Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 1, 147–148

BOUNDEDNESS WEIGHTED CRITERIA FOR MULTILINEAR RIEMANN-LIOUVILLE INTEGRAL OPERATORS

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Abstract. The necessary and sufficient condition on a weight function v governing the weighted inequality

$$\|R_\alpha(f_1,\dots,f_m)\|_{L^q_v(\mathbb{R}_+)} \le C\prod_{k=1}^m \|f_k\|_{L^{p_k}(\mathbb{R}_+)}$$

for the one-sided multilinear fractional integral operator

$$R_\alpha(f_1,\dots,f_m)(x) = \int\limits_{(0,x)^m} \frac{f_1(x-t_1)\dots f_m(x-t_m)}{(t_1+\dots+t_m)^{m-\alpha}}dt_1\dots dt_n$$

is established.

In this note, we present the weighted criteria for the weighted inequality

$$\|R_\alpha(f_1,\dots,f_m)\|_{L^q_v(\mathbb{R}_+)} \le C\prod_{k=1}^m \|f_k\|_{L^{p_k}(\mathbb{R}_+)}, \quad f_k\in L^{p_k}(\mathbb{R}_+),\ \ k=1,\dots,m, \tag{1}$$

and for the one-sided multilinear fractional integral operator

$$R_\alpha(f_1,\dots,f_m)(x) = \int\limits_{(0,x)^m} \frac{f_1(x-t_1)\dots f_m(x-t_m)}{(t_1+\dots+t_m)^{m-\alpha}}dt_1\dots dt_m.$$

Taking $\alpha=m$, we have multilinear Hardy operator $H := R_m$. The two-weight criteria for H in the bilinear case (i.e., for $m=2$) were derived in [1].

The operator R_α is the one-sided variant of the multilinear fractional integral operator

$$\mathscr{I}_\gamma(f_1,\dots,f_m)(x) = \int\limits_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\dots+|x-y_m|)^{mn-\gamma}}dy_1\dots dy_m, \quad x\in\mathbb{R}^n,\ \ 0<\gamma<nm$$

(see also [2] for R_α).

The operator $\mathscr{I}_\gamma$ is a very natural intermediate operator (written in an m-linear form) between $(I_{\alpha_1}f_1)(I_{\alpha_2}f_2)$ and $B_{\alpha_1+\alpha_2}(f_1,f_2)$ (see [6]), where

$$I_\gamma f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}}\,dy, \quad x\in\mathbb{R}^n, \quad 0<\gamma<n,$$

is the Riesz potential, and

$$B_\alpha(f_1,f_2)(x) = \int\limits_{\mathbb{R}^n} \frac{f_1(x+t)f_2(x-t)}{|t|^{n-\alpha}}dt, \quad 0<\alpha<n,$$

is the bilinear fractional integral operator introduced and studied in [3,4].

The one-weight characterization for I_γ in terms of vector type Muckenhoupt–Wheeden condition was given in [9], while the trace inequality criterion for I_γ in terms of D. Adams condition was found in [7].

2020 *Mathematics Subject Classification.* 26A33, 47G10, 47H60.
Key words and phrases. Multilinear one-sided potentials; Weighted inequality; Boundedness.

Let $1 < s < \infty$, Ω be a domain in $\mathbb{R}^n$ and w be an a positive a.e. locally integrable function on Ω. Denote by $L^s_w(\Omega)$ the weighted Lebesgue space defined with respect to the norm $\|f\|_{L^s_w(\Omega)} := \Big(\int\limits_{\Omega} |f(x)|^s w(x)dx\Big)^{1/s}$.

We assume that p is defined by the identity

$$\frac{1}{p} = \sum_{j=1}^{m} \frac{1}{p_j}. \tag{2}$$

Theorem 1. *Let* $1 < \min\{p_1, \dots, p_m\} \leq \max\{p_1, \dots, p_m\} \leq q < \infty$ *and let* $\alpha > 1/p$*, where* p *is defined by* (2)*. Then inequality* (1) *holds if and only if*

$$\sup_k \Bigg(\int\limits_{2^k}^{2^{k+1}} v(x)dx \Bigg)^{1/q} 2^{k(\alpha - 1/p)} < \infty. \tag{3}$$

This statement for the linear case (i.e., for $m = 1$) was proved in [8] (see also [10]).

In the bilinear ($m = 2$) case, we have the following statement.

Theorem 2. *Let* $m = 2$*,* $1 < \min\{p_1, p_2\} \leq q < \infty$ *and let* $\alpha > 1/p$*, where* p *is defined by* (2)*. Then inequality* (1) *holds if and only if condition* (3) *holds.*

Finally, we mention that some one-weight estimates for R_α were derived in [5].

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(Received 12.08.2023)

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