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DUALITY AND INTERPOLATION FOR WEIGHTED GRAND MORREY SPACES

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Abstract. Complex interpolation and duality problems for two-weighted grand Morrey spaces are studied. Some interpolation statements are applied to obtain the boundedness of linear operators of harmonic analysis in the aforementioned spaces.

1. Introduction

In this note, we present our results regarding the duality and complex interpolation of weighted grand Morrey spaces $L^{p),\lambda,\varphi(\cdot)}_{w,v}$ defined on quasi-metric measure spaces (X,d,μ). Characterization of the predual space of $L^{p),\lambda,\varphi(\cdot)}_{w,v}$ is also given. If $\varphi(\cdot)$ is a constant, then the function space is the classical weighted Morrey space denoted by $L^{p,\lambda}_{w,v}$. The results in this note are new even for $L^{p,\lambda}_{w,v}$. The spaces are defined on a quasi-metric measure space (X,d,μ), where X is an abstract set, d is a quasi-metric on $X\times X$, and μ is a finite measure on X, i.e., $\mu(X)<\infty$, such that all balls are μ-measurable. We write $B(x,r)$ for the ball with center x and radius r. If μ satisfies the doubling condition, i.e., $\mu(B(x,2r)\le D_\mu B(x,r)$, where D_μ is the doubling constant, then (X,d,μ) is called a space of homogeneous type (SHT for short). Examples of SHT include the following:

(a) domains Ω in $\mathbb{R}^n$ satisfying the condition $|B(x,r)\cap\Omega|\ge Cr^n$, $x\in\Omega$, with the positive constant C independent of x and r, where $|E|$ denotes the Lebesgue measure of E;

(b) rectifiable regular (Carleson) curves in $\mathbb{C}$ with the Euclidean metric and arc-length measure;

(c) nilpotent Lie groups with Haar measure (homogeneous groups).

For additional properties on SHT see, e.g., [3,19,30].

Let w and v be weight functions on X, i.e., w and v are μ-a.e. positive integrable functions on X. Recalling that $\|f\|_{L^p_w(E)}:=(\int_E|f(x)|^pw(x)d\mu(x))^{1/p}$, the spaces $L^{p,\lambda}_{w,v}(X)$ and $L^{p),\lambda,\varphi(\cdot)}_{w,v}$ are defined via the norms

$$\|f\|_{L^{p,\lambda}_{w,v}(X)}:=\sup_B\frac{1}{\big(v(B)\big)^{\lambda/p}}\|f\|_{L^p_w(B)},\quad \|f\|_{L^{p),\lambda,\varphi(\cdot)}_{w,v}(X)}:=\sup_{0<\varepsilon<p-1}\varphi(\varepsilon)\|f\|_{L^{p-\varepsilon,\lambda}_{w,v}(X)},$$

where $1<p<\infty$, $0<\lambda<1$, and $\varphi(\cdot)$ is a non-increasing function on $(0,p-1)$ satisfying the following condition: there are the positive constants C_1 and C_2 such that

$$C_1\varphi(t/2)\le\varphi(t)\le C_2\varphi(t/2).$$

We denote the class of such φ functions by $\mathcal{W}$ (cf., [17]).

For simplicity, if $\varphi(x)=x^\theta$, then we denote $\|f\|_{L^{p),\lambda,\varphi(\cdot)}_{w,v}(X)}$ by $\|f\|_{L^{p),\lambda,\theta}_{w,v}(X)}$. If $v=w$, we write $L^{p,\lambda}_{w,v}(X)$ and $L^{p),\lambda,\varphi(\cdot)}_{w,v}(X)$ as $L^{p,\lambda}_{w}(X)$ and $L^{p),\lambda,\varphi(\cdot)}_{w}(X)$, respectively. If $v\equiv1$, then we use the notation $\mathcal{L}^{p,\lambda}_w(X)$ and $\mathcal{L}^{p),\lambda,\varphi(\cdot)}_w(X)$, respectively. The space $L^{p,\lambda}_w(\mathbb{R}^n)$ was introduced and studied in [15], while $\mathcal{L}^{p,\lambda}_w(\mathbb{R}^n)$ was introduced and discussed in [28].

We now list some properties of weighted grand Morrey spaces:

(i)

$$L^{p,\lambda}_{w,v}(X)\hookrightarrow L^{p,\lambda,\varphi(\cdot)}_{w,v}(X)\hookrightarrow L^{p-\varepsilon,\lambda}_{w,v}(X),\quad \varepsilon\in(0,p-1);$$

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(ii) if the condition

$$\sup_B \frac{w(B)}{v(B)^\lambda} < \infty \tag{1.1}$$

is satisfied, then

$$\mathcal{L}^{p,\lambda}_{w,v}(X) \hookrightarrow \mathcal{L}^{p,\lambda,\varphi(\cdot)}_{w,v}(X) \hookrightarrow \mathcal{L}^{p-\varepsilon,\lambda}_{w,v}(X), \quad \varepsilon \in (0, p-1).$$

Condition (1.1) was introduced in [21]. In that note, Rubio de Francia's extrapolation statements were presented in the framework of the weighted Morrey spaces.

It can be checked that, for example, if $v \equiv 1$ and the measure μ satisfies the condition

$$c_1 r^N \leq \mu B(x,r) \leq c_2 r^N \tag{1.2}$$

with some positive constants c_1, c_2 and N, then the weight function $w(x) = d(x_0, x)^\beta$, where x_0 is a point of X and $\beta \geq N - \lambda N$, satisfies condition (1.1) (see [21]).

Mapping properties of operators of harmonic analysis in the spaces $L^{p,\lambda,\varphi(\cdot)}_w(X)$ were studied in [11, 12, 14] under the Muckenhoupt condition on w. The weighted extrapolation and boundedness in $\mathcal{L}^{p,\lambda,\theta}_w(X)$ for weights beyond the Muckenhoupt range were established in [21].

For $w \equiv 1$, the space $\mathcal{L}^{p),\lambda,\theta}_w(X)$, denoted by $\mathcal{L}^{p),\lambda,\theta}(X)$, was introduced and studied in [20]. Later, in [26], the author introduced generalized grand Morrey spaces $\mathcal{L}^{p),\lambda),\theta}(X)$, where the aggrandization is simultaneously applied to p and λ. Grand Morrey spaces $\mathcal{L}^{p),\lambda,\theta}$ are generalizations of grand Lebesgue spaces $L^{p)}$ introduced in 1992 by Iwaniec and Sbordone [9] in their studies associated with the integrability properties of the Jacobian in a bounded open set Ω. A generalized version of them, $L^{p),\theta}(\Omega)$, $\theta > 0$, appeared in Greco, Iwaniec and Sbordone [7], where the authors investigated the solvability of nonhomogeneous n-harmonic equation div $A(x, \nabla u) = \mu$. The space associate to $L^{p),\theta}$ is called a small Lebesgue space (see [4]). The grand Lebesgue space, as proved in [4], is a non-separable and non-reflexive Banach space.

Morrey spaces $\mathcal{L}^{p,\lambda}$ were introduced in 1938 by Morrey [22] in connection with the regularity problems of solutions to PDEs and provided a useful tool in the regularity theory of PDEs. In recent years, there is an increasing interest in applications of Morrey spaces in various areas of analysis such as partial differential equations, potential theory and harmonic analysis.

The study of interpolation on classical Morrey spaces started with Stampacchia in [29], Campanato and Murthy [2], and Peetre [25]. In 1990's, Ruiz and Vega [27] and Blasco, Ruiz and Vega [1] showed that, in general, Morrey spaces have no interpolation properties. Lemarié–Rieusset in [16] further pointed out explicitly that Morrey spaces have no interpolation properties if the parameters of two Morrey spaces are different. It was shown in Lu, Yang, Yuan [18] that by using $\pm$ interpolation method, the interpolation space of two Morrey spaces is also a Morrey space. The same property was proved in [17] for unweighted grand Morrey spaces. In view of this, it is natural to ask whether we can interpolate the generalized weighted grand Morrey spaces, as well.

2. Main Results

2.1. **Predual spaces.** First, we present the results dealing with the predual space of weighted grand Morrey spaces $L^{p),\lambda,\theta}_{w,v}(X)$ (see [17] for an unweighted case). To this end, we need some definitions.

Definition 2.1. Let $p \in (1, \infty)$ and $\lambda \in [0, 1)$. We say that a μ-measurable function b on X is a *(p', λ, v, w)-block* if b is supported on a ball B and satisfies

$$\left(\int_B |b(x)|^{p'} w^{1-p'} d\mu(x) \right)^{1/p'} [v(B)]^{\lambda/p} \leq 1.$$

Furthermore, let $\varphi \in \mathcal{W}$. We say that a μ-measurable function b on X is a *$(p', \lambda, \varphi(\cdot), v, w)$-block* if b is supported on a ball B and additionally satisfies the condition

$$\inf_{0<\varepsilon<p-1} [\varphi(\varepsilon)]^{-1} \left(\int_B |b(x)|^{(p-\varepsilon)'} w^{1-(p-\varepsilon)'} d\mu(x) \right)^{1/(p-\varepsilon)'} [v(B)]^{\lambda/(p-\varepsilon)} \leq 1.$$

Definition 2.2. Suppose that $p \in (1,\infty)$ and $\lambda \in [0,1)$. The space $B^{p',\lambda}_{w,v}(X)$ is defined to be the collection of all μ-measurable functions f which can be represented as $f = \sum\limits_{i\in\mathbb{Z}} t_i b_i$ μ-a.e., where $\{t_i\}_i \in l^1$ and $\{b_i\}_i$ is a sequence of (p',λ,v,w)-blocks. Moreover, let

$$\|f\|_{B^{p',\lambda}_{w,v}(X)} := \inf\left\{\|\{t_i\}_i\|_{l^1} : \; f = \sum_{i\in\mathbb{Z}} t_i b_i\right\},$$

where the infimum is taken over all possible decompositions of f. Moreover, $B^{p',\lambda,\varphi(\cdot)}_{w,v}(X)$ is the collection of all μ-measurable functions f on X which can be represented as $f = \sum\limits_{i\in\mathbb{Z}} t_i b_i$ μ-a.e., where $\{t_i\}_i \in l^1$ and $\{b_i\}_i$ is a sequence of $(p',\lambda,\varphi(\cdot),v,w)$-blocks. In this case, we set

$$\|f\|_{B^{p',\lambda,\varphi(\cdot)}_{w,v}(X)} := \inf\left\{\|\{t_i\}_i\|_{l^1} : \; f = \sum_{i\in\mathbb{Z}} t_i b_i\right\}.$$

Now, we give the characterization of the predual space for the classical two-weighted Morrey space $L^{p,\lambda}_{w,v}(X)$:

Theorem 2.3. *Let $p \in (1,\infty)$ and $\lambda \in [0,1)$. Assume that w, v are the weights on X and (1.1) is satisfied. Then the dual space of $B^{p',\lambda}_{w,v}(X)$ is the Morrey space $L^{p,\lambda}_{w,v}(X)$ in the following sense: for any $g \in L^{p,\lambda}_{w,v}(X)$, the integral $\int\limits_X f(x)g(x)d\mu(x)$ induces a bounded linear functional on $B^{p',\lambda}_{w,v}(X)$; conversely, for any $L \in \left(B^{p',\lambda}_{w,v}(X)\right)^*$, there exists $g \in L^{p,\lambda}_{w,v}(X)$ such that $L(f) = \int\limits_X f(x)g(x)d\mu(x)$ for all $f \in B^{p',\lambda}_{w,v}(X)$.*

The next statement deals with the predual space of the two-weighted grand Morrey space.

Theorem 2.4. *Let $p \in (1,\infty)$, $\lambda \in [0,1)$ and $\varphi \in \mathcal{W}$. Suppose that w, v are weight functions on X and (1.1) is satisfied. Then the dual space of $B^{p',\lambda,\varphi(\cdot)}_{w,v}(X)$ is $L^{p),\lambda,\varphi(\cdot)}_{w,v}(X)$ in the following sense: for any $g \in L^{p),\lambda,\varphi(\cdot)}_{w,v}(X)$, the functional $\int\limits_X f(x)g(x)d\mu(x)$ induces a bounded linear functional on $B^{p',\lambda,\varphi(\cdot)}_{w,v}(X)$; conversely, for any $L \in \left(B^{p',\lambda,\varphi(\cdot)}_{w,v}(X)\right)^*$, there exists $g \in L^{p),\lambda,\varphi(\cdot)}_{w,v}(X)$ such that $L(f) = \int\limits_X f(x)g(x)d\mu(x)$ for all $f \in B^{p',\lambda,\varphi(\cdot)}_{w,v}(X)$.*

2.2. **Interpolation results.** We have investigated interpolation of two-weighted (grand) Morrey spaces via the complex interpolation.

We now recall some definitions from the general interpolation theory.

Let X_0, X_1 be a couple of quasi-Banach spaces which are continuously embedded into a Hausdorff topological vector space Y. The space $X_0 + X_1$ is defined by

$$X_0 + X_1 := \left\{y \in Y : \text{ there exists } y_i \in X_i, \; i \in \{0,1\} \text{ such that } y = y_0 + y_1\right\},$$

and its norm by

$$\|y\|_{X_0+X_1} := \inf\left\{\|y_0\|_{X_0} + \|y_1\|_{X_1} : \; y = y_0 + y_1, \;\; y_i \in X_i, \; i \in \{0,1\}\right\}.$$

Furthermore, the space $X_0 \cap X_1$ is defined with respect to the norm

$$\|y\|_{X_0\cap X_1} = \max\left\{\|y_0\|_{X_0}, \|y_1\|_{X_1} : \; y = y_0 + y_1, \;\; y_i \in X_i, \; i \in \{0,1\}\right\}.$$

A quasi-Banach space X is called an *intermediate space* with respect to X_0+X_1 if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ with continuous embeddings. If X is an intermediate space with respect to $X_0 + X_1$, let X^0 denote the closure of $X_0 \cap X_1$ in X, and let the *Gagliardo closure* of X with respect to X_0+X_1, written $X^{\sim}$, be defined as follows: $a \in X^{\sim}$ if and only if there exists a sequence $\{a_i\}_{i\in\mathbb{N}}$ such that $a_i \to a$ in $X_0 + X_1$ and $\|a_i\|_X \le \lambda < \infty$. Moreover, $\|a\|_{X^{\sim}} := \inf\{\lambda\}$.

Definition 2.5. Suppose that (X_0, X_1) is a pair of quasi-Banach spaces and $t \in (0,1)$.

(i) (The Gagliardo-Peetre method). We say that $a \in \langle X_0, X_1\rangle_t$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ in $X_0 + X_1$ and, for any bounded sequence $\{\varepsilon_i\}_{i\in Z} \subset \mathbb{C}$, the series $\sum_{i\in\mathbb{Z}} \varepsilon_i 2^{i(j-t)a_i}$ converges in X_j, $j \in \{0,1\}$. Moreover, for $j \in \{0,1\}$,

$$\Big\| \sum_{i\in\mathbb{Z}} \varepsilon_i 2^{i(j-t)} a_i \Big\|_{X_j} \le C \sup_{i\in\mathbb{Z}} |\varepsilon_i|$$

for a constant $C \ge 0$, independent of $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and $\{a_i\}_{i\in\mathbb{Z}}$. Define $\|a\|_{\langle X_0,X_1\rangle_t} := \inf\{C\}$.

(ii) (*The* $\pm$ *method*). We say that $a \in \langle X_0, X_1, t\rangle$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ in $X_0 + X_1$ and, for any finite subset $F \in \mathbb{Z}$ and bounded sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$, and $j \in \{0,1\}$,

$$\Big\| \sum_{i\in F} \varepsilon_i 2^{i(j-t)} a_i \Big\|_{X_j} \le C \sup_{i\in\mathbb{Z}} |\varepsilon_i|$$

for a constant $C \ge 0$, independent of F, $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and $\{a_i\}_{i\in\mathbb{Z}}$. Let $\|a\|_{\langle X_0,X_1,t\rangle} := \inf\{C\}$.

Definition 2.6. A quasi-Banach space X of complex-valued measurable functions is called a *quasi-Banach lattice* if, for any $f \in X$ and g satisfying $|g| \le |f|$, we have that $g \in X$ and $|g|_X \le |f|_X$.

Given two quasi-Banach lattices X_0, X_1 and $t \in (0,1)$, their *Calderón product* $X_0^{1-t}X_1^t$ is defined by

$$X_0^{1-t}X_1^t := \Big\{ f \text{ is a complex-valued measurable function: there exist } f^0 \in X_0,\ f^1 \in X_1 \text{ such that } |f| \le |f^0|^{1-t}|f^1|^t \Big\}$$

and its norm is given by $\|f\|_{X_0^{1-t}X_1^t} := \inf\{\|f^0\|_{X_0}^{1-t}\|f^1\|_{X_1}^t\}$, where the infimum is taken over all $f^0 \in X_0$ and $f^1 \in X_1$ such that $|f| \le |f^0|^{1-t}|f^1|^t$.

The following statement for the unweighted case is well-known (see [18, Theorem 1.2] and also [32]).

Theorem A. *Let* $t \in (0,1)$, $\lambda \in [0,1)$, *and* $p, p_0, p_1 \in (1,\infty]$ *be such that* $(1-t)/p_0 + t/p_1 = 1/p$. *Then*

$$\langle L^{p_0,\lambda}(X), L^{p_1,\lambda}(X)\rangle_t = \Big(\big[L^{p_0,\lambda}(X)\big]^{1-t} \big[L^{p_1,\lambda}(X)\big]^t \Big)^\circ = L^{p,\lambda}(X)^\circ$$

and

$$\langle L^{p_0,\lambda}(X), L^{p_1,\lambda}(X), t\rangle = \big[L^{p_0,\lambda}(X)\big]^{1-t} \big[L^{p_1,\lambda}(X)\big]^t = L^{p,\lambda}(X).$$

Our result regarding the interpolation is the following statement.

Theorem 2.7. *Let* $t \in (0,1)$, $\lambda \in [0,1)$, *and* $p, p_0, p_1 \in (1,\infty]$ *be such that* $(1-t)/p_0 + t/p_1 = 1/p$. *Suppose that* $\varphi, \varphi_0, \varphi_1 \in \mathcal{W}$ *so,* $\varphi, \varphi_0^{p/p_0}$ *and* φ_1^{p/p_1} *are equivalent. Then the following equalities:*

$$\langle L_{w,v}^{p_0),\lambda,\varphi_0(\cdot)}(X), L_{w,v}^{p_1),\lambda,\varphi_1(\cdot)}(X)\rangle_t = L_{w,v}^{p),\lambda,\varphi(\cdot)}(X)^\circ,$$

and

$$\langle L_{w,v}^{p_0),\lambda,\varphi_0(\cdot)}(X), L_{w,v}^{p_1),\lambda,\varphi_1(\cdot)}(X), t\rangle = L_{w,v}^{p),\lambda,\varphi(\cdot)}(X),$$

hold, where $L_{w,v}^{p),\lambda,\varphi(\cdot)}(X)^\circ$ *denotes the closure of* $L_{w,v}^{p_0),\lambda,\varphi_0(\cdot)}(X) \cap L_{w,v}^{p_1),\lambda,\varphi_1(\cdot)}(X)$ *in* $L_{w,v}^{p),\lambda,\varphi(\cdot)}(X)$.

As a consequence, we have interpolation of linear operators in two-weighted grand Morrey spaces.

Corollary 2.8. *Let* (X_0, X_1) *be a couple of quasi-Banach spaces. Under the hypotheses of Theorem* 2.7*, we have:*

(i) *If a linear operator* T *is bounded from* $L_{w,v}^{p_j),\lambda,\varphi_j}(X)$ *to* A_j *with operator norms* M_j, $j \in \{0,1\}$, *then* T *is also bounded from* $L_{w,v}^{p),\lambda,\varphi}(X)$ *to* $\langle A_0, A_1, t\rangle$ *with the operator norm not greater than a positive constant multiple of* $M_0^{1-t}M_1^t$.

(ii) *If a linear operator T is bounded from A_j to $L^{p_j),\lambda,\varphi_j}_{w,v}(X)$ with operator norms M_j, $j \in \{0,1\}$, then T is also bounded from $\langle A_0, A_1, t\rangle$ to $L_{w,v),\lambda,\varphi}(X)$ with the operator norm not greater than a positive constant multiple of $M_0^{1-t}M_1^t$.*

We say that a weight function w belongs to the Muckenhoupt class $A_p(X)$ (A_p for short), $1 < p < \infty$, if

$$\big[w\big]_{A_p} := \sup_B \left(\frac{1}{\mu(B)}\int_B w(x)d\mu(x)\right)\left(\frac{1}{\mu(B)}\int_B w^{1-p'}(x)d\mu(x)\right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset X$. The symbol $\big[w\big]_{A_p}$ is called the characteristic of w. Moreover, a weight w belongs to the Muckenhoupt class $A_1(X)$ if $Mw(x) \le Cw(x)$ a.e., where M is the so-called *Hardy–Littlewood maximal operator* defined by

$$Mw(x) := \sup_{B\ni x}\frac{1}{\mu(B)}\int_B w(y)d\mu(y). \tag{2.1}$$

It is easy to see that for $1 \le p \le s < \infty$, the inclusion $A_p \subset A_s$ holds. One of the important properties of the Muckenhoupt class is the so-called "openness property" (see, e.g., [30]): if $w \in A_p$, then there is a positive constant σ such that $w \in A_{p-\sigma}$. It is worth noting that the value of such constant σ is known and can be found in [8].

It is known (see, e.g., [30] for the case of SHT and [6] for Euclidean spaces) that for many operators of harmonic analysis, the A_p condition is a criterion for their boundedness in the weighted Lebesgue space $L^p_w(X)$, $1 < p < \infty$. Furthermore, the A_p-condition guarantees the boundedness of operators of harmonic analysis in weighted Morrey spaces $L^{p,\lambda}_w(X)$ (see [15] for $\mathbb{R}^n$ and [10] for SHT). We refer also to [11, 12, 14] for the boundedness of various integral operators in weighted grand Morrey spaces $L^{p),\lambda,\theta}_w(X)$ under the A_p-condition.

It is also known that the Hardy–Littlewood maximal and singular integral operators are bounded in $\mathcal{L}^{s,\lambda}_{|x|^\beta}(\mathbb{R}^n)$ for $\lambda n - n < \beta < n\lambda + n(p-1)$ (see [23, 28, 31]). We refer, e.g., to [21] for similar results in the case of SHT with condition (1.2) on a measure and weight $d(x_0, x)^\beta$, where x_0 is a point in X.

If we use Corollary 2.8 for the classical weighted Morrey spaces $L^{p,\lambda}_w(X)$ for $w \in A_p(X)$, and take into account the openness property of the $A_p(X)$ weights, we have

Corollary 2.9. *Let $1 < p < \infty$, $\lambda \in (0,1)$ and $w \in A_p$. If a linear operator T is bounded in $L^{p,\lambda}_w(X)$ for all $p \in (1,\infty)$ and $\lambda \in (0,1)$, then T is bounded in $L^{p),\lambda,\varphi}_w(X)$ for $1 < p < \infty$, $\lambda \in (0,1)$, $\varphi(\cdot) \in \mathcal{W}$ and $w \in A_p(X)$.*

Corollary 2.10. *Let $1 < p < \infty$ and $\lambda \in (0,1)$. Suppose that x_0 is a point in X and let $\lambda n - n < \beta < n\lambda + n(p-1)$. If a linear operator T is bounded in $\mathcal{L}^{p,\lambda}_{d(x_0,x)^\beta}(X)$ for all $p \in (1,\infty)$ and $\lambda \in (0,1)$, then T is bounded in $\mathcal{L}^{p),\lambda,\varphi(\cdot)}_{d(x_0,x)^\beta}(X)$ for $1 < p < \infty$, $\lambda \in (0,1)$, and $\varphi(\cdot) \in \mathcal{W}$, and $w \in A_p(X)$.*

Definition 2.11 ([17])**.** Let $t \in (0,1)$, $\lambda \in [0,1)$, and $p, p_0, p_1 \in (1,\infty]$ be such that $(1-t)/p_0 + t/p_1 = 1/p$. For $\varepsilon \in [0, p-1]$ and $s, t \in [0,\infty]$, define

$$\widetilde{p_i}(\varepsilon) := p_i - \frac{\varepsilon}{p-1}(p_i - 1), \;\; i \in \{0,1\}, \quad H(s,\tau) := \frac{1}{(1-t)/s + t/\tau},$$
$$h(\varepsilon) := H(\widetilde{p_0}(\varepsilon), \widetilde{p_1}(\varepsilon)), \quad p_i(\varepsilon) := \widetilde{p_i}(h^{-1}(p-\varepsilon)), \;\; i \in \{0,1\}.$$

To prove the main result, we will need some auxiliary statements:

Proposition 2.12 ([17])**.** *Let all the notations be as in Definition 2.11. Then $p_i(\varepsilon)$ is continuous, strictly decreasing and satisfies the following conditions:*

$$p_i(0) = p_i, \quad p_i(p-1) = 1, \quad \frac{1-t}{p_0(\varepsilon)} + \frac{t}{p_1(\varepsilon)} = \frac{1}{p-\varepsilon}, \quad \lim_{\varepsilon\to 0+}\frac{p_i - p_i(\varepsilon)}{\varepsilon} = \frac{p_0p_1(p_i-1)}{p(p+p_0p_1-p_0-p_1)}.$$

Corollary 2.13 ([17])**.** *There exist positive constants K_1 and K_2 such that for all $\varepsilon \in [0, p-1]$, $K_1\varepsilon \le p_i - p_i(\varepsilon) \le K_2\varepsilon$ and, if $\varphi \in \mathcal{W}$, then $\varphi(p_i - p_i(\cdot)) \in \mathcal{W}$ and is equivalent to φ.*

Moreover, we have

Theorem 2.14. *Let $t \in (0,1)$, $\lambda \in [0,1)$ and $p, p_0, p_1 \in (1,\infty]$ be such that $(1-t)/p_0 + t/p_1 = 1/p$. Assume additionally that $\varphi, \varphi_0, \varphi_1 \in \mathcal{W}$ so, $\varphi, \varphi_0^{p/p_0}$ and φ_1^{p/p_1} are equivalent. Then*

$$\left[L^{p_0),\lambda,\varphi_0}_{w,v}(X)\right]^{1-t}\left[L^{p_1),\lambda,\varphi_1}_{w,v}(X)\right]^{t} = L^{p),\lambda,\varphi(\cdot)}_{w,v}(X).$$

Definition 2.15. A quasi-Banach lattice X is said to be of type $\mathcal{E}$ if there exists an equivalent lattice quasi-norm on X such that for some $p \geq 1$, $X^{(p)}$ is a Banach lattice in this norm.

We use the symbol $X^{(p)}$ to denote the p-convexification of X. Notice that for $\delta \in (0, \min(1,p)]$, the $1/\delta$-convexification $\left(L^{p),\lambda,\varphi(\cdot)}_{w,v}(X)\right)^{(1/\delta)}$ of the generalized grand Morrey space is a Banach space, namely, the generalized grand Morrey space $L^{p),\lambda,\varphi(\cdot)}_{w,v}(X)$ is of type $\mathcal{E}$.

Theorem 2.16. *Let X_0 and X_1 be two quasi-Banach lattices of type $\mathcal{E}$. Then*

$$\langle X_0, X_1\rangle_t = (X_0^{1-t}, X_1^t)^\circ$$

and

$$X_0^{1-t}, X_1^t \subset \langle X_0, X_1, t\rangle \subset (X_0^{1-t}, X_1^t)^\sim.$$

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