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CONTINUITY OF SPECTRAL FACTORS OF MULTIVARIATE FUNCTIONS

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Dedicated to the Memory of Edem Lagvilava

Abstract. Lagvilava and colleagues developed very efficient algorithms for spectral factorization of matrix-valued functions on the circle group. These have vast applications in engineering and science. Since their algorithms apply a sequence of transforms to the matrix using spectral factors of diagonal entries, the matrix spectral factors are continuous iff the spectral factors of the diagonal entries are continuous. In this paper we discuss continuity of spectral factors of continuous functions on more general compact abelian groups whose Pontryagin duals are equipped with an archimedian order. We focus on factors of band limited functions since this is applicable to image compression and segmentation. Our approach reduces the factorization to the factorization of a uniformly almost periodic function on the group of real numbers and then applies classical results about entire functions.

1. Introduction

Lagvilava and colleagues proposed the algorithmization of spectral factorization of semipostivie definite matrix-valued functions on the circle group in numerous publications including [9–12, 19]. Their algorithms apply a sequence of transforms to the original function to obtain a sequence of such functions terminating in a diagonal matrix. Each step uses spectral factors of diagonal entries of the current matrix. Therefore properties of the matrix spectral factor, such as integrability, continuity, and smoothness, are determined by corresponding properties of the scalar spectral factors. The well known Riesz–Fejer lemma shows that the spectral factor of a univariate nonnegative trigonometric polynomial is a trigonometric polynomial but this rarely holds for multivariate polynomials because they are generically irreducible. It follows from the relationship between conjugate functions and spectral factors that Tsereteli's paper [32] and references therein imply that spectral factorization usually does not preserve function properties. Riemann's mapping theorem gives a plethora of continuous positive functions on the circle group whose spectral factors are discontinuous.

We are interested in spectral factorization of continuous

$$F : G \mapsto [0, \infty)$$

where G is a compact connected abelian groups whose Pontryagin dual $\widehat{G}$ is equipped with an archimedian order as explained in Chapter 8 in [28], or equivalently, there exists an injective order preserving homomorphism

$$\psi : \widehat{G} \mapsto \mathbb{R}.$$

By the Pontryagin duality theorem, explained in Chapter 1 in [28], this is equivalent to

$$\widehat{\psi} : \widehat{\mathbb{R}} \simeq \mathbb{R} \mapsto \widehat{\widehat{G}} = G$$

having a dense image. Then S is a spectral factor of F iff $\widehat{\psi} \circ S$ is a spectral factor, in the sense of Ahiezer, Krein and Levin [1, 21], of the uniformly almost periodic function $\widehat{\psi} \circ F$. Therefore S is continuous iff $\widehat{\psi} \circ S$ is uniformly almost periodic. In 1985 Meemong Lee and I used spectral factorization of functions on the two dimensional torus group to segment gray level images based on textures of subregions, and noted that the matrix extension of our algorithm could be used to segment colored images [23]. In such applications the function on the torus is smooth as well as continuous

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so can with neglible error be replaced by a multivariate positive trigonometric polynomial and then the corresponding uniform almost periodic function is band limited, or equivalently, has a bounded spectrum.

Bohr proved that a uniformly almost periodic function f has a bounded spectrum if and only if it extends to an entire function F of exponential type $\tau(F) < \infty$. If $f \geq 0$ then a result of Krein implies that f admits a factorization $f = |s|^2$ where s extends to an entire function S of exponential type $\tau(S) = \tau(F)/2$ having no zeros in the open upper half plane. The spectral factor s is unique up to a multiplicative factor having modulus 1. Krein and Levin constructed f such that s is not uniformly almost periodic and proved that if $f \geq m > 0$ has absolutely converging Fourier series then s is uniformly almost periodic and has absolutely converging Fourier series. We derive neccesary and sufficient conditions on $f \geq m > 0$ for s to be uniformly almost periodic, we construct an $f \geq m > 0$ with non absolutely converging Fourier series such that s is uniformly almost periodic, and we suggest research questions.

Section 2 introduces notation, Sections 3 and 4 record classical results about entire and almost periodic functions, Section 5 derives two modest results intended to illustrate the synergy of concepts in the preceding two sections, and Section 6 asks questions.

2. Notation

$:=$ means 'is defined to equal' and iff means 'if and only if'. $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are the natural, integer, rational, real, and complex numbers. For $z \in \mathbb{C}$, $x := \Re z; y := \Im z$ are its real; imaginary parts. $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ is the closed unit disk, $\mathbb{D}^o := \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disk, and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is the circle. $\mathbb{U} := \{\, z \in \mathbb{C} \,:\, \Im z \geq 0 \,\}$ is the closed upper half–plane and $\mathbb{U}^o := \{\, z \in \mathbb{C} \,:\, \Im z > 0 \,\}$ is the open upper half plane. For $z \in \mathbb{C}$, $\chi_z(x) := e^{izx}$. For closed $K \subset \mathbb{C}$, $C_b(K)$ is the C^*–algebra of bounded continuous complex–valued functions on K with norm $||f|| := \sup_{z \in K} |f(z)|$. For $\rho > 0$, $\mathfrak{D}_\rho : C_b(\mathbb{R}) \to C_b(\mathbb{R})$ is the dilation operator $(\mathfrak{D}_\rho f)(x) := f(\rho\, x)$. Zeros of nonzero entire functions are denoted by sequences $z_n, n \geq 1$ of finite (possibly zero) or infinite length.

3. Entire Functions

Lemma 1. *If $E : \mathbb{U} \to \mathbb{C}$ is continuous, holomorphic on $\mathbb{U}^o$, $\Re E$ is bounded above, and $\Re E|_{\mathbb{R}} = 0$, then $E(z) = i(a + bz)$ for a real and $b \geq 0$.*

Proof. For some $c > 0$ the function $E_1 : \mathbb{D}\backslash\{-1\} \to \mathbb{R}$ defined by

$$E_1(z) := c - \Re E\left(\frac{i - iz}{1 + z}\right), \quad z \in \mathbb{D}\backslash\{-1\}$$

is positive and continuous and its restriction to $\mathbb{D}^o$ is harmonic. Herglotz [17], [29, Theorem 11.30] proved that there exists a unique positive Borel measure μ on $\mathbb{T}$ such that

$$E_1(z) := \Re \int\limits_{w \in \mathbb{T}} \frac{w + z}{w - z}\, d\mu(w), \quad z \in \mathbb{D}^o.$$

Since $\Re E|_{\mathbb{R}} = 0$ implies $E_1(z) = c$ for $z \in \mathbb{T}\backslash\{-1\}$, it follows that there exists $b \geq 0$ with $d\mu(w) = cd\sigma + b\delta_{-1}$ where σ is normalized Haar measure on $\mathbb{T}$ and δ_{-1} is the point measure at -1. Therefore

$$E_1(z) = c + b\, \Re\, \frac{1 - z}{1 + z}, \quad z \in \mathbb{D}\backslash\{-1\} \implies \Re E(z) = \Re ibz, \quad z \in \mathbb{U}$$

and the result follows from the Cauchy–Riemann equations. □

Throughout this section F denotes a nonconstant entire function. Its order [4, Definition 2.1.2], defined by

$$\rho(F) := \limsup_{r \to \infty} \frac{\log \log M(r)}{\log r}$$

where

$$M(r) := \max\{\, |F(z)| \,:\, |z| \leq r \,\}, \quad 0 \leq r,$$

is in $[0, \infty]$. For integers $k \geq 1$ the entire function $z^k; e^{z^k}$ has order $0; k$. The convergence exponent [4, Definition 2.5.2] $\rho_1 \in [0, \infty]$ of the zeroes $z_j, j \geq 1$ of F is

$$\rho_1 := \inf \left\{ \alpha \in (0, \infty] : \sum_n |z_n|^\alpha < \infty \right\}.$$

If $\rho_1 < \infty$ the genus p of its zeros is the smallest nonnegative integer with

$$\sum_n |z_n|^{p+1} < \infty.$$

Clearly $p \in [\rho_1 - 1, \rho_1]$. Finite sequences have $\rho_1 = p = 0$. The infinite sequence $\frac{1}{n}; \frac{1}{n \log^2(n+1)}$ has $\rho_1 = 1; 1$ and $p = 1; 0$.

The following result shows how to construct certain entire functions with finite order [4, (2.6.3), Theorem 2.6.4].

Lemma 2. *If $z_j, j \geq 1$, has finite convergence exponent ρ_1 and genus p, then the canonical product of genus p defined by*

$$P(z) := \prod_{0 < |z_n|} E(z/z_n, p),$$

where

$$E(z, p) := (1 - z) \exp \left[z + \frac{z^2}{2} + \cdots + \frac{z^p}{p} \right]$$

is the Weierstrass primary factor, converges uniformly on compact subsets to an entire function P that has order ρ_1 and zeros $\{z_j : 0 < |z_n|\}$.

Hadamard's factorization [4, Theorem 2.7.1] gives

Lemma 3. *If F has finite order $\rho(F)$ then its zeros $z_j, j \geq 0$ have finite convergence exponent $\rho_1 \leq \rho(F)$. If $m \geq 0$ is the multiplicity of 0 as a root of F, then there exists a polynomial Q having degree $q \leq \rho(F)$ such that*

$$F(z) = z^m e^{Q(z)} P(z),$$

where P is the canonical product of genus p. Furthermore $\rho(F) = \max\{q, \rho_1\}$.

If F has finite order ρ we define its type

$$\tau(F) := \limsup_{r \to \infty} r^{-\rho} \log M(r)$$

and say it has finite type if $\tau(F) < \infty$. Define

$$n_F(r) := \text{cardinality of } \{j : |z_j| \leq r\}.$$

Lindelöf [25], [4, Theorem 2.10.1] proved

Lemma 4. *If F is an entire function whose order ρ is a positive integer, then F has finite type iff both $n_F(r) = O(r^\rho)$ and*

$$\sup_{r \geq 0} \left| \sum_{0 < |z_n| \leq r} z_n^{-\rho} \right| < \infty.$$

We say F has exponential type if $\rho(F) = 1$ and $\tau(F) < \infty$. For $\alpha \in \mathbb{C} \backslash \{0\}$, the function $e^{\alpha z}$ has exponential type $\tau = |\alpha|$.

Krein [20] proved

Lemma 5. *If F is of exponential type and $f := F|_{\mathbb{R}}$ is bounded and nonegative, then f admits a factorization $f = |s|^2$ where s extends to an entire function S of exponential type that has no zeros in $\mathbb{U}^o$. Moreover $\tau(S) = \tau(F)/2$ and s is unique up to multiplication by a constant having modulus 1.*

Levin [24, p. 437], mentions that Krein used approximation of f by Levitan trigonometric polynomials [24, Appendix I, Section 4]. We observe that $F = S\overline{S}$ where $\overline{S}(z) := \overline{S(\overline{z})}$. Levin [24, Chapter V] defines

$$A := \Big\{ F \text{ entire } : \sum_{0<|z_n|} \Big| \Im \frac{1}{z_n} \Big| < \infty \Big\}.$$

Boas [4, p. 134] proved:

Lemma 6. *If F is of exponential type, then $F \in A$ iff*

$$\sup_{L \geq 1} \int_1^L x^{-2} |\log |f(x)f(-x)|\,\}\, dx < \infty.$$

Moreover, this condition holds whenever

$$\int_{-\infty}^{\infty} (1+x^2)^{-1} \max\{0, \log|f(x)|\} dx < \infty.$$

Ahiezer [1] extended Krein's result by proving:

Lemma 7. *If $f = F|_{\mathbb{R}} \geq 0$, then $F \in A$ iff f admits a factorization $f = |s|^2$ where s extends to an entire function S of exponential type $\tau(S) = \tau(F)/2$ having no zeros in $\mathbb{U}^o$.*

We summarize Levin's proof in [24, p. 437]. The if part follows from Lemma 4. The roots of F occur in conjugate pairs so we order $\{z_j\}$ so j odd $\implies \Im z_j \leq 0$ and $z_{j+1} = \overline{z}_j$. Lemma 3 implies there exists an integer $m \geq 0$ and $a, b \in \mathbb{R}$ such that

$$F(z) = z^{2m} e^{2az+2b} \prod_{j \geq 1} \Big(1 - \frac{z}{z_j}\Big) e^{\frac{z}{z_j}}.$$

Define $\gamma := -\sum_{j=odd} \Im \frac{1}{z_j}$, and $s := S|_{\mathbb{R}}$ where

$$S(z) := z^p e^{az+b+i\gamma z} \prod_{j=odd} \Big(1 - \frac{z}{z_j}\Big) e^{\frac{z}{z_j}}.$$

Then $F = S\overline{S}$ and S has no zeros in $\mathbb{U}^o$. Since $S \in A$, Lemma 4 implies that S is of exponential type. The rest of the proof shows that $\tau(S) = \tau(F)/2$.

$\xi(\mathbb{R})$ is the space of smooth complex valued functions on $\mathbb{R}$ with the topology of uniform convergence of derivatives of every order on compact subsets. Its dual space $\xi'(\mathbb{R})$ is the space of compactly supported distributions [18, Theorem 1.5.2]. For $u \in \xi'(\mathbb{R})$ we define $\alpha(u), \beta(u) \in \mathbb{R}$ so that $[\alpha(u), \beta(u)]$ is the smallest closed interval containing the support of u, and we define its Fourier-Laplace transform by $\widehat{u}(z) := u(\chi_{-z}), z \in \mathbb{C}$.

Lemma 8. *Conditions 1 and 2 are equivalent and imply Condition 3*

(1) *F is of exponential type and $\exists\, c > 0, N \in \mathbb{Z}$ with $|F(x)| \leq c\,(1+|x|)^N,\ x \in \mathbb{R}$.*
(2) *$F = \widehat{u}$ for $u \in \xi'(\mathbb{R})$ with $\tau(F) = \max\{|\alpha(u)|, |\beta(u)|\}$.*
(3) $\lim_{r \to \infty} r^{-1} n_F(r) = \frac{\beta(u)-\alpha(u)}{2\pi}$.

Schwartz [30] proved the equivalence of Conditions 1 and 2 and Titchmarsh [31] proved they imply Condition 3.

4. Almost Periodic Functions

$T(\mathbb{R})$ is the algebra of trigonometric polynomials spanned by $\{\chi_\omega : \omega \in \mathbb{R}\}$, and the C^*–algebra $U(\mathbb{R})$ of uniformly almost periodic functions is its closure with respect to $||\cdot||$. Bohr [5] proved that if $f \in U(\mathbb{R})$ its mean value

$$M(f) := \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} f(x)dx$$

exists, defined its Fourier transform $\widehat{f} : \mathbb{R} \to \mathbb{C}$ by

$$\widehat{f}(\omega) := M(f\,\chi_{-\omega}),\ \omega \in \mathbb{R},$$

and proved that its spectrum $\Omega(f) := \{\omega \in \mathbb{R} : \widehat{f}(\omega) \neq 0\}$ is countable. For $f \in U(\mathbb{R})$ its Fourier series is the formal sum

$$f \sim \sum_{\omega\in\Omega(f)} \widehat{f}(\omega)\,\chi_\omega,$$

its bandwidth $b(f) := \sup\Omega(f) - \inf\Omega(f) \in [0,\infty]$, and

$$||f||_A := \sum_{\omega\in\Omega(f)} |\widehat{f}(\omega)| \in [0,\infty].$$

We define the following subsets of $U(\mathbb{R})$:

(1) Bandlimited algebra $B(\mathbb{R}) := \{\, f \in U(\mathbb{R}) \,:\, b(f) < \infty \,\}$.
(2) Wiener Banach algebra $A(\mathbb{R}) := \{f \in U(\mathbb{R}) \,:\, ||f||_A < \infty\}$.
(3) Hardy Banach algebra $H(\mathbb{R}) := \{f \in U(\mathbb{R}) \,:\, \Omega(f) \subset [0,\infty)\}$.
(4) Invertible Hardy functions $IH(\mathbb{R}) := \{f \in H(\mathbb{R}) : \frac{1}{f} \in H(\mathbb{R})\}$.

Bohr [5], [24, Chapter VI, p. 268, Corollary to Theorem 1] proved:

Lemma 9. *If $f \in U(\mathbb{R})$ then $f \in B(\mathbb{R})$ iff f extends to an entire function F of exponential type. Then $\tau(F) = \max\{|\inf\Omega(f)|, |\sup\Omega(f)|\}$.*

The Fourier series for $f \in U(\mathbb{R})$ converges absolutely iff $f \in A(\mathbb{R})$. Cameron [8] and Pitt [26], [13, Section 29, Corollary 2 to Theorem 2] proved

Lemma 10. *If $f \in A(\mathbb{R})$ and Φ is holomorphic in an open region containing the closure of $f(\mathbb{R})$, then the composition $\Phi \circ f \in A(\mathbb{R})$.*

If $f \in B(\mathbb{R})$ and $f \geq 0$ then Lemma 9 implies that f extends to an entire function F of exponential type $\tau(F) = b(f)/2$ so Lemma 5 implies that $f = |s|^2$ where s extends to an entire function S of exponential type $\tau(S) = b(f)/4$. Levin [24, Appendix 2, Theorem 2] used Lemma 10 to prove

Lemma 11. *If $f \in A(\mathbb{R}) \cap B(\mathbb{R})$ and there exists $m > 0$ with $f \geq m$, then the spectral factor $s \in A(\mathbb{R}) \cap B(\mathbb{R})$.*

For $\Delta > 0$ let $[\Delta]$ denote the set of entire functions F of exponential type $\tau(F) = \Delta$ such that $f := F|_{\mathbb{R}} \in B(\mathbb{R})$ and $-\Delta, \Delta \in \Omega(f)$. Krein and Levin obtained a precise characterization of the zeros of functions in $[\Delta]$ and published these results without proofs in [21]. In [24, Appendix VI] for the first time they gave proofs for these results and used them [24, Appendix VI, p. 463] to prove

Lemma 12. *There exists $\Delta > 0$ and $F \in [\Delta]$ with $f \geq 0$ whose spectral factor $s \notin U(\mathbb{R})$.*

Levin's result [24, Chapter VI, Section 2, Lemma 3] implies:

Lemma 13. *If $h \in U(\mathbb{R})$ then $-\infty < \Delta := \inf\Omega(h)$ iff h extends to a continuous function H on $\mathbb{U}$ which is holomorphic on $\mathbb{U}^o$ and satisfies*

$$\lim_{y\to\infty} e^{-i\Delta(x+iy)}\,H(x+iy) = \widehat{h}(-\Delta)$$

where convergence is uniform in x. Therefore $h \in H(\mathbb{R})$ iff $H \in C_b(\mathbb{U})$.

The Poisson kernel functions $P_y : \mathbb{R} \to \mathbb{R}, y > 0$ are

$$P_y(x) := \frac{y}{\pi}\frac{1}{x^2+y^2}, \quad x+iy \in \mathbb{U}^o.$$

For $f \in C_b(\mathbb{R})$ its Poisson integral $P[f] : \mathbb{U} \to \mathbb{C}$ is $P[f](x) := f(x)$ and

$$P[f](x+iy) := \begin{cases} f(x), & y = 0 \\ \int\limits_{-\infty}^{\infty} P_y(x-s)\, f(s)\, ds, & y > 0 \end{cases}$$

Lemma 14. *If $f \in C_b(\mathbb{R})$, then $P[f] \in C_b(\mathbb{U})$, its restriction $P[f]|_{\mathbb{U}^o}$ is harmonic, and*

$$\sup_{x\in\mathbb{R}} \Re f(x) \geq \Re P[f](z) \geq \inf_{x\in\mathbb{R}} \Re f(x), \quad z \in \mathbb{U}^o.$$

If $f \in U(\mathbb{R})$ then $f \in H(\mathbb{R})$ iff $P[f]|_{\mathbb{U}^o}$ is holomorphic.

Proof. The first assertion follows since $P_y(x)$ is harmonic, positive valued, and $\int P_y(x)dx = 1, y > 0$. The second assertion follows since

$$P[\chi_\omega](z) = \begin{cases} e^{i\omega z}, & \omega \geq 0 \\ e^{i\omega \overline{z}}, & \omega < 0. \end{cases} \qquad \square$$

Bohr [5], [24, Chapter VI, Theorem 2] proved:

Lemma 15. *If $h \in H(\mathbb{R})$ is nonzero, $H := P[h]$, and $z_n, n \geq 1$ are the zeros of H, then $\{\Im z_n\}$ is bounded iff* $\inf \Omega(h) \in \Omega(h)$.

Bohr [6], [24, p. 274, footnote] proved:

Lemma 16. *If $h \in U(\mathbb{R})$ and $|h|^2 \geq m$ for some $m > 0$, then there exists $c \in \mathbb{R}$ and $\theta \in U(\mathbb{R})$ such that*

$$(\arg h)(x) = cx + \theta(x), \quad x \in \mathbb{R}.$$

Lemma 17. *If $h \in IH(\mathbb{R})$ then $|P[h]|$ is bounded below by a positive number, $\Re \log P[h] \in C_b(\mathbb{U})$, and $\widehat{h}(0) \neq 0$.*

Proof. Lemma 13 implies $P[h], P[1/h], P[h]P[1/h] \in C_b(\mathbb{U})$. Lemma 14 implies $P[h]P[1/h]$ is holomorphic on $\mathbb{U}^o$. Since $P[h]P[1/h]|_{\mathbb{R}} = 1$, the Schwarz reflection principle [29, Theorem 11.14] implies $P[h]P[1/h] = 1$. Therefore $|P[h]|$ is bounded below by a positive number, so $\log P[h]$ exists, is unique up to addition by an integer multiple of $2\pi i$, and $\Re \log P[h] = \log |P[h]| \in C_b(\mathbb{U})$. Since $P[h]$ and $P[1/h]$ have no zeros, Lemma 15 implies $\inf \Omega(h) \in \Omega(h)$ and $\inf \Omega(1/h) \in \Omega(1/h)$. Since $\{0\} = \Omega(1) \subset \Omega(h) + \Omega(1/h)$ it follows that $\inf \Omega(h) = \inf \Omega(1/h) = 0$ so $\widehat{h}(0) \neq 0$. $\square$

Lemma 18. *If $h \in IH(\mathbb{R})$, $f := |h|^2$, and $f \in B(\mathbb{R})$, then $h \in B(\mathbb{R})$ and $\chi_{-b(f)/4}\, h$ is a spectral factor of f.*

Proof. Lemma 5 implies $f = |s|^2$ where the spectral factor s extends to an entire function S of exponential type $\tau(S) = b(f)/4$ which has no zeros in $\mathbb{U}^0$. Define $S_1(z) := e^{ib(f)z/4}\, S(z)|_{\mathbb{U}}$ and $H := P[h]$. Then $S_1 \in C_b(\mathbb{U})$ is bounded and holomorphic with no zeros in $\mathbb{U}^o$ and Lemma 17 implies that H is holomorphic on $\mathbb{U}^o$ and $|H|$ is bounded below by a positive number. Therefore $G := S_1/H \in C_b(\mathbb{U})$ is holomorphic with no zeros on $\mathbb{U}^o$, and $|G(x)| = |s(x)|/|h(x)| = 1$ for x real. Therefore $E := \log G$ exists and satisfies the hypothesis of Lemma 1 hence $E(z) = i(a + bz)$ for some a real and $b \geq 0$. Therefore $e^{ib(f)z/4}\, S(z) = e^{i(a+bz)}\, H(z)$ hence $s = e^{ia} \chi_{b-b(f)/4}\, h$ so $h \in B(\mathbb{R})$. Lemma 17 implies $\inf \Omega(h) = 0$. Since $\inf \Omega(s) = -b(f)/4$ it follows that $b = 0$ hence $\chi_{-b(f)/4}\, h$ is a spectral factor. $\square$

Boas [4, Theorem 11.1.2] proved this generalization of Sergei Bernstein's classic theorem [2], [4, Theorem 11.1.1] for polynomials:

Lemma 19. *If F is an entire function of exponential type and $f := F|_{\mathbb{R}}$ is bounded, then $f' := \frac{df}{dx}$ is bounded and $||f'|| \leq \tau(F)\, ||f||$.*

Lemma 20. *If $f \in B(\mathbb{R})$, $m > 0$, $f \geq m$, $g := \frac{1}{2} \log f$, $u := P[g]$, $u_o := u|_{\mathbb{U}^o}$, and $v_o : \mathbb{U}^o \to \mathbb{R}$ is a harmonic function conjugate to u_o, then v_o is uniformly continuous so extends to a continuous function $v : \mathbb{U} \to \mathbb{R}$.*

Proof. Define $\gamma := \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|s^2-1|}{(s^2+1)^2}\,ds$. A computation gives

$$\left|\frac{\partial u_o}{\partial y}(x+iy)\right| \leq \frac{\gamma}{y}||g|| \leq \frac{\gamma}{2y}\max\{|\log m|,\, |\log ||f||\,|\} < \infty, \quad x+iy \in \mathbb{U}^o.$$

Lemma 19 gives

$$\left|\frac{\partial u_o}{\partial x}(x+iy)\right| \leq ||\,P[g']\,|| \leq \frac{||f'||}{2m} < \infty, \quad x+iy \in \mathbb{U}^o.$$

The Cauchy–Riemann equations imply v_o is uniformly continuous. □

5. Results

Theorem 1. *If f, g, u and v are as in Lemma 20, then f has a spectral factor $s \in B(\mathbb{R})$ iff $v|_{\mathbb{R}}$ has the form in Lemma 16.*

Proof. Let $H := e^{u+iv}$, $h := H|_{\mathbb{R}}$. If $v|_{\mathbb{R}} = \arg h$ has the form in Lemma 16 then $h = \chi_c \sqrt{f}\, e^{i\theta} \in U(\mathbb{R})$. Since $P[h] = H$ is holomorphic on $\mathbb{U}^o$, Lemma 14 implies $h \in H(\mathbb{R})$. A similar argument implies $\frac{1}{h} \in H(\mathbb{R})$ so $h \in IH(\mathbb{R})$. Since $|h|^2 = f$, Lemma 18 implies $h \in B(\mathbb{R})$ and $s := \chi_{-b(f)/4}\, h$ is a spectral factor of f in $B(\mathbb{R})$. Conversely, if f has a spectral factor $s \in B(\mathbb{R})$ then $h := \chi_{b(f)/4}\, s \in H(\mathbb{R})$ hence $P[h] = e^{u+iv}$. Since $|h|^2 = f \geq m > 0$, Lemma 16 implies that $v|_{\mathbb{R}} = \arg h$ has the form in Lemma 16. □

The following result shows that the assumption $f \in A(\mathbb{R})$ in Lemma 11 is not necessary to ensure that f has a spectral factor $s \in B(\mathbb{R})$.

Theorem 2. *For $m > 0$ there exists $f \in B(\mathbb{R}) \setminus A(\mathbb{R})$ with $f \geq m$ whose spectral factor $h \in B(\mathbb{R}) \setminus A(\mathbb{R})$.*

Proof. Rudin [29, Theorem 5.12] gave a proof, based on the Banach–Steinhaus theorem [29, Theorem 5.8], that the subset of $U(\mathbb{R})$ of period 2π–periodic functions with non absolutely convergent Fourier series is nonempty. Zygmund [33, Chapter VI, 3.7] gave the example

$$\phi(x) \sim \sum_{n=2}^{\infty} \frac{\sin nx}{n \log n}.$$

Fejér [33, Chapter III, Theorem 3-4] proved its Cesàro sums

$$p_n(x) = \sum_{k=2}^{n} \frac{n+1-k}{n}\,\frac{\sin kx}{k \log k}, \quad n \geq 2$$

converge uniformly to ϕ therefore there exists an integer sequence $2 \leq n_1 < n_2 < \cdots$ with $||\phi - p_{n_j}|| \leq 2^{-j}/3$. Define $q_1 = p_{n_1}$ and $q_j = p_{n_{j+1}} - p_{n_j}, j \geq 2$. Then $||q_j|| \leq 2^{-j}$ and $\min \Omega(q_j) = -n_{j+1}$ and $\max \Omega(q_j) = n_{j+1}$. Construct a sequence $\rho_j \in (0, n_{j+1}^{-1})$ with $\{\rho_j\}$ lineary independent over $\mathbb{Q}$ and define

$$g_j := \mathfrak{D}_{\rho_j} q_j, \quad j \geq 1.$$

Then $||g_j|| = ||q_j||, ||g_j||_A = ||q_j||_A$, and $\Omega(g_j) = \rho_j \Omega(q_j)$ are pairwise disjoint subsets of $(-1,1)$. Therefore

$$g := \sum_{j=1}^{\infty} g_j.$$

satisfies $g \in B(\mathbb{R})$, $\Omega(g) \subset (-1,1)$, and

$$||g||_A = \lim_{k\to\infty} \sum_{j=1}^{k-1} ||q_j||_A \geq \lim_{k\to\infty} ||\sum_{j=1}^{k-1} q_j||_A = \lim_{k\to\infty} ||p_{n_k}||_A = \infty$$

so $g \in B(\mathbb{R}) \setminus A(\mathbb{R})$, g is real valued, and $\Omega(g) \subset (-1,1)$. Define $\Delta = -\inf \Omega(g)$, $h_1 := \chi_\Delta g$, and $H_1 = P[h_1]$. Lemma 13 implies that H_1 is bounded and $h_1 \in B(\mathbb{R}) \cap H(\mathbb{R})$. Define $c := \sqrt{m} - \inf \Re H_1$, $h := c + h_1$, $f := |h|^2$, and $H := P[h]$. Lemma 14 implies that $|H(z)| \geq |\Re H(z)| \geq \sqrt{m}$ so H has no zeros in $\mathbb{U}$. Then $s := \chi_{-\Delta}h$ is a spectral factor of f. Since $s \notin A(\mathbb{R})$, Lemma 11 implies that $f \notin A(\mathbb{R})$. □

6. Questions

We suggest the following questions for future research:

(1) If the hypothesis $f \geq m > 0$ in Lemma 11 and Theorem 1 is replaced by the weaker hypothesis $f \geq 0$, what conclusions about the spectral factor s of f can be deduced?

(2) If $f \in B(\mathbb{R})$ is nozero and $f \geq 0$, Bohr showed that it lifts to a function $\widetilde{f} \in C(\mathbb{R}_B)$, where $\mathbb{R}_B$ is the Bohr compactification of $\mathbb{R}$, and we proved in [22] that $\log \widetilde{f} \in L^1(\mathbb{R})$. Helson and Lowdenslager [16] proved that $\widetilde{f} = |\widetilde{h}|^2$ where $\widetilde{h}$ is an outer function in the Hardy space $H^2(\mathbb{R}_B)$ (with respect to the linear order on the Pontryagin dual of $\mathbb{R}_B$ which equals the discrete real group). What is the relationship between $\widetilde{h}$ and the lift $\widetilde{s}$ of the spectral factor s of f given by Lemma 5?

(3) Ahiezer's result in Lemma 7 holds for operator valued and matrix valued functions [9, 10, 14, 15, 27]. What analogues do the results in this paper have in this context besides the already established ones in the almost periodic setting [7]?

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