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ON THE FACTORIZATION AND PARTIAL INDICES OF PIECEWISE CONSTANT MATRIX FUNCTIONS

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Dedicated to the memory of Edem Lagvilava

Abstract. In this paper, we give algorithm for calculating L_p-partial indices for some classes of the piecewise constant matrix functions. We reduce this problem to calculation of partial indices of a rational matrix function, which leads Fuchsian system of differential equations to a system of a standard form.

1. Φ-factorization and Fundamental Matrix of Solutions of the Fuchsian System

Let Γ be a smooth closed positively oriented loop in $\mathbb{CP}^1$ which separates $\mathbb{CP}^1$ into two connected domains U_+ and U_-. Suppose $0 \in U_+$ and $\infty \in U_-$. Let $L^p(\Gamma)$, $p > 1$, be the space of p-integrable functions on Γ and $L^p_\pm(\Gamma)$ be the space of analytic functions $U^\pm$, respectively, defined by the standard projector operator [5, 7]. Let $L^\infty(\Gamma)$ be the Banach space of essentially bounded functions.

Denote by Ω the space of all Hölder-continuous matrix functions $f : \Gamma \to GL_n(\mathbb{C})$ with the natural topology. For a set $\mathcal{S}$, let $\mathcal{S}^{n\times n}$ be the set of $n \times n$-matrices with entries from $\mathcal{S}$.

Φ-*Factorization* of a matrix-function $G \in L^\infty(\Gamma)^{n\times n}$ in the space $L^p(\Gamma)$ is its representation in the form

$$G(t) = G_+(t)\Lambda(t)G_-(t), \quad t \in \Gamma, \tag{1}$$

where $\Lambda(t) = \operatorname{diag}(t^{k_1}, \dots, t^{k_n})$, $k_i \in \mathbb{Z}$, $i = 1, \dots, n$, $G_+ \in L^p_+(\Gamma)^{n\times n}$ and $G_+^{-1} \in L^q_+(\Gamma)^{n\times n}$, $G_- \in L^q_-(\Gamma)^{n\times n}$ and $G_-^{-1} \in L^p_-(\Gamma)^{n\times n}$, $\frac{1}{p} + \frac{1}{q} = 1$, and the operator $G_-^{-1} Q_\Gamma G_+^{-1}$ is bounded in the space $L^p(\Gamma)^n$. Here, we use the standard notation

$$Q_\Gamma = \frac{1}{2}(\mathbf{1} - S_\Gamma), \quad (S_\Gamma f)(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{f(\tau)}{\tau - t}d\tau, \quad t \in \Gamma,$$

and $\mathbf{1}$ is an identity operator.

The integers $k_1, \dots, k_n$, $k_1 \geq \dots \geq k_n$ are called L^p-*partial indices* of the piecewise constant matrix function $G(t)$.

Let us consider the particular subspace $PC(\Gamma)^{n\times n}$ of piecewise constant matrix-functions on Γ. For elements of this subspace, there exist the one-sided limits $G(t + 0)$ and $G(t - 0)$ for each $t \in \Gamma$. For such matrix-functions, a necessary and sufficient condition for the existence of Φ-factorization is given by the following

Theorem 1 ([5]). *A matrix-function $G \in PC(\Gamma)^{n\times n}$ is Φ-factorizable in the space $L^p(\Gamma)$ if and only if*

a) *the matrices $G(t + 0)$ and $G(t - 0)$ are invertible for each $t \in \Gamma$;*
b) *for each $j = 1, \dots, n$ and $t \in \Gamma$, one has*

$$\theta_j = \frac{1}{2\pi}\arg\lambda_j(t) + \frac{1}{p} \notin \mathbb{Z}. \tag{2}$$

Here, $\lambda_1(t), \dots, \lambda_n(t)$ are the eigenvalues of the matrix $G(t - 0)G(t + 0)^{-1}$.

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If a matrix-function G is Φ-factorizable, then $\zeta_j(\tau) = -\frac{1}{2\pi}\arg\lambda_j(\tau)$ is a single-valued function taking values in the interval $\left(\frac{1}{p}-1, \frac{1}{p}\right)$.

Suppose G has m singular (discontinuity) points $s_1, \dots, s_m \in \Gamma$, then

$$\kappa = \sum_{k=1}^{m}\left[\frac{1}{2\pi}\arg\det G(t)\right]_{t=s_k+0}^{s_{k+1}-0} + \sum_{k=1}^{m}\sum_{j=1}^{n}\zeta_j(s_k). \tag{3}$$

The quantity κ is called the *index* of the matrix function which is equal to the sum of partial indices: $\kappa = k_1 + k_2 + \cdots + k_n$. It can be seen from (3) that κ depends on $L^p(\Gamma)$. If $\lambda_j(\tau)$ are positive real numbers, then $\zeta_j(\tau) = 0$ and, consequently, κ does not depend on the space $L^p(\Gamma)$.

Suppose $G \in PC(\Gamma)^{n\times n}$ is a piecewise constant matrix function with singular points $s_1, \dots, s_m \in \Gamma$ occurring in the given order, which is factorizable in the space $L^p(\Gamma)$. Let $M_k = G(s_k - 0)G(s_k + 0)^{-1}$, $k = 1, \dots, m$. Thus G is constant on the arc (s_k, s_{k+1}), and we are assume that, $s_{m+1} = s_1$, $M_1M_2\dots M_m = \mathbf{I}$, where $\mathbf{I}$ is an identity matrix. Suppose that the matrices $M_k, k = 1, \dots m$, are similar to the matrices $\exp(-2\pi i E_k)$ and eigenvalues of E_k belong to the interval $\left(\frac{1}{p}-1, \frac{1}{p}\right)$ which determines the matrices E_k uniquely. The numbers $\zeta_1(s_k), \dots, \zeta_n(s_k)$ are equal to the real parts of eigenvalues of E_k. This implies that for the index κ one has the formula $\kappa = \sum\limits_{k=1}^{m} \mathtt{tr} E_k$. Thus the matrices $E_1, \dots, E_m$ depend on the space $L^p(\Gamma)$. They also depend on the choice of a logarithm of eigenvalues of the matrices M_k. Thus $G \in PC(\Gamma)^{n\times n}$ produces two m-tuples $(M_1, \dots, M_m)$ and $(E_1, \dots, E_m)$ of matrices.

Let

$$\frac{df}{dz} = A(z)f(z) \tag{4}$$

be an n-system of differential equations with regular singularities, having $s_1, \dots, s_m$ as singular points, and ∞ as an apparent singular point. It is known that such system has n linearly independent solutions in the neighborhood of a regular point.

Let us denote such a fundamental system of solutions by $F(\tilde{z})$. It is possible to characterize the behavior of $F(\tilde{z})$ near the singular points $s_1, \dots, s_m$ by the matrices $M_1, \dots, M_m$ which are determined by $E_1, \dots, E_m$ and by the behavior at ∞ which is characterized by partial indices $k_1, \dots, k_m$. Therefore it is said that system (4) has the standard form with respect to the matrices $(M_1, \dots, M_m)$ and $(E_1, \dots, E_m)$ satisfying the condition $M_1 \dots M_m = 1$, where M_k are similar to $\exp(-2\pi i E_k)$, $k = 1, \dots, m$, and E_j are not resonant and with singular points $s_1, \dots, s_m$, and with partial indices $k_1 \geq \cdots \geq k_n$, if

i) $s_1, \dots, s_m$ are the only singular points of (4), with ∞ as an apparent singular point;

ii) the monodromy group of (4) is conjugate to the subgroup of $GL_n(\mathbb{C})$ generated by the matrices $M_1, \dots, M_m$;

iii) in a neighborhood U_j of the point s_j, the solution has the form

$$F(\tilde{z}) = Z_j(z)(\tilde{z} - s_j)^{E_j}C,$$

where $Z_j(z)$ is an analytic and invertible matrix-function on $U_j \cup \{s_j\}$ and C is a nondegenerate matrix;

iv) the solution of the system in the neighborhood U_∞ of ∞ has the form

$$F(z) = \operatorname{diag}(z^{k_1}, \dots, z^{k_n})Z_\infty(z)C, \quad z \in U_\infty,$$

where $Z_\infty(z)$ is holomorphic and invertible in U_∞.

In particular, every coordinate function $f_k(\tilde{z})$ of a solution $F(\tilde{z})$ of system (4) is

$$f_k(\tilde{z}) = \sum_{p,q} \tilde{z}^{\tau_p} h_{p,q}(z) \ln^{l_q} \tilde{z}, \ \text{ where } \quad 0 \leq Re\tau_q < 1, l_q \in Z, l_q \geq 0. \tag{5}$$

Theorem 2 ([5]). *Suppose $G \in PC(\Gamma)^{n\times n}$ is a piecewise constant function with jump points $s_1, \dots, s_m$ which admits Φ-factorization in the space $L_p(\Gamma)$, $1 < p < \infty$, and $(M_1, \dots, M_m)$, $(E_1, \dots, E_m)$ are the matrices associated to G.*

Suppose there exists a system of differential equations in the standard form (4) *with singular points* $s_1, \dots, s_m$ *and partial indices* $k_1, \dots, k_n$*. Let* $F_1(z)$*,* $F_2(z)$ *be fundamental systems of its solutions in* U_+ *and* $U_- \setminus \{\infty\}$*, respectively.*

Then there exist nondegenerate $n \times n$*-matrices* C_1 *and* C_2 *such that*

$$G(t) = G_+(t)\Lambda(t)G_-(t)$$

is a Φ*-factorization of* G *in* $L^p(\Gamma)$*, where* $\Lambda(t) = \operatorname{diag}(t^{k_1}, \dots, t^{k_n})$*,*

$$G_+(z) = C_1^{-1}F_1^{-1}(z),\ z \in U_+,\ G_-(z) = \Lambda^{-1}(z)F_2(z)C_2,\ z \in U_- \setminus \{\infty\}.$$

We use the above factorization of the piecewise constant matrix function $G(t)$ for a proof of the main result of this paper.

2. L_p-factorization of Piecewise Constant Matrix Function

We consider the regular system of the differential equations of the form

$$dF = \omega F, \tag{6}$$

where $F : U \to \mathbb{C}^n$ is an unknown vector-valued function defined in a complex domain $U \subset \mathbb{C} \cup \{\infty\}$ and ω is the given matrix-valued meromorphic 1-form on $\mathbb{CP}^1$, with an $S = \{s_1, s_2, \dots, s_m\}$ set of singular points. If the above system is Fuchsian, then

$$dF = \left(\sum_j^m \frac{A_j}{z - s_j} dz\right) F, \tag{7}$$

where A_j, $j = 1, \dots, m$ are constant matrices: $A_j = \operatorname{res}_{z=s_j}\omega(z)$ and $\sum\limits_{j=1}^m A_j = 0$ if $s_j \neq \infty$ for all $j = 1, \dots, m$.

Let $X_m = \mathbb{CP}^1 \setminus S$ and let $\mathcal{X}_m$ be a universal covering of X_m. Then the group of deck transformations Γ of the covering $p : \mathcal{X}_m \to X_m$ is the infinite cyclic group generated by the deck transformations g_j corresponding to one counterclockwise trip around s_j, $j = 1, \dots, m$.

If we identify the fundamental group $\pi_1(X_m, z_0)$, $z_0 \notin S$, of the noncompact Riemann surface X_m with Γ, we obtain the monodromy representation of systems (6), (7),

$$\rho : \pi_1(X_m, z_0) \to GL_n(\mathbb{C}) \tag{8}$$

defined by the correspondence

$$g_j \to M_j.$$

Here, we preserve the notation for the deck transformation $g_j \in \Gamma$ for the generators of the fundamental group $\pi_1(X_m, z_0)$ and in this new notation, g_j are the homotopy classes of simple loops starting and ending at z_0 which goes around s_j counterclockwise. Therefore the generators of the fundamental group $\pi_1(X_m, z_0)$ are the homotopy classes g_j, $j = 1, \dots, m$, with the relation $g_1 \dots g_m = 1$. The monodromy representation (8) is homomorphism of groups and the monodromy matrices $M_j, j = 1, \dots, m$ are generators of the monodromy group of Fuchsian system (7) and they satisfy the relation $M_1 \dots M_m = \mathbf{I}$.

By Levelt's theory [3], the fundamental matrix of solutions of (6) in the neighborhood of s_j is

$$\Phi_j\,(\tilde{z}) = U_j(z)(z - s_j)^{\Psi_j}(\tilde{z} - s_j)^{E_j}, \tag{9}$$

where $\tilde{z}$ is a local variable on $\mathcal{X}_m$, $\Psi_j = \operatorname{diag}(\varphi_j^1, \dots, \varphi_j^m)$ are diagonal matrices with integer entries, $E_j = \frac{1}{2\pi i}\ln M_j$ are upper triangular matrices with eigenvalues $\mu_j^k = \frac{1}{2\pi i}\ln m_j^k$, $k = 1, \dots, n$ satisfying the conditions $0 \le \operatorname{Re}\mu_j^k < 1$. The numbers $\beta_j^k = \varphi_j^k + \mu_j^k$ will be called *exponents* of the solution space $\mathcal{V}$ at the point s_j (or j-exponents). The integers $\varphi_j^k = [\operatorname{Re}\beta_j^k]$, are called *valuations* at s_j and Ψ_j is called a *matrix of valuations* of the solution space $\mathcal{V}$ at the singular points $s_j, j = 1, \dots, m$. It is clear that $\beta_j^1, \dots, \beta_j^n$ are eigenvalues of the matrices A_j, $j = 1, \dots, m$, from system (7) and $\sum\limits_{j=1}^m \sum\limits_{k=1}^n \beta_j^k = 0$.

For our goals, we use a version of the Riemann–Hilbert monodromy problem known as *Riemann-Hilbert problem with the given asymptotics* [4]:

For the given representation (8) *and a collection of diagonal matrices* $\Psi = (\Psi^1, \dots, \Psi^m)$ *such that*

$$\sum_{k=1}^{m} \Psi^k = 0, \tag{10}$$

construct Fuchsian system (7) *with the given monodromy* ρ *and with the given* Ψ *as a collection of matrices of valuations.*

In general, this problem has a negative solution for the class of Fuchsian systems. Moveover, the following theorem holds.

Theorem 3 (see [3])**.** *There exists a system of differential equations*

$$\frac{dy(z)}{dz} = B(z)y(z) \tag{11}$$

with the given matrix of valuations $\Psi = (\Psi^1, \dots, \Psi^m)$*, whose monodromy representation coincides with* ρ *defined by* (8) *which is Fuchsian at the given points* $s_1, \dots, s_m$ *and have apparent singularity at* ∞.

The existence of system (11) with the given monodromy and singular points $s_1, \dots, s_m$ follows from a positive solution of the Riemann–Hilbert monodromy problem for regular systems (Plemelj Theorem [3]). The first part of the theorem follows from Proposition 1 below.

Proposition 1 ([4,6,8])**.** *Let* $\tilde{\Psi} = (\tilde{\Psi}^1, \dots, \tilde{\Psi}^m)$ *be given. Then there exists a rational matrix function* $T(z)$ *such that the Fuchsian system* (7) *is a gauge equivalent to system* (11)*:*

$$B(z) = T(z)\left(\sum_{j=0}^{m} \frac{A_j}{z - s_j}\right)T^{-1}(z) + \frac{dT(z)}{dz}T^{-1}(z).$$

In addition, the fundamental matrix of solutions of this system at the singular points s_j *has the form*

$$\Phi_j\left(\tilde{z}\right) = U_j(z)(z - s_j)^{\tilde{\Psi}_j}(\tilde{z} - s_j)^{E_j} \tag{12}$$

for each $j = 1, \dots, m$.

We call a rational matrix function $T(z)$ a *transform matrix.*

Proposition 2 ([1,2])**.** *The transform matrix* $T(z)$ *can be calculated algorithmically.*

Consider *a canonical extension* (E^0, ∇^0) of the holomorphic vector bundle induced from system (11) and suppose that the splitting type of this bundle is $K = (k_1, \dots, k_n), k_1 \geq \dots \geq k_n$:

$$E^0 \cong \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_n). \tag{13}$$

Note that the canonical extension is induced from the regular system of the form (11) with valuations $\Psi^j = (0, \dots, 0)$ at each singular points s_j, $j = 1, \dots, m$. As mentioned above (see [5]), such system is the system of standard form with indices $k_1, \dots, k_n$. It follows from (12) and (13) that the fundamental matrix of solutions of the system of standard form at the points s_j and ∞ has the forms

$$\Phi_j\left(\tilde{z}\right) = U_j(z)(\tilde{z} - s_j)^{E_j} \tag{14}$$

and

$$\Phi_\infty(z) = \operatorname{diag}(k_1, \dots, k_n)U_\infty(z), \tag{15}$$

where $U_j(z)$ and $U_\infty(z)$ are holomorphic invertible matrix functions at $s_j, j = 1, \dots, m$ and ∞, respectively.

Proposition 3 ([4, 6, 8])**.** *The splitting type of the canonical vector bundle* E^0 *coincides with the partial indices of the transform matrix* $T(z)$.

Suppose that two m-tuples $(M_1, \dots, M_m)$ and $(E_1, \dots, E_m)$ of matrices are produced from the piecewise constant matrix function $G(t) \in PC(\Gamma)^{n\times n}$. Let (11) be the system of standard form induced from the following data: collection of singular points $\{s_1, \dots, s_m\}$, monodromy matrices $(M_1, \dots, M_m)$ and nonresonant matrices $(E_1, \dots, E_m)$.

Suppose $\theta = \min_{s_k, j}\{\theta_j\}$, $j = 1, \dots, n$, $k = 1, \dots, m$, where θ_j is defined by (2).

Theorem 4. *A piecewise constant matrix function $G(t) \in PC(\Gamma)^{n\times n}$ is factorisable in the space $L^p(\Gamma)$, $p > \theta$ and L^p-partial indices of $G(t)$ coincide with partial indices of the transform matrix $T(z)$.*

Proof. Let $(M_1,\dots,M_m)$ and $(E_1,\dots,E_m)$ be the matrices associated with G. Let (11) be the system of standard form with unknown partial indices $k_1,\dots,k_n$ induced from the given data. Let $F_1(z)$, $F_2(z)$ be the fundamental systems of their solutions in U_+ and $U_- \setminus \{\infty\}$, respectively. Then by Theorem 2, there exist nondegenerate $n\times n$-matrices C_1 and C_2 such that

$$G(t) = C_1^{-1}F_1^{-1}(t)\Lambda(t)\Lambda^{-1}(t)F_2(t)C_2,$$

and if we take

$$G_+(z) = C_1^{-1}F_1^{-1}(z),\ z\in U_+,\ G_-(z) = \Lambda^{-1}(z)F_2(z)C_2,\ z\in U_-\setminus\{\infty\},$$

we obtain the Φ-factorization of $G(t)$ in L_p :

$$G(t) = G_+(t)\Lambda(t)G_-(t),$$

where $\Lambda(t) = \mathrm{diag}(t^{k_1},\dots,t^{k_n})$. On the other hand, it follows from (13) and (15) that n-tuple $(k_1,\dots,k_n)$ coincides with the splitting type of the canonical vector bundle induced from this system of equations. Therefore by Proposition 3, the partial indices $k_1,\dots,k_n$ coincide with the partial indices of the transform matrix $T(z)$. This completes the proof.

Example. Consider the second order Fuchsian system with four singular points

$$\frac{dy}{dz} = \left(\frac{\begin{pmatrix} -\frac{7}{4} & i \\ 0 & -\frac{1}{2}\end{pmatrix}}{z-(-\frac{1}{2}+\frac{3i}{2})} + \frac{\begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ 0 & -\frac{1}{3}\end{pmatrix}}{z-(\frac{1}{3}+\frac{3i}{2})} + \frac{\begin{pmatrix} \frac{5}{4} & -i \\ 0 & -\frac{1}{3}\end{pmatrix}}{z-(\frac{1}{2}-\frac{i}{4})} + \frac{\begin{pmatrix} \frac{3}{4} & -\frac{1}{2} \\ 0 & -\frac{1}{4}\end{pmatrix}}{z-(-\frac{1}{3}-\frac{i}{3})} \right) y.$$

The transform matrix for valuations $\Psi = ((0,0),(0,0),(0,0),(0,0))$ which corresponds to canonical extension of vector bundle or, equivalently, to the system of standard form, is

$$T(z) = \frac{1}{z-1/2+i/4}\begin{pmatrix} T_{11} & T_{12} \\ 0 & -T_{22}\end{pmatrix}.$$

Here,

$$T_{11} = (z+1/2-3i/4)^2(z-1/3-3i/2),$$
$$T_{12} = (-12/11-12i/11)(z+1/2-3i/4)(z-1/3-3i/2),$$
$$T_{22} = (z-1/2+i/4)(z+1/2-3i/4)(z-1/3-3i/2).$$

The partial indices of $T(z)$ are $k_1 = 1$, $k_2 = 1$ and therefore the splitting type of the associated vector bundle is $(1,1)$.

The local monodromy matrices corresponding to singular points $1/2-3i/2$, $1/3+3i/2$, $1/2-i/4$, $-1/3-i/3$ are

$$M_1 = \begin{pmatrix} i & \frac{4}{5}-\frac{4}{5}i \\ 0 & -1\end{pmatrix},\quad M_2 = \begin{pmatrix} -i & 3i(-2+\sqrt{3}) \\ 0 & -\frac{1}{2}-\frac{\sqrt{3}i}{2}\end{pmatrix},$$
$$M_3 = \begin{pmatrix} i & \frac{6\sqrt{3}}{19}+\frac{2}{19}-\frac{6}{19}i \\ 0 & -\frac{1}{2}-\frac{i\sqrt{3}i}{2}\end{pmatrix},\quad M_4 = \begin{pmatrix} -i & 0 \\ 0 & -i\end{pmatrix},$$

respectively.

Consider the quadrilateral with vertices at points $s_1 = 1/2-3i/2$, $s_2 = 1/3+3i/2$, $s_3 = 1/2-i/4$ and $s_4 = -1/3-i/3$ as a closed contour on the complex plane. Let $f(t)$ be a piecewise constant matrix function defined as follows: $f(t) = M_1$ if $t\in(s_1,s_2]$; $f(t) = M_2M_1$, if $t\in(s_2,s_3]$, $f(t) = M_3M_2M_1$, if $t\in(s_3,s_4]$ and $f(t) = M_4M_3M_2M_1$, if $t\in(s_4,s_1]$. The eigenvalues of the monodromy matrices are $\pm i, 1, \frac{1}{2}+\frac{\sqrt{3}i}{2}$. The quantities θ_j are $\frac{1}{2\pi}\arg(i) = \frac{1}{4}$, $\frac{1}{2\pi}\arg(-i) = \frac{3}{4}$, $\frac{1}{2\pi}\arg(1) = 0$, $\frac{1}{2\pi}\arg(\frac{1}{2}+\frac{\sqrt{3}i}{2}) = \frac{1}{6}$. Therefore $\theta = \frac{6}{5}$. Consequently, L^p-partial indices of piecewise constant matrix function $f(t)$ are $k_1 = 1$ and $k_2 = 1$, where $p > \frac{6}{5}$.

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