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DARBOUX TYPE MULTI-DIMENSIONAL PROBLEM FOR A CLASS OF HIGHER-ORDER NONLINEAR HYPERBOLIC EQUATIONS

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Abstract. Darboux type multi-dimensional problem for a class of higher–order nonlinear hyperbolic equations is considered. The theorems on the existence, uniqueness and nonexistence of solutions of this problem are proved.

1. Introduction

In the Euclidean space $\mathbb{R}^{n+1}$ of variables $x = (x_1, \dots, x_n)$ and t, we consider the following nonlinear hyperbolic equation with an iterated wave operator in the main part:

$$\Box^2 u + f(\Box u) + g(u) = F(x,t), \tag{1.1}$$

where $\Box := \frac{\partial^2}{\partial t^2} - \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, f, g and F are given, while u is an unknown scalar function, $n \geq 2$.

Denote by $D_T : |x| < t < T$, $x_n > 0$, a conical domain which is bounded by the half $S_T : t = |x|$, $x_n \geq 0$, $0 \leq t \leq T$ of the truncated characteristic conoid $S : t = |x|$, temporal orientation surface $\varGamma_{1,T} : x_n = 0$, $|x| \leq t \leq T$ and the plane $t = T$.

For equation (1.1) in the domain D_T consider the following boundary value problem: find the solution $u = u(x,t)$ of the equation (1.1) in the domain D_T, which satisfies the following homogeneous conditions on the parts of the boundary S_T and $\varGamma_{1,T}$

$$u|_{S_T} = 0, \quad \frac{\partial u}{\partial \nu}|_{S_T} = 0, \tag{1.2}$$

$$u|_{\varGamma_{1,T}} = 0, \quad \frac{\partial^2 u}{\partial x_n^2}|_{\varGamma_{1,T}} = 0, \tag{1.3}$$

where ν is a unit vector of outer normal with respect to the boundary ∂D_T.

It should be noted that other boundary value problems posed for equation (1.1) have been investigated in papers [1–3].

Remark 1.1. If u, where $u, \Box u \in C^2(\overline{D}_T)$, represents the classical solution of problem (1.1)–(1.3), then by introducing the function $\nu = \Box u$, this problem can be reduced with respect to the unknown functions u and ν to the following boundary value problem:

$$L_1(u,\nu) := \Box u - \nu = 0, \quad (x,t) \in D_T, \tag{1.4}$$

$$L_2(u,v) := \Box u + f(v) + g(u) = F(x,t), \quad (x,t) \in D_T, \tag{1.5}$$

$$u|_{S_T} = 0, \quad u|_{\varGamma_{1,T}} = 0, \tag{1.6}$$

$$v|_{S_T} = 0, \quad v|_{\varGamma_{1,T}} = 0. \tag{1.7}$$

Vice versa, if $u, v \in C^2(\overline{D}_T)$ represents the classical solution of problem (1.4)–(1.7), then the function u will be the classical solution of problem (1.1)–(1.3).

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Definition 1.1. Let $f, g \in C(\mathbb{R})$, $F \in L_2(D_T)$. The system of functions u and v is called a generalized solution of the class W_2^1 of problem (1.4)–(1.7) if

$$u, v \in \overset{0}{W_2^1}(D_T, S_T, \Gamma_{1,T}) := \left\{ w \in W_2^1(D_T) : w|_{S_T \cup \Gamma_{1,T}} = 0 \right\}$$

and there exists the sequence

$$u_m, v_m \in \overset{0}{C^2}\left(\overline{D}_T, S_T, \Gamma_{1,T}\right) := \left\{ w \in C^2\left(\overline{D}_T\right) : w|_{S_T \cup \Gamma_{1,T}} = 0 \right\}$$

such that

$$\lim_{m\to\infty} \|u_m - u\|_{W_2^1(D_T)} = 0, \quad \lim_{m\to\infty} \|v_m - v\|_{W_2^1(D_T)} = 0,$$

$$\lim_{m\to\infty} \|L_1(u_m, v_m)\|_{L_2(D_T)} = 0, \quad \lim_{m\to\infty} \|L_2(u_m, v_m) - F\|_{L_2(D_T)} = 0,$$

where $W_2^1(D_T)$ is the well-known Sobolev space.

Remark 1.2. It is clear that the classical solution $u, v \in \overset{0}{C^2}\left(\overline{D}_T, S_T, \Gamma_{1,T}\right)$ of problem (1.4)–(1.7) represents the generalized solution of the class $W_2^1(D_T)$ of this problem.

2. Main Results

The following lemma holds.

Lemma 2.1. *Let $f, g \in C(\mathbb{R})$, $F \in L_2(D_T)$. Then for any generalized solution u, v of the class W_2^1 of problem (1.4)–(1.7), the following inequality:*

$$\|u\|_{W_2^1(D_T)} \le c\|v\|_{L_2(D_T)}, \tag{2.1}$$

where the constant $c > 0$ does not depend on the functions u, v and F, is true.

Consider the following conditions imposed on the functions f and g

$$\int_0^s f(\tau)\, d\tau \ge -M_1 - M_2 s^2 \quad \forall s \in \mathbb{R}, \quad M_i = \text{const} \ge 0, \;\; i = 1, 2, \tag{2.2}$$

$$g \in C(\mathbb{R}), \quad |g(s)| \le N_1 + N_2 |s| \quad \forall s \in \mathbb{R}, \quad N_i = \text{const} \ge 0, \;\; i = 1, 2. \tag{2.3}$$

Based on the inequalities (2.1)–(2.3), we prove the following lemma about the a priori estimate for the generalized solution of the class W_2^1 of problem (1.4)–(1.7).

Lemma 2.2. *Let $f, g \in C(\mathbb{R})$, $F \in L_2(D_T)$, and the functions f and g satisfy conditions (2.2), (2.3). Then for any generalized solution u, v of the class W_2^1 of problem (1.4)–(1.7), the following a priori estimate*

$$\|u\|_{W_2^1(D_T)} \le c_1\|F\|_{L_2(D_T)} + c_2, \quad \|v\|_{W_2^1(D_T)} \le c_3\|F\|_{L_2(D_T)} + c_4$$

is valid; here, the constants $c_i = \text{const} \ge 0$, $i = 1, \dots, 4$, do not depend on the functions u, v and F, and at the same time $c_1 > 0$, $c_3 > 0$.

Using Lemma 2.2, the following theorem is proved.

Theorem 2.1. *Let the functions $f, g \in C(\mathbb{R})$ satisfy conditions (2.2), (2.3) and*

$$|f(v)| \le \gamma_1 + \gamma_2 |v|^\alpha \;\; \forall v \in \mathbb{R}, \quad 0 \le \alpha = \text{const} < \frac{n+1}{n-1}, \tag{2.4}$$

where $\gamma_i = \text{const} \ge 0$, $i = 1, 2$. Then for any function $F \in L_2(D_T)$, problem (1.4)–(1.7) has at least one generalized solution of the class W_2^1 in the sense of Definition 1.1; besides, if, in addition, the functions f and g satisfy the conditions

$$f \epsilon C^1(\mathbb{R}), \quad |f'(s)| \le d_1 + d_2 |s|^\gamma \quad \forall s \in \mathbb{R},$$

$$g \epsilon C^1(\mathbb{R}), \quad |g'(s)| \le d_3 + d_4 |s|^\gamma \quad \forall s \in \mathbb{R},$$

where $d_i = \text{const} \geq 0$*,* $i = 1, \dots, 4$*;* $0 \leq \gamma = \text{const} < \frac{2}{n-1}$*, then we have also the uniqueness of the solution.*

Now, let us consider one case in which problem (1.4)–(1.7) does not have a solution.

Theorem 2.2. *Let* $g = 0$ *and the function* $f \in C(\mathbb{R})$ *satisfy condition* (2.4) *and*

$$f(s) \leq -|s|^p \quad \forall s \in \mathbb{R}, \quad p = \text{const} > 1, \tag{2.5}$$

$F = \lambda F_0$*, where* $F_0|_{D_T} > 0$*,* $F_0 \in L_2(D_T)$*. Then there exists a number* $\lambda_0 = \lambda_0(F_0, p) > 0$ *such that for* $\lambda > \lambda_0$*, problem* (1.4)–(1.7) *does not have a generalized solution of the class* W_2^1 *in the sense of Definition* 1.1.

Note that when condition (2.5) holds, then condition (2.2) violates.

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