0+hy)(t) - f(c_0+hz)(t)\big]\operatorname{sgn}(y(t)-z(t)) \le p_\rho(t)\nu(\|y-z\|)(t)$$

is fulfilled. Then problem (1), (2) *has no more than one solution.*

Corollary 2. *If along with the conditions of Corollary* 1, *equality* (5) *is likewise fulfilled, then problem* (1), (2) *has one and only one non-continuable solution.*

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