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ON THE SPECTRAL FACTORIZATION OF SINGULAR, NOISY, AND LARGE MATRICES BY JANASHIA–LAGVILAVA METHOD

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Dedicated to the memory of Edem Lagvilava

Abstract. Janashia–Lagvilava algorithm is a relatively new method of matrix spectral factorization. In our previous publications on this topic, we demonstrated that the algorithm is capable to compete with other existing methods of factorization. In the present paper, we provide further refinements of the algorithm emphasizing that it might have a significant advantage in many scenarios arising in practical applications.

1. Introduction

Spectral factorization plays a prominent role in a wide range of applications in Communications, System Theory, Control Engineering and so on. It is the process by which a positive definite matrix-valued function S on the unit circle $\mathbb{T}=\{t\in\mathbb{C}:|t|=1\}$ is expressed in the form

$$S(t)=S^+(t)\big(S^+(t)\big)^*,$$

for a certain matrix-valued function S^+ which can be extended analytically into the open unit disc. It is assumed that the entries of S are integrable functions, $S_{ij}\in L_1(\mathbb{T})$, and therefore the entries of S^+ are square integrable functions i.e. S^+_{ij} belong to the Hardy space of analytic functions H_2.

In the scalar case arising for single input and single output systems, the factorization problem is relatively easy, and several classical algorithms exist to tackle it (see the survey paper [20]) together with the reliable information on their software implementations [11]. Matrix spectral factorization (MSF) which arises for multidimensional systems is essentially more difficult. Therefore, starting with Wiener's original efforts [21] to create a sound computational method of MSF, dozens of different algorithms have appeared in the literature (see the survey papers [18, 20] and references therein; for more recent results, see [2, 13]). Of course, the problem remains hard and none of existing algorithms claim to factorize arbitray matrix within any accuracy in reasonable time.

A different approach to the solution of the MSF problem, without imposing any additional restriction on S, besides the necessary and sufficient condition

$$\int\limits_{\mathbb{T}}\log\det S(t)\,dt>-\infty$$

for the existence of spectral factorization, was originally developed by Janashia and Lagvilava in [14] for 2×2 matrices. This approach was subsequently extended to matrices of arbitrary size in [15]. The effective numerical implementation of the method was proposed in [7] where three different algorithms of MSF were described suitable for different situations. It was illustrated in [7] by numerical simulations that Janashia–Lagvilava algorithm is comparable in accuracy and speed with Wilson algorithm [22] which is widely used nowadays in neuroscience [3, 12]. However, this comparison has been performed under the assumption that the exact result was known beforehand. Of course, this condition is not fulfilled in practical situations.

In the present paper, we provide further improvements and ramifications of existing implementations of Janashia–Lagvilava algorithm which reveal significant advantages of the method in many scenarios encountered in practical applications. In particular, we demonstrate that Janashia–Lagvilava

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method is capable to identify the difficulties of the factorization of a given matrix and react adequately according to the needs and requirements of the applier. Consequently, the algorithm can make trade-offs between the available time and desired accuracy which exist in the solution of every practical problem.

2. General Description of Janashia–Lagvilava Method

In this section we outline Janashia–Lagvilava matrix spectral factorization method. The details can be found in [7, 15].

For a positive definite $r \times r$ matrix function $S(t) = [S_{ij}]_{i,j=\overline{1r}}$ with integrable determinant, at first lower-upper triangular factorization is performed,

$$S(t) = M(t)M^*(t). \tag{1}$$

Here

$$M(t) = \begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0 \\ \xi_{21}(t) & f_2^+(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{r-1,1}(t) & \xi_{r-1,2}(t) & \cdots & f_{r-1}^+(t) & 0 \\ \xi_{r1}(t) & \xi_{r2}(t) & \cdots & \xi_{r,r-1}(t) & f_r^+(t) \end{pmatrix}, \tag{2}$$

where $\xi_{ij} \in L_2(\mathbb{T})$ and diagonal entries f_i^+ are stable analytic functions.

Next, each ξ_{ij} is approximated in L_2-norm by its Fourier series:

$$\xi_{ij}(t) \approx \sum\nolimits_{n=-N}^{\infty} c_n\{\xi_{ij}\}t^n =: \xi_{ij}^{[N]}(t) \tag{3}$$

and (2) is approximated by

$$M_N(t) = \begin{pmatrix} f_1^+(t) & 0 & \cdots & 0 & 0 \\ \xi_{21}^{[N]}(t) & f_2^+(t) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \xi_{r-1,1}^{[N]}(t) & \xi_{r-1,2}^{[N]}(t) & \cdots & f_{r-1}^+(t) & 0 \\ \xi_{r1}^{[N]}(t) & \xi_{r2}^{[N]}(t) & \cdots & \xi_{r,r-1}^{[N]}(t) & f_r^+(t) \end{pmatrix}. \tag{4}$$

Then $S_N = M_N M_N^*$, which obviously approximates (1) in L_1-norm, is factorized explicitly:

$$S_N(t) = S_N^+(t)\big(S_N^+(t)\big)^*.$$

In particular, S_N^+ is represented as

$$S_N^+(t) = M_N(t)\mathbf{U}_2(t)\mathbf{U}_3(t)\dots\mathbf{U}_r(t), \tag{5}$$

where each $\mathbf{U}_m$, $m = 2, 3, \dots, r$, has the form

$$\mathbf{U}_m(t) = \begin{pmatrix} U_m(t) & 0 \\ 0 & I_{r-m} \end{pmatrix}. \tag{6}$$

In (6), U_m, $m = 2, 3, \dots, r$, are unitary matrix functions of special structure which are closely related to so called wavelet matrices (see [6, 16]). They are constructed recurrently with respect to m, relying on coefficients $c_n\{\xi_{ij}\}$, $n = -N, \dots, -1$, in (3), by solving corresponding $N \times N$ linear system of algebraic equations. The convergence

$$S_N^+ \to S^+ \tag{7}$$

in L_2-norm is proved in [5].

For practical applications, the most valuable advantage of the proposed method over the existing algorithms is that it is not of iterative type. The single tuning parameter N, which determines the level of approximation in (7), can be selected well in advance of main computations directly after performing (1). Accordingly, no convergence estimations or stopping rules are required afterwards.

Other advantages of the method are highlighted in the following sections.

3. Factorization of Singular Matrices

It is well known that all spectral factorization methods encounter severe difficulties in the cases where the underlying power spectral density matrix S is singular at some points on the unit circle, i.e., $\det S(t_0) = 0$ for some $t_0 \in \mathbb{T}$. This flaw is in the nature of the problem, as spectral factorization is unstable for singular functions even in the scalar case (see [8]). On the other hand, in practical applications, one should always expect that S is singular since the related processes whose behaviour characterizes S are in most cases unstable. Of course, Janashia–Lagvilava method also works more efficiently in the non-singular cases, where the factorization is stable (see [1,5,9]), than in the singular case. However, as numerical simulations confirm, the method actually reduces the difficulty of singular matrix spectral factorization to the problem of scalar spectral factorization for singular functions. This is not a panacea, as numerical spectral factorization is still difficult in the singular scalar case, however, this reduction is rather promising, as the existing formula for factorization of positive functions

$$f^+(z) = \exp\left(\frac{1}{4\pi}\int_{\mathbb{T}} \frac{t+z}{t-z}\log f(t)\,dt\right), \tag{8}$$

no analogue of which exists in the matrix case, rises a hope that the performance of current methods of spectral factorization will be improved along with technological improvements more significantly in the scalar case than in the matrix case. In addition, in many situations encountered in practice, e.g. in the construction of certain wavelets and multiwavelets, the functions appearing on the diagonal in the representation (1) have a simple form and can be factorized easily.

To demonstrate the above claim, we consider the following matrix

$$S(t) = \begin{pmatrix} -\frac{1}{64}t^{-2} + \frac{17}{32} - \frac{1}{64}t^2 & \frac{1}{256}t^{-4} - \frac{3}{256}t^{-2} + \frac{1}{2}t^{-1} + \frac{3}{256} - \frac{1}{256}t^2 \\ -\frac{1}{256}t^{-2} + \frac{3}{256} + \frac{1}{2}t - \frac{3}{256}t^2 + \frac{1}{256}t^4 & \frac{1}{1024}t^{-4} + \frac{3}{256}t^{-2} + \frac{243}{512} + \frac{3}{256}t^2 + \frac{1}{1024}t^4 \end{pmatrix} \tag{9}$$

whose spectral factorization is required for the construction of certain multiwavelets.[1] The determinant of (9) is

$$\det S(t) = \frac{1}{4096}(t^{-2}-1)^2(t^2-1)^2, \tag{10}$$

so S is singular. The exact factorization $S(t) = S^+(t)\big(S^+(t)\big)^*$ of (9), where

$$S^+(t) = \begin{pmatrix} \frac{2}{5} + \frac{2}{5}t - \frac{1}{10}t^2 & \frac{13}{40} - \frac{3}{10}t + \frac{3}{40}t^2 \\ \frac{1}{40} + \frac{2}{5}t + \frac{7}{20}t^2 - \frac{1}{10}t^3 + \frac{1}{40}t^4 & -\frac{3}{160} + \frac{13}{40}t - \frac{21}{80}t^2 + \frac{3}{40}t^3 - \frac{3}{160}t^4 \end{pmatrix}$$

has been achieved by elementary methods described in [4] and hence no specific algorithm was used. However, the exact answer helps to perform a detailed analysis when the different numerical methods are applied.

Since $S_{11}(t) = -\frac{1}{64}t^{-2} + \frac{17}{32} - \frac{1}{64}t^2$ is not singular, it can be factorized within a machine accuracy using the formula (8) and FFT. The determinant (10) can also be easily factorized manually. Therefore, the triangular factor (2) for (9) has the form

$$M(t) = \begin{pmatrix} S_{11}^+(t) & 0 \\ S_{21}(t)/\overline{S_{11}^+(t)} & (t^2-1)^2/64S_{11}^+(t) \end{pmatrix}.$$

Consequently, in this way, a machine accuracy can be achieved by the Janashia–Lagvilava method. In particular, one can take $N \approx 40$ in the Fourier approximation of $M_{21}(t)$, $S_{21}(t)/\overline{S_{11}^+(t)} \approx \sum_{n=-N}^{\infty} c_n t^n$, in order to obtain more than 100 correct digits in $S^+(t)$ by using the MultiPrecision Computation (MPC) toolbox of Matlab.

Another striking example of factorization of singular matrices by the Janashia–Lagvilava method is

$$S(t) = \begin{pmatrix} -\frac{1-4\tilde{\alpha}}{64}t^{-3} + \frac{1+4\alpha}{64}t^{-1} + 1 + \frac{1+4\alpha}{64}t - \frac{1-4\tilde{\alpha}}{64}t^3 & \frac{\tilde{\alpha}}{16}t^{-3} - \frac{\alpha}{16}t^{-1} + \frac{\alpha}{16}t - \frac{\tilde{\alpha}}{16}t^3 \\ -\frac{\tilde{\alpha}}{16}t^{-3} + \frac{\alpha}{16}t^{-1} - \frac{\alpha}{16}t + \frac{\tilde{\alpha}}{16}t^3 & \frac{1-4\tilde{\alpha}}{64}t^{-3} - \frac{1+4\alpha}{64}t^{-1} + 1 - \frac{1+4\alpha}{64}t + \frac{1-4\tilde{\alpha}}{64}t^3 \end{pmatrix}$$

[1] We are grateful to Prof. Vasil Kolev, Bulgarian Academy of Sciences, for providing this example.

with determinant

$$\det S(t) = \frac{8\tilde{\alpha}-1}{4096}(t+1)^4(t-1)^4(t+i)^2(t-i)^2, \tag{11}$$

where $\alpha = 4+\sqrt{15}$ and $\tilde{\alpha} = 4-\sqrt{15}$. The accurate spectral factorization of (11) is required for the construction of the so-called SA4 multiwavelet [17]. Although the exact answer $S^+(t)$ is not available in this case, its coefficients exhibit certain symmetry and therefore the number of correct digits can be identified in the approximated result $\hat{S}^+(t)$ by comparing the corresponding coefficients. (It should be carefully noticed that the estimation of $\|S-\hat{S}^+(\hat{S}^+)^*\|_\infty$ can be misleading in obtaining the number of correct digits in $\hat{S}^+$).

As authors of [17] report, 5-8 correct digits in coefficients of $\hat{S}^+$ can be obtained by traditional methods, while they are making the additional efforts to increase this number to 11. However, the Janashia–Lagvilava method, factorizing again (11) manually, can achieve the machine accuracy in computation of S^+, say, more than 100 correct digits by using the Matlab MPC toolbox.

Of course, this very high accuracy drops drastically if we avoid the exact factorization of the determinant and use the existing numerical methods for scalar factorization. However, the results are anyway better than the ones obtained by other methods. Furthermore, the convergence of iterative methods of spectral factorization is proved under the natural theoretical assumption that all intermediate terms in the iteration are stable. However, in actual computations, if S is singular, some terms of the iteration may develop zeros inside the unit circle due to round-off errors. This might cause a fluctuation or even divergence of consequent terms. On the other hand, proposed algorithm preserves convergent properties even in the case where the diagonal entries in (4) are approximated in unstable way due to round-off errors (see Theorem 5 in [10]). In addition, the stable factorization is easier in the scalar case because of formula (8).

In conclusion, the claim that the Janashia–Lagvilava method reduces the complexity of matrix spectral factorization to the complexity of scalar spectral factorization is justified.

4. Factorization of Noisy Matrices

In many applications where the estimation of an unknown signal is required, a power spectral density matrix S, which has to be factorized, is constructed from empirical observations. Such a procedure is always subject to numerical errors and instead of theoretically existing exact values of S, we obtain its noisy version $\hat{S}$. (If, in addition, S is singular, $\hat{S}$ may occur not to be even positive semi-definite on $\mathbb{T}$. How to deal with such a situation is described in [10].) Usually, in such cases, the spectral factor S^+ provides a probabilistic answer to the imposed estimation problem. Under these circumstances, we intend to identify a rough approximation of $\hat{S}^+$ in the shortest possible time.

Mathematically we can formulate the following general optimization problem: for a given positive definite matrix function $S(t)$ and small $\varepsilon > 0$, find a causal stable $\hat{S}^+$ with minimal possible computations such that

$$\|S(t) - \hat{S}^+(t)\big(\hat{S}^+(t)\big)^*\| < \varepsilon.$$

The Janashia–Lagvilava method can tackle the problem by replacing S with its smooth (non-singular) approximation $\hat{S}$, so that $\|S-\hat{S}\| < \varepsilon/2$ and set

$$\|\hat{S}(t) - M_N(t)M_N^*(t)\| < \frac{\varepsilon}{2}$$

with minimal possible N in the triangular factorization (4). Since, for relatively small N, the triangular matrix $M_N(t)$ can be promptly converted into $S_N^+(t)$ by formula (5), the method has the potential to be adjusted to specific applications.

5. Factorization of Large Matrices

Current neuroscience research might involve factorization of matrices as large as the order of 1000. Entries of these matrices are functions defined on the unit circle as discrete values of high-frequency resolution. Numerical spectral factorization of such matrices is indeed a challenging problem. Janashia–Lagvilava's method can contribute significantly to the solution of this problem.

First of all, we would mention that for such large matrices the modification of the JLE-algorithms 1-3 presented in [7] is necessary. This modification was proposed and utilized in [19] and we will label it as JLE-algorithm 4. This algorithm produces the triangular factorization (2) by pointwise Cholesky factorization in frequency domain, followed by multiplication with the diagonal unitary matrix $D = \mathrm{diag}(u_1, u_2, \dots, u_r)$, where $|u_k| = 1$ for $k = 1, 2, \dots, r$, which makes diagonal entries in $M(t)$ causal. Then $M(t)$ is transformed into the time domain by IFFT, it is approximated by $M_N(t)$, and then returned back to the frequency domain by FFT. Next, recursively with respect to $m = 2, 3, \dots, r$, the m-th row of $M_N\mathbf{U}_2, \dots, \mathbf{U}_{m-1}$ is computed (in time domain), the unitary matrix function (6) is constructed (in time domain) and it is post multiplied by $M_N\mathbf{U}_2 \dots \mathbf{U}_{m-1}$ (in frequency domain) to obtain $M_N\mathbf{U}_2 \dots \mathbf{U}_{m-1} \cdot \mathbf{U}_m$. As it is noticed already in [19], in these procedures, the most time-consuming step is the last multiplication in the frequency domain as it has to be performed $r-1$ times. However, we can perform these multiplications in the time domain at least in the beginning for some $\mathbf{U}_2\mathbf{U}_3 \cdots \mathbf{U}_k$ and then switch to the frequency domain. In addition, as numerical simulations confirm, to achieve the desired accuracy, it is frequently sufficient to compute M (i.e. to perform the pointwise Cholesky factorization and make diagonal entries analytic) with high resolution while continuing multiplications by $\mathbf{U}_k$ with lower resolution.

The main advantage of the Janashia–Lagvilava method is that while processing such large data, the method can analyze the intermediate results and provide a user with corresponding recommendations for optimization of the process. For example, if an $r \times r$ matrix is too large and "heavily" singular, then the factorization of this matrix within acceptable accuracy might be impossible. This negative result might be revealed after a few days of computations by other iterative algorithms losing all the time and getting no necessary information. The user-friendly design of JLE-algorithm 4 can identify this problem within hours and recommend increasing the resolution for obtaining a satisfactory result. Furthermore, JLE-4 can extract as much information as possible even without increasing the resolution. Namely, it can satisfactorily factorize the maximal possible principle $m \times m$ submatrix of the original matrix, where $m < r$. It might not necessarily be the leading principal submatrix. A slightly modified JLE-4 automatically searches for the maximal possible m for which the $m \times m$ submatrix (with the same indexed columns and rows) can be factorized. This modification consists of optimal rearrangement of columns and rows during the Cholesky factorization. Of course, if $r \times r$ spectral density matrix is designed in order to extract some useful information from its spectral factor (mostly for connectivity analysis between the different channels), the factorization of its $m \times m$ submatrix for m close to r will contain a large portion of the necessary information.

6. Conclusions

Properties of the Janashia–Lagvilava method described in this paper were identified by intensive numerical simulations carried out mostly during theoretical research. In the real signal processing applications of the algorithm, when data arrival and transfer of the results are incorporated, further characteristics of the method can be investigated which is the task of future research.

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