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ON MEASURABLE HULLS AND MEASURABLE KERNELS OF ALMOST INVARIANT SETS

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Abstract. In this paper, some applications of measurable hulls and measurable kernels of almost invariant sets are given to the measure extension problem for σ-finite invariant (quasi-invariant) measures.

In the present paper, we discuss one approach to non-measurable sets in the general theory of invariant (quasi-invariant) measures. Namely, we show that some properties of almost invariant sets and their measurable hulls and kernels are useful for the measure extension problem and for certain constructions of non-measurable sets in a measure space (E, G, S, μ), where E is a ground set, G is a group of transformations of E, and S is some σ-algebra of subsets of E, and μ is a σ-finite measure on S.

Throughout this article, we use the following standard notation:

$\mathbf{R}$ is the set of all real numbers;

$\mathbf{c}$ is the cardinality of the continuum (i.e., $\mathbf{c} = 2^{\omega}$);

μ_* is the inner measure associated with the given measure μ;

λ is the standard Lebesgue measure on $\mathbf{R}$.

There are various possibilities to define almost invariant sets in a space (E, G), where E is a ground set and G is a group of transformations of E. We will consider below three variants of the corresponding definitions (see [9]).

Let E be an infinite set, G be a group of transformations of E, and let X be a subset of E.

We say that X is almost G-invariant (in the set-theoretical sense) if for each transformation $g \in G$, we have

$$\mathrm{card}(g(X) \triangle X) < \mathrm{card}(E),$$

where the symbol $\triangle$ denotes the operation of symmetric difference of two sets.

Suppose now that E is additionally endowed with some topology.

We say that a set $Y \subset E$ is almost G-invariant (in the topological sense) if for each transformation $g \in G$, the set $g(Y \triangle Y)$ is of the first Baire category in E.

Analogously, suppose that E is endowed with some measure μ.

We say that a set $Z \subset E$ is almost G-invariant (in the measure-theoretical sense or, more precisely, with respect to μ,) if for each transformation $g \in G$, we have the equality

$$(\forall g \in G)(\mu(g(Z) \triangle Z) = 0).$$

Notice that the three introduced concepts of almost invariance of subsets of a ground set E (equipped with a group G of its transformations) are mutually independent (see, e.g., [9]).

In the sequel, we will be focused only on the measure-theoretical concept of almost invariance.

For our further purpose, we will need two auxiliary notions from the general measure theory.

Let μ be a σ-finite measure on E and let X be any subset of E.

A measurable hull of a set X is a minimal μ-measurable set (with exactness to a μ-measure zero subset of E) which covers X.

A measurable kernel of a set X is a maximal μ-measurable set (with exactness to a μ-measure zero subset of E) which is contained in X.

The following two statements are valid.

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Theorem 1. *If a set X is almost G-invariant, then its μ-measurable hull is also almost G-invariant.*

Theorem 2. *If a set X is almost G-invariant, then its μ-measurable kernel is also almost G-invariant.*

Remark 1. The above-mentioned definitions of measurable hull and measurable kernel are dual to each other. Therefore, if one of these two theorems is proved, then the other one is trivially true.

Theorem 3. *The next two statements are valid:*

(a) *If a set X is not almost G-invariant, but its μ-measurable hull is an almost G-invariant set, then X is a non-measurable set.*

(b) *If a set X is not almost G-invariant, but its μ-measurable kernel is an almost G-invariant set, then X is a non-measurable set.*

Remark 2. It is well known that there are different constructions of non-measurable sets on the real lane $\mathbf{R}$ with respect to the Lebesgue measure (Vitali's sets, Bernstein's sets, non-measurable sets associated with a Hamel basis, sets participating in the Banach-Tarski paradox, etc.). For more information about the above-mentioned topics see, e.g., [1–4, 8, 14].

Let again E be a nonempty set, G be a group of transformations of E and X be a subset of E. We shall say that X is almost non-invariant under the group G if for each transformation $g \in G$, we have

$$\operatorname{card}\{g : g(X) \cap X \neq \emptyset\} < \operatorname{card}(G).$$

In connection with Theorem 3, we give the following example.

Example 1. According to Kunen's result, if the cardinal $\mathbf{c}$ is real-valued measurable, then there exists a λ-nonmeasurable set $X \subset \mathbf{R}$ with $\operatorname{card}(X) < \mathbf{c}$. Notice that X is an almost $\mathbf{R}$-invariant set in the set-theoretical sense, but is not an almost $\mathbf{R}$-invariant set with respect to λ (see [8, 11]).

Let D be a λ-measurable hull of X. Since the measure λ is complete, we obtain $\lambda(D) > 0$.

In view of the ergodicity of the measure λ, there exists a countable family of transformations $\{h_n : n \in N\} \subset \mathbf{R}$ such that

$$\lambda\Big(\mathbf{R} \setminus \bigcup_{n \in N} (h_n + D)\Big) = 0.$$

It is not hard to prove that the set

$$\bigcup_{n \in N} (h_n + D)$$

is a measurable hull of the set

$$\bigcup_{n \in N} (h_n + X).$$

Simultaneously, the set $\bigcup\limits_{n \in N} (h_n + X)$ is almost non-invariant under the group $\mathbf{R}$.

Example 2. It is known that in the infinite-dimensional vector space $\mathbf{R}^\omega$ there exists a nontrivial, σ-finite metrically transitive complete Borel measure χ on the Borel σ-algebra $B(\mathbf{R}^\omega)$, which is invariant with respect to the group S_ω, where S_ω is the group generated by s_0 and G. Here, s_0 is the central symmetry of $\mathbf{R}^\omega$ with respect to the origin and

$$G = \{x : x \in \mathbf{R}^\omega, \operatorname{card}\{i : i \in \omega : x_i \neq 0\} < \omega\}$$

(see [7]).

Then there exists a partition $\{A, B\}$ of $\mathbf{R}^\omega$, where the sets A or B are not almost invariant sets with respect to the group S_ω, but their measurable hulls are almost invariant sets with respect to the same group (see [5, 10]).

Remark 3. Let Y be the μ-measurable kernel of an almost invariant set X, i.e., Y be a μ-measurable subset of X such that

$$\mu_*(X \setminus Y) = 0.$$

According to Theorem 2, Y is also an almost invariant set with respect to μ. Consequently, the set $X \setminus Y$ is almost invariant. If $\mathfrak{I}$ is the invariant σ-ideal generated by all subsets of $X \setminus Y$, then applying Marczewski's method, we can extend measure μ to an invariant measure (see [12, 13]).

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