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PROPERTY (az) THROUGH TOPOLOGICAL NOTIONS AND SOME APPLICATIONS

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In memorial to mathematician Jorge Medina Sancho, a great teacher of the ESPOL

Abstract. In this article, for a bounded linear operator defined on a complex infinite-dimensional Banach space, through classical methods of local spectral theory, we study an important variant of a-Browder's theorem, the variant known as the property (az). Among other new results, are obtained some characterization with topological terms showing that a set of operators that verify this property (az) is closed. Some existing results are generalized, and from a new perspective, the stability of this property (az) is studied under the classic perturbations that commute with the operator.

1. Introduction and Preliminaries

The spectral theory in the last three decades has got several developments through the diverse use of its techniques, for example, the single-valued extension property (SVEP) has allowing to develop several works, many of them are mentioned in [3]. In this theory, we have the upper semi-Weyl operators, i.e., those with the closed range, the difference between the dimension of their kernel and the co-dimension of their range is negative. These operators have an impact on the theory of linear operators, since the classical a-Browder's theorem is defined through the upper semi-Weyl spectrum. There is a variant of the a-Browder's theorem called the property (az) [19], which means that the set of all spectral points λ of T for which $\lambda I - T$ is upper semi-Weyl, coincides with the set of all eigenvalues λ of T for which $\lambda I - T$ is upper semi-Fredholm with a finite ascent. This variant (az) has been studied in [4], where it is verified that the properties (az) and (gaz) are equivalent, and also, through the classical methods of local spectral theory, a relevant study of the property (gaz) (equivalent for the property (az)) has been made. In [18], we see that the property (az) is equivalent to the property (Sab), if the upper semi-Berkani Weyl spectrum is equal to the upper semi-Weyl spectrum, in this case, the property (az) is equivalent to the other 15 properties, in particular, to the a-Browder's theorem, or equivalently, to the property (Bv), this property (Bv) is studied in [8], where it is noted that the upper semi-Fredholm spectrum and Fredholm spectrum are coincident, if the operator verifies property (az). The property (gaz) has been considered in [10], where by using algebraic techniques, it is demonstrated that this property is transmitted from T to its Drazin inverse, in case that T is a Drazin invertible operator. So, in this paper, we shall give other new results for the property (az), or equivalently, we continue the study for the property (gaz). Actually, in Section 2, the property (az) is characterized by using the SVEP and then, in Section 3, through the topological notions. In Section 4, we studied the property (az) under classic perturbations, and some applications are studied in Section 5.

In what follows, we use the following terminology (see [3], for details).

Let $L(X)$ be the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. Let $T \in L(X)$, denote by $\alpha(T)$ the dimension of kernel of T denoted by $\ker T$ and by $\beta(T)$ the co-dimension of the range $R(T) := T(X)$. Below, we give the following classical notations:

- Spectrum: $\sigma(T)$,
- Approximate point spectrum: $\sigma_a(T)$,
- Fredholm spectrum: $\sigma_e(T)$,

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- Upper semi-Fredholm spectrum: $\sigma_{usf}(T)$,
- B-Fredholm spectrum: $\sigma_{bf}(T)$,
- Upper semi B-Fredholm spectrum: $\sigma_{ubf}(T)$,
- Weyl spectrum: $\sigma_w(T)$,
- Upper semi-Weyl spectrum: $\sigma_{uw}(T)$,
- Lower semi-Weyl spectrum: $\sigma_{lw}(T)$,
- Upper semi B-Weyl spectrum: $\sigma_{ubw}(T)$,
- Browder spectrum: $\sigma_b(T)$,
- Upper semi-Browder spectrum: $\sigma_{ub}(T)$,
- Lower semi-Browder spectrum: $\sigma_{lb}(T)$,
- Drazin invertible spectrum: $\sigma_d(T)$,
- Left Drazin invertible spectrum: $\sigma_{ld}(T)$,
- Quasi-nilpotent part: $H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}$,
- Subspace hyper-range: $T^\infty(X) := \bigcap_{n=1}^{\infty} T^n(X)$,
- $p(T)$ the ascent of T,
- $q(T)$ the descent of T.

The boundary of the spectrum is always contained in the approximate point spectrum (see [3, Theorem 1.12]).
Let $X^* := L(X, \mathbb{C})$ be the *dual* of X. By $T^* \in L(X^*)$ we denote the classical *dual operator* of T defined by

$$(T^* f)(x) := f(Tx) \quad \text{for all} \quad x \in X, \;\; f \in X^*.$$

By $\mathcal{H}(\sigma(T))$ we denote the set of all analytic functions defined on an open neighborhood of $\sigma(T)$, and $f(T)$ is defined by the classical functional calculus, for every $f \in \mathcal{H}(\sigma(T))$. The single-valued extension property at $\lambda \in \mathbb{C}$ was introduced by Finch in [14].

Definition 1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (SVEP at λ_0) if for every open disc $\mathbb{D}$ with $\lambda_0 \in \mathbb{D}$, the only analytic function $f : \mathbb{D} \to X$ satisfying the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in \mathbb{D}$, is the function $f \equiv 0$. An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

It is easy to prove that $T \in L(X)$ has SVEP at every isolated point of the spectrum, we also have

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{1}$$

and dually,

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{2}$$

(see [1, Theorem 3.8]). Furthermore, it is easily seen that

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually,

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{4}$$

Note that $H_0(T)$ is not generally closed, by ([1, Theorem 2.31])

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{5}$$

Remark 1. The converse of the implications (1)–(5) holds also when $\lambda I - T$ is semi-Fredholm, or semi B-Fredholm (see [2]).

Let M, N be two closed linear subspaces of X and define

$$\delta(M, N) := \sup\{\operatorname{dist}(u, N) : u \in M, \|u\| = 1\},$$

in the case $M \neq \{0\}$, otherwise the set $\delta(\{0\}, N) = 0$ for any subspace N. According to [15, §2, Chapter iv], the *gap* between M and N is defined by

$$\widehat{\delta}(M, N) := \max\{\delta(M, N), \delta(N, M)\}.$$

The function $\widehat{\delta}$ is a *gap metric* on the set of all linear closed sub-spaces of X, and the convergence $M_n \to M$ is, obviously, defined by $\widehat{\delta}(M_n, M) \to 0$, as $n \to \infty$.

The *reduced minimal modulus* of T is defined as $\gamma(T) := \inf_{x \notin \ker T} \frac{\|Tx\|}{\operatorname{dist}(x, \ker T)}$. We set $\gamma(0) = \infty$. It is well known that: $\gamma(T) > 0$ if and only if $T(X)$ is closed. Also, $\gamma(T) = \gamma(T^*)$.

Remark 2 ([16, Chapter 10])**.** If $T(X)$ and $T_n(X)$, $(1 \leq n)$ are closed in X, and $\lim\limits_{n \to +\infty} \| T_n - T \| = 0$, then as a result, $\widehat{\delta}(\ker(T_n), \ker(T)) \to 0$ as $n \to \infty$, and also, $\widehat{\delta}(\mathrm{R}(T_n), \mathrm{R}(T)) \to 0$, as $n \to \infty$. Hence $\dim(\ker(T_n)) = \dim(\ker(T))$ and $\dim(\mathrm{R}(T_n)) = \dim(\mathrm{R}(T))$, for $n \geq N_0$. Note that $\lim\limits_{n \to +\infty} \gamma(T_n) = \gamma(T)$.

2. Other Characterizations of the Property (az)

In this section, we consider two important sets linked to the operator $T \in L(X)$, these are:

$$\Delta^+(T) := \sigma(T) \setminus \sigma_{uw}(T), \quad p_{00}^a(T) := \sigma_a(T) \setminus \sigma_{ub}(T).$$

Recently in [9], a study was carried out linking the set $\Delta^+(T)$ through the property (V_Π) given for T if $\Delta^+(T) = \sigma(T) \setminus \sigma_d(T)$. We continues the study linking to $\Delta^+(T)$ through equality $\Delta^+(T) = p_{00}^a(T)$, which implies that

$$\sigma(T) = \sigma_a(T) \quad \text{and} \quad \sigma_{uw}(T) = \sigma_{ub}(T).$$

Truly, if $\lambda \notin \sigma_a(T)$, then $\lambda \notin \sigma_{uw}(T)$, but if $\lambda \in \sigma(T)$, then $\lambda \in \Delta^+(T) = p_{00}^a(T)$, hence $\lambda \in \sigma_a(T)$ which is a contradiction. Thus $\sigma(T) = \sigma_a(T)$, and then $\sigma_{uw}(T) = \sigma_{ub}(T)$.

In general, $p_{00}^a(T) \subseteq \Delta^+(T)$ and under certain conditions, the equality is obtained. In this section, we give some conditions to obtain such equality, referring precisely to the property (az).

Definition 2 ([19])**.** $T \in L(X)$ verifies property (az) if $\Delta^+(T) = p_{00}^a(T)$.

Next, let us consider several consequences and characterizations for the property (az), some through the set $\Delta^+(T)$.

Clearly, if $T \in L(X)$ verifies the property (az), so $\sigma_a(T) = \sigma(T)$, hence by [4, Lemma 2.4] and [9, Lemma 2.1], we have the following

Theorem 1. *If $T \in L(X)$ verifies the property (az), then*

$$\sigma_d(T) = \sigma_{ld}(T), \ \textit{equivalently} \ \sigma_a(T) = \sigma(T), \ \textit{equivalently} \ \sigma_b(T) = \sigma_{ub}(T).$$

Dually, if T^* verifies the property (az), then

$$\sigma_d(T) = \sigma_{rd}(T), \ \textit{equivalently} \ \sigma_s(T) = \sigma(T), \ \textit{equivalently} \ \sigma_b(T) = \sigma_{lb}(T).$$

In general $\sigma_{uw}(T) \subseteq \sigma_w(T)$, but under the effects of the property (az), it is ensured that $\sigma_{uw}(T) = \sigma_w(T)$. Formally, we have the following result.

Theorem 2. *Let $T \in L(X)$. Then*

i) *T verifies the property (az) if and only if $\sigma_b(T) = \sigma_{ub}(T) = \sigma_w(T) = \sigma_{uw}(T)$.*

ii) *T^* verifies the property (az) if and only if $\sigma_b(T) = \sigma_{lb}(T) = \sigma_w(T) = \sigma_{lw}(T)$.*

Proof. i) Directly. If T verifies the property (az), then $\sigma(T) = \sigma_a(T)$ and $\sigma_{uw}(T) = \sigma_{ub}(T)$. By Theorem 1, if $\sigma_{ub}(T) = \sigma_b(T)$, then $\sigma_{uw}(T) = \sigma_b(T)$. Hence

$$\sigma_w(T) = \sigma_{uw}(T) = \sigma_{ub}(T) = \sigma_b(T).$$

Reciprocally, by Theorem 1, as $\sigma_{ub}(T) = \sigma_b(T)$, so $\sigma(T) = \sigma_a(T)$. Since $\sigma_{uw}(T) = \sigma_{ub}(T)$, as a result T verifies the property (az).

ii) Is obtained similarly to (i) by the duality between spectra. □

Next, the property (az) is characterized through the local SVEP.

Theorem 3. *Let $T \in L(X)$. Then*
i) *T^* has the SVEP at $\lambda \notin \sigma_{uw}(T)$ if and only if T verifies the property (az).*
ii) *T has the SVEP at $\lambda \notin \sigma_{lw}(T)$ if and only if T^* verifies the property (az).*

Proof. i) Directly. If $\lambda \notin \sigma_{uw}(T)$, then T^* has the SVEP at λ, by Remark 1, $q(\lambda I - T) < \infty$, this implies that ind $(\lambda I - T) \geq 0$, hence $\lambda \notin \sigma_w(T)$ and this way $q(\lambda I - T) = p(\lambda I - T) < \infty$, whereby $\lambda \notin \sigma_{ub}(T)$, note also that if $\lambda \notin \sigma_a(T)$, then $\lambda \notin \sigma_{uw}(T)$, consequently, $\lambda \notin \sigma(T)$. Hence $\sigma_{uw}(T) = \sigma_{ub}(T)$ and $\sigma_a(T) = \sigma(T)$. It follows that T verifies the property (az).

Reciprocally, if T verifies the property (az), then by Theorem 2, $\sigma_b(T) = \sigma_{uw}(T)$. Hence T^* has the SVEP at $\lambda \notin \sigma_{uw}(T)$, since $q(\lambda I - T) < \infty$ (see equation (2)).

ii) This proof for the duality between spectra is similar to that of part (i), remember that $\sigma_b(T) = \sigma_b(T^*)$ and $\sigma_{lw}(T) = \sigma_{uw}(T^*)$. □

Clearly, if T^* (resp., T) has the SVEP, then (az) is given for T (resp., T^*). It is known that $\sigma_{ubw}(T) \subseteq \sigma_{uw}(T) \subseteq \sigma(T)$, and the property (gaz) is given for T if and only if $\sigma_{ubw}(T) = \sigma_{ld}(T)$ and $\sigma_a(T) = \sigma(T)$. As the properties (az) and (gaz) are equivalents, so the following corollary generalizes [4, Theorem 3.6].

Corollary 1. *Let $T \in L(X)$. Then*
i) *T^* has the SVEP at $\lambda \notin \sigma_{uw}(T)$ if and only if (gaz) is given for T.*
ii) *T has the SVEP at $\lambda \notin \sigma_{lw}(T)$ if and only if (gaz) is given for T^*.*

Example 1. Consider the classical Hardy space $H^2(T)$ and let P denote the projection of $L^2(T)$ onto $H^2(T)$. The Toeplitz operator T_ϕ on $H^2(T)$, with symbol ϕ, is defined by $T_\phi f := P(\phi f)$ for $f \in H^2(T)$.

In [7], it is shown that if orientation of the curve $\phi(T)$ is traced clockwise, then T_ϕ^* has SVEP and, analogously, if orientation of the curve $\phi(T)$ is traced counterclockwise, then T_ϕ has SVEP. The SVEP for T_ϕ^* or T_ϕ, entails that the property (az) holds for T_ϕ or T_ϕ^*, respectively.

Next, a characterization through the quasi-nilpotent part.

Theorem 4. *Let $T \in L(X)$. Then T verifies the property (az) if and only if $H_0(\lambda I - T)^*$ is finite-dimensional, $\forall \lambda \in \Delta^+(T)$.*

Proof. Directly. Suppose that T verifies the property (az), thus by Theorem 2, we have $\sigma_{uw}(T) = \sigma_b(T)$, then by [3, Theorem 4.3], as a result, we have $H_0(\lambda I - T)^*$ is finite-dimensional $\forall \lambda \in \Delta^+(T)$.

Reciprocally, if $\forall \lambda \in \Delta^+(T)$ is $H_0(\lambda I - T)^*$ finite-dimensional so, T^* has SVEP at λ (see equation 5). Hence by Theorem 3, as a result, T verifies the property (az). □

Corollary 2. *Let $T \in L(X)$. Then T^* verifies the property (az) if and only if $H_0(\lambda I - T)$ is finite-dimensional $\forall \lambda \in \Delta^+(T^*)$.*

Proof. The proof is obtained by Theorem 4 and the duality. □

The spectral mapping theorem is known to be invalid for the Weyl-type spectra, however, for spectra, $\sigma(T)$, $\sigma_a(T)$, $\sigma_{ld}(T)$, i.e., $\sigma(f(T)) = f(\sigma(T))$, the same is given for the other two spectra, with $f \in \mathcal{H}(\sigma(T))$. Next, the property (az) is characterized by the spectral mapping theorem.

Theorem 5. *Let $f \in \mathcal{H}(\sigma(T))$ and $T \in L(X)$ checking the property (az). Then $\sigma_{ubw}(f(T)) = f(\sigma_{ubw}(T))$ if and only if $f(T)$ verifies the property (az).*

Proof. Directly, by the hypothesis, as a result, T verifies the property (gaz) and $\sigma_a(T) = \sigma(T)$. But $\sigma_{ubw}(f(T)) = f(\sigma_{ubw}(T))$ so, $\sigma_{ubw}(f(T)) = f(\sigma_{ld}(T)) = \sigma_{ld}(f(T))$. Clearly, $\sigma_a(f(T)) = \sigma(f(T))$. Therefore $f(T)$ verifies the property (gaz) or, equivalently, $f(T)$ verifies the property (az).

Reciprocally, if $f(T)$ verifies the property (az), or equivalently $f(T)$ verifies the property (gaz), then $\sigma_{ubw}(f(T)) = \sigma_{ld}(f(T))$, also, by the hypothesis if $\sigma_{ubw}(T) = \sigma_{ld}(T)$, then $f(\sigma_{ubw}(T)) = \sigma_{ld}(f(T))$. Hence $\sigma_{ubw}(f(T)) = f(\sigma_{ubw}(T))$. □

3. Property (az) and Topological Notions

In this section, by $Cl(A)$, $\text{int}(A)$ and $\partial(A)$ we denote the closure, interior and border, respectively, of $A \subseteq \mathbb{C}$. With this notation, some notions of topology are used to characterize the property (az). If $\Delta^+(T)$ has an empty interior, several results are obtained, among which the property (az) is given for T. To begin, note that $\sigma_{uw}(T) \subseteq \sigma_a(T)$, and recall that a set of the upper semi-Weyl operators is open in $L(X)$.

Theorem 6. *Let $T \in L(X)$. If $\text{int}\,(\Delta^+(T)) = \emptyset$, then $\sigma(T) = \sigma_{\text{a}}(T)$.*

Proof. Let $\lambda_0 \notin \sigma_{\text{a}}(T)$ and suppose that $\lambda_0 \in \sigma(T)$, then $\lambda_0 I - T$ is upper semi-Weyl, and hence there exists an open disc $\mathbb{D}(\lambda_0, \varepsilon)$, centered at λ_0, such that $\lambda I - T$ is upper semi-Weyl and bounded below for all $\lambda \in \mathbb{D}(\lambda_0, \varepsilon)$. Notice that $\mathbb{D}(\lambda_0, \varepsilon) \subseteq \sigma(T)$, if not, $\emptyset \neq \mathbb{D}(\lambda_0, \varepsilon) \cap \partial\sigma(T) \subseteq \sigma_a(T)$. Therefore $\mathbb{D}(\lambda_0, \varepsilon) \subseteq \text{int}\Delta^+(T)$, which is a contradiction, hence $\lambda_0 \notin \sigma(T)$ and then $\sigma(T) = \sigma_{\text{a}}(T)$. □

The following theorem will allow us to prove that a set of operators that verify the property (az) is closed in $L(X)$. Since $\sigma_{ubw}(T) \subseteq \sigma_{uw}(T)$, this theorem generalizes a part of Theorem 3.2 stated in [4].

Theorem 7. *For $T \in L(X)$, the following statements are equivalent:*
i) *T verifies the property (az).*
ii) *$\Delta^+(T) \subseteq \text{iso}\,\sigma_a(T) \subseteq \partial\sigma_a(T)$.*
iii) *$\text{int}\,(\Delta^+(T)) = \emptyset$.*
iv) *$\Delta^+(T) \bigcap \text{acc}(\sigma_a(T)) = \emptyset$, and $\sigma(T) = \sigma_a(T)$.*

Proof. (i) ⇔ (ii) In one direction, $\Delta^+(T) = p_{00}^a(T) \subseteq \Pi_a(T) \subseteq \text{iso}\,\sigma_a(T) \subseteq \partial\sigma_a(T)$. In the reverse direction, if $\lambda \in \Delta^+(T)$, then $\lambda \in \text{iso}\,\sigma_a(T)$, and by equation 3, T verifies the SVEP at λ, and then by Remark 1, $p(\lambda I - T) < \infty$, so $\lambda \in p_{00}^a(T)$. Then T verifies the property (az).

(ii) ⇒ (iii) Is clear.

(iii) ⇒ (i) If $\text{int}\,(\Delta^+(T)) = \emptyset$, then by Theorem 6, $\sigma(T) = \sigma_a(T)$. On the other hand, if $\lambda_0 \in \Delta^+(T)$, there exists an open disc $\mathbb{D}(\lambda_0, \epsilon)$ such that $\lambda \notin \sigma_{uw}(T)$ for all $\lambda \in \mathbb{D}(\lambda_0, \epsilon)$, but $\lambda_0 \in \partial(\sigma(T))$, otherwise there exists an open disc $\mathbb{D}(\lambda_0, \epsilon_1)$ such $\mathbb{D}(\lambda_0, \epsilon_1) \subseteq \sigma(T)$, then λ_0 is an interior point of $\Delta^+(T)$ which is not possible. Hence, $\lambda_0 \in \partial(\sigma(T))$ and so T has the SVEP at λ_0, and then $\sigma_{uw}(T) = \sigma_{ub}(T)$, hence T verifies the property (az).

(ii) ⇒ (iv) It is clear, since $\text{iso}\,\sigma_a(T) \cap \text{acc}\,(\sigma_a(T)) = \emptyset$, and as (ii) ⇒ (i), then $\sigma(T) = \sigma_a(T)$.

(iv) ⇒ (ii) If $\lambda \in \Delta^+(T)$, then $\lambda \notin \text{acc}\,(\sigma_a(T))$, so then $\sigma_a(T)$ does not cluster at λ, thus by equation 3, T has SVEP at λ, hence $p(\lambda I - T) < \infty$ (see Remark 1), thus $\lambda \in \sigma(T) \setminus \sigma_{ub}(T) = \sigma_a(T) \setminus \sigma_{ub}(T) \subseteq \Pi_a(T) \subseteq \text{iso}\,\sigma_a(T) \subseteq \partial\sigma_a(T)$. Hence $\Delta^+(T) \subseteq \text{iso}\,\sigma_a(T) \subseteq \partial\sigma_a(T)$. □

Corollary 3. *Let $T \in L(X)$. Then T verifies the property (az) if and only if $\sigma_{uw}(T) = \sigma_b(T)$.*

Proof. Directly. If T verifies the property (az), then by Theorem 2, we have $\sigma_{uw}(T) = \sigma_b(T)$.

Reciprocally, if $\sigma_{uw}(T) = \sigma_b(T)$, then $\Delta^+(T) \subseteq \text{iso}\,\sigma_a(T)$, thus by Theorem 7, we have that T verifies the property (az). □

Corollary 4. *Let $T \in L(X)$ and $f \in \mathcal{H}(\sigma(T))$. If $\text{acc}\,(\sigma(T)) = \emptyset$, then $f(T)$ verifies the property (az).*

Proof. If $\text{acc}\,(\sigma(T)) = \emptyset$, then $\text{acc}\,(\sigma(f(T))) = \emptyset$, so $\text{int}\,(\Delta^+(f(T))) = \emptyset$. Hence by Theorem 7, $f(T)$ verifies the property (az). □

Example 2. An operator $T \in L(X)$ is said to be algebraic if there exists a complex nontrivial polynomial h such that $h(T) = 0$. Every algebraic operator T has a finite spectrum, whereby $\text{acc}\,(\sigma(T)) = \emptyset$. Then for $f \in \mathcal{H}(\sigma(T))$, $f(T)$ verifies the property (az).

Corollary 5. *Let $T \in L(X)$, if T^* has SVEP, then $\text{int}\,(\Delta^+(f(T))) = \emptyset$, for each $f \in \mathcal{H}(\sigma(T))$.*

Proof. By [1, Theorem 2.40], we find that $f(T^*)$ has SVEP, thus $f(T)$ verifies the property (az), whereby by Theorem 7, we get $\text{int}\,(\Delta^+(f(T))) = \emptyset$. □

Example 3. Let $L \in L(\ell_2(N))$ be the left shift given by

$$L(x_1, x_2, \dots) := (x_2, x_3, \dots), \quad \text{for all } \ x = (x_k)_{k\in N} \in \ell_2(N).$$

Hilbert's adjoint is known to $L^* = R$, also $R^* = L$, where R is the right shift defined as $R(x_1, x_2, \dots) := (0, x_1, x_2, \dots)$ for all $x = (x_k)_{k\in N} \in \ell_2(N)$. It is known that R has SVEP. Thus if $f \in \mathcal{H}(\sigma(L))$, then $\text{int}\,(\Delta^+(f(L))) = \emptyset$.

The following theorem generalizes Theorem 3.15 from [4].

Theorem 8. *Let $T \in L(X)$. Then T verifies the property (az) if and only if the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous at λ, for each $\lambda \in \Delta^+(T)$.*

Proof. Directly. If T verifies the property (az), then T verifies the property (gaz). Now, if $\lambda_0 \in \Delta^+(T)$, then $\lambda_0 \in \Delta_g^+(T)$. By [4, Theorem 3.15], the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous at λ_0 in a gap metric, because $\gamma(\lambda_0 I - T) > 0$, since the range of $\lambda_0 I - T$ is closed, and by [1, Theorem 1.38], the mapping $\lambda \to \gamma(\lambda I - T)$ is not continuous in λ_0.

Reciprocally. Let $\lambda_0 \in \Delta^+(T)$ so, the range of $\lambda_0 I - T$ is closed, whereby $\gamma(\lambda_0 I - T) > 0$, thus by [1, Theorem 1.38], the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous in λ_0 in a gap metric. But $\lambda_0 I - T$ is upper semi-Fredholm so, by [1, Theorem 1.64], there exist $\epsilon_0 > 0$ and a disc $\mathbb{D}_{\epsilon_0}(\lambda_0)$ such that $\forall \lambda \in \mathbb{D}_{\epsilon_0}(\lambda_0) - \{\lambda_0\}$ is $k := \alpha(\lambda I - T) \le \alpha(\lambda_0 I - T) < \infty$.

Note that the mapping $\lambda \to \ker(\lambda I - T)$ is not continuous $\forall \lambda \in \Delta^+(T)$ in a gap metric, whereby $\alpha(\lambda I - T) < \alpha(\lambda_0 I - T)$, $\forall \lambda \in \mathbb{D}_{\epsilon_0}(\lambda_0) - \{\lambda_0\}$.

We known that $\rho_{uw}(T) := \sigma(T) \setminus \sigma_{uw}(T)$ is open, so $\exists\, \epsilon_1 > 0$ and a disc $\mathbb{D}_{\epsilon_1}(\lambda_0)$ such that $\mathbb{D}_{\epsilon_1}(\lambda_0) \subseteq \rho_{uw}(T)$. We consider $\epsilon = \min\{\epsilon_0, \epsilon_1\}$, then $\forall \lambda \in \mathbb{D}_{\epsilon}(\lambda_0) - \{\lambda_0\}$ implies that $\lambda \in \rho_{uw}(T)$ and $\alpha(\lambda I - T) < \alpha(\lambda_0 I - T)$.

Now, we suppose there exists λ_1 such that $\alpha(\lambda_1 I - T) > 0$ and $\lambda_1 \in \mathbb{D}_\epsilon(\lambda_0) - \{\lambda_0\}$, so $\alpha(\lambda_1 I - T) < \alpha(\lambda_0 I - T)$. But notice that $\lambda_1 \in \Delta^+(T)$, how is it made for λ_0, by the discontinuity at λ_1, $\exists \lambda_2 \in \mathbb{D}_\epsilon(\lambda_0) - \{\lambda_0, \lambda_1\}$ such that $k = \alpha(\lambda_2 I - T) < \alpha(\lambda_1 I - T) = k$, which is absurd. Hence $\alpha(\lambda_1 I - T) = 0$.

Thus $\forall \lambda \in \mathbb{D}_\epsilon(\lambda_0) - \{\lambda_0\}$ we have $\alpha(\lambda I - T) = 0$ and so, $\lambda_0 \in \text{iso}\,\sigma_a(T)$, since $\alpha(\lambda_0 I - T) > 0$. Hence $\Delta^+(T) \subseteq \text{iso}\,\sigma_a(T)$, so by Theorem 7, T verifies the property (az). □

The set of operators that verify the property (az) is closed in $L(X)$. In fact.

Theorem 9. *Let $T \in L(X)$ and T_n be a sequence of operators in $L(X)$ such that T_n verifies the property (az), $n \ge 1$, if $\lim\limits_{n\to+\infty} \| T_n - T \| = 0$. Then:*

i) *For some $N_3 \in \mathbb{N}$, the mapping $\lambda \to \gamma(\lambda I - T_n)$ is not continuous at λ in gap a metric, $\forall \lambda \in \Delta^+(T)$ and $\forall n \ge N_3$.*

ii) *T verifies the property (az).*

iii) $\lim\limits_{n\to+\infty} \gamma(\lambda I - T_n) = \gamma(\lambda I - T)$, *$\forall \lambda \in \Delta^+(T)$.*

Proof. i) The set of upper semi-Weyl operators is open. Let $\lambda_0 \in \Delta^+(T)$, so by the convergence of T_n to T, $\exists N_0 \in \mathbb{N}$ such that $\lambda_0 \notin \sigma_{uw}(T_n)$, $\forall n \ge N_0$. Since $\lambda_0 \in \sigma(T)$, by Remark 2, we have $\exists\, N_1 \in \mathbb{N}$ such that $\lambda_0 \in \sigma(T_n)$, $\forall n \ge N_1$. Hence if $N_3 := \max\{N_0, N_1\}$, then

$$\Delta^+(T) \subseteq \Delta^+(T_n), \quad \forall n \ge N_3.$$

By Theorem 8, $\forall n \ge N_3$, the mapping $\lambda \to \gamma(\lambda I - T_n)$ is not continuous at $\lambda_0 \in \Delta^+(T_n)$ in a gap metric. Hence $\forall n \ge N_3$, the mapping $\lambda \to \gamma(\lambda I - T_n)$ is not continuous at each $\lambda \in \Delta^+(T)$ in a gap metric.

ii) As in the previous part, there exists $N_3 \in \mathbb{N}$ such that $\forall n \ge N_3$, $\Delta^+(T) \subseteq \Delta^+(T_n)$. But by Theorem 7, $\text{int}\,\Delta^+(T_n) = \emptyset$. Hence $\text{int}\,\Delta^+(T) = \emptyset$. Again, by Theorem 7, we conclude that T verifies the property (az).

iii) Let $\lambda \in \Delta^+(T)$ be arbitrary, then $(\lambda I - T)(X)$ is closed and so, $\gamma(\lambda I - T) > 0$, also, by the hypothesis, $\lim\limits_{n\to+\infty} \| T_n - T \| = 0$, by Remark 2, it follows that $\forall \lambda \in \Delta^+(T)$ is $\lim\limits_{n\to+\infty} \gamma(\lambda I - T_n) = \gamma(\lambda I - T)$. □

4. Property (az) Under Perturbations

In this section, we study (in a summary way) mainly the stability of the property (az) under commuting perturbations which are nilpotent, quasi-Nilpotent, Riesz and algebraic. First is justified, and then establish the result.

An operator $N \in L(X)$ is nilpotent if there is an $n \in \mathbb{N}$ such that $N^n = 0$. We know that if N commutes with $T \in L(X)$, then $\sigma(T) = \sigma(T+N)$ and by [1, Theorem 3.65], $\sigma_{uw}(T) = \sigma_{uw}(T+N)$, thus $\Delta^+(T) = \Delta^+(T+N)$. Therefore by Theorem 7, we havethe next

Theorem 10. *Let $T \in L(X)$ and $N \in L(X)$ be a nilpotent operator that commutes with T. Then T verifies the property (az) if and only if $T+N$ verifies the property (az).*

Recall that an operator $Q \in L(X)$ is quasi-Nilpotent if for all $\lambda \neq 0$, $\lambda I - Q$ is invertible, that is, $\sigma(Q) = 0$. We suppose that Q commutes with $T \in L(X)$. Thus by [3, Corollary 3.24], $\sigma(T) = \sigma(T+Q)$ and by [3, Corollary 3.18], $\sigma_{uw}(T) = \sigma_{uw}(T+Q)$. Hence $\Delta^+(T) = \Delta^+(T+Q)$. By Theorem 7, we have the next

Theorem 11. *Let $T \in L(X)$, and let $Q \in L(X)$ be a quasi-Nilpotent operator that commutes with T. Then T verifies the property (az) if and only if $T+Q$ verifies the property (az).*

Recall that $R \in L(X)$ is a Riesz operator if for all $\lambda \neq 0$, $\lambda I - R$ is an Fredholm operator. If R^n for some $n \in N$ is a Riesz operator, then by [1, Theorem 3.4], we find that R is a Riesz operator, also, if R commutes with $T \in L(X)$, then by [3, Corollary 3.18], we have $\sigma_{uw}(T) = \sigma_{uw}(T+R)$, by [3, Theorem 3.16], T has a finite ascent if and only if $T+R$ has finite ascent, so then $\sigma_{ub}(T) = \sigma_{ub}(T+R)$. Analogous result is that $\sigma_{lb}(T) = \sigma_{lb}(T+R)$, hence $\sigma_b(T) = \sigma_b(T+R)$. Now, applying Corollary 3, we get the next.

Theorem 12. *Let $T \in L(X)$ such that $RT = TR$, being R^n a Riesz operator, for some $n \in N$. Then T verifies the property (az) if and only if $T+R$ verifies the property (az).*

An operator $T \in L(X)$ is *hereditarily polaroid* if every isolated point of the spectrum is a pole of the resolvent of $T_{|M}$, where M is a closed T-invariant subspace of X, by [3, Theorem 4.31], we have that T has SVEP. On the other hand, if $K \in L(X)$ is algebraic and commutes with T, so by [5, Theorem 2.3], $T+K$ has the SVEP, then $f(T+K)$ has SVEP for $f \in \mathcal{H}(\sigma(T+K))$. Therefore we have the following

Theorem 13. *If $T \in L(X)$ is hereditarily polaroid, then T^*, $T^* + K^*$ and $f(T+K)^*$ verify the property (az), where $f \in \mathcal{H}(\sigma(T+K))$ and $K \in L(X)$ is algebraic and commutes with T.*

Corollary 6. *If $T \in L(X)$ is hereditarily polaroid, then $f(T)^* + K^*$ verifies the property (az), where $f \in \mathcal{H}(\sigma(T))$ and $K \in L(X)$ is algebraic commuting with T.*

5. Some Applications

In this section, we obtain some applications of the results obtained with the property (az).

i) We consider $\Delta_+(T) := \sigma(T) \setminus \sigma_{usf}(T)$, if $\operatorname{int} \Delta_+(T) = \emptyset$, then proceeding in a similar way as in proving (iii) $\Rightarrow$ (i) of Theorem 7, we get that T has SVEP at $\lambda \notin \sigma_{usf}(T)$, by [17, Theorem 2.2], we have that $\sigma_{usf}(T) = \sigma_{ub}(T)$ or, equivalently, T verifies the property (bz), but, $\Delta^+(T) \subseteq \Delta_+(T)$, thus T verifies the property (az) (see Theorem 7), for that $\sigma_a(T) = \sigma(T)$, and then the applied Theorems 1 and 2 result in

$$\sigma_{usf}(T) = \sigma_e(T) = \sigma_{uw}(T) = \sigma_w(T) = \sigma_{ub}(T) = \sigma_b(T).$$

On the other hand, note that T verifies the property (gaz) so, $\sigma_{ubw}(T) = \sigma_{ld}(T)$ and $\sigma(T) = \sigma_a(T)$ (then $\sigma_d(T) = \sigma_{ld}(T)$) and by [17, Theorem 2.4], also, T verifies the property (gbz), i.e., $\sigma_{ubf}(T) = \sigma_{ld}(T)$. Therefore

$$\sigma_{ubf}(T) = \sigma_{bf}(T) = \sigma_{ubw}(T) = \sigma_{bw}(T) = \sigma_{ld}(T) = \sigma_d(T).$$

For instance, if S is a left m-invertible contraction such that $\sigma(S) \subseteq \Gamma$, then $\lambda \in \sigma(S)$ is a polo of S if and only if $(\lambda I - S)(X)$ is closed, (see [13], for definition and details). Hence as $\forall \lambda \in \Delta_+(S)$, we have that $(\lambda I - S)(X)$ is closed, then $\Delta^+(S) \subseteq \Delta_+(S) \subseteq p_{00}(S) \subseteq \Pi(S) \subseteq \operatorname{iso}\sigma_a(S)$, whereby:

$$\sigma_{usf}(S) = \sigma_e(S) = \sigma_{uw}(S) = \sigma_w(S) = \sigma_{ub}(S) = \sigma_b(S).$$
$$\sigma_{ubf}(S) = \sigma_{bf}(S) = \sigma_{ubw}(S) = \sigma_{bw}(S) = \sigma_{ld}(S) = \sigma_d(S).$$

ii) Recall that an operator $T \in L(X)$ is said to be *Drazin invertible* if there exist an operator $S \in L(X)$ (called the *Drazin inverse* of T) and an integer $n \geq 0$ such that

$$TS = ST,\ STS = S,\ T^nST = T^n. \tag{6}$$

The operator S described in (6) is unique and also is Drazin invertible (see [11]).
By [9, Theorem 4.2], we have

$$0 \in \sigma(S) \setminus \sigma_{uw}(S) \Leftrightarrow 0 \in \sigma(T) \setminus \sigma_{uw}(T).$$

Also, from [6], if $\lambda \neq 0$, we have

$$\sigma_{uw}(S) \setminus \{0\} = \{\frac{1}{\lambda} : \lambda \in \sigma_{uw}(T) \setminus \{0\}\}.$$

Note that $\operatorname{int}\Delta^+(T) = \emptyset$ if and only if $\operatorname{int}\Delta^+(S) = \emptyset$. Therefore by Theorem 7,

T verifies the property (az) if and only if S verifies the property (az).

On the other hand, in [10], it is proved that T verifies the property (gaz) if and only if S verifies the property (gaz), but using other algebraic methods.

iii) Let Y be an infinite-dimensional Banach space, we considered that $S \in L(Y)$ and $T \in L(X)$. Note that $\sigma_{uw}(T \oplus S) \subseteq \sigma_{uw}(T) \cup \sigma_{uw}(S)$ and how $p(T \oplus S) = \max(p(S), p(T))$, the same for descent, we get that

$$\sigma_{ub}(T \oplus S) = \sigma_{ub}(T) \cup \sigma_{ub}(S) \text{ and } \sigma_b(T \oplus S) = \sigma_b(T) \cup \sigma_b(S).$$

Now, if both T and S satisfy the property (az), then

$T \oplus S$ verifies property (az) if and only if $\sigma_{uw}(T) \cup \sigma_{uw}(S) = \sigma_{uw}(T \oplus S)$.

In fact, in the direct sense, the result shows that $\sigma_{ub}(T \oplus S) = \sigma_{uw}(T \oplus S)$. Clearly, if $\lambda \notin \sigma_{uw}(T \oplus S)$, then $\lambda \notin (\sigma_{ub}(T) \cup \sigma_{ub}(S))$, but T and S satisfy the property (az), whereby $\lambda \notin (\sigma_{uw}(T) \cup \sigma_{uw}(S))$. Hence $\sigma_{uw}(T) \cup \sigma_{uw}(S) = \sigma_{uw}(T \oplus S)$. On the other hand, in the reciprocal sense, how T and S satisfy the property (az) so, by Corollary 3, we get that $\sigma_{uw}(T) = \sigma_b(T)$ and $\sigma_{uw}(S) = \sigma_b(S)$, whereby $\sigma_{uw}(T \oplus S) = \sigma_b(T \oplus S)$, therefore again, by Corollary 3, the result shows that $T \oplus S$ verifies the property (az).

iv) Let W be a proper closed subspace of X, and we consider the following set:

$$\mathcal{P}(X, W) := \{T \in L(X) : T(W) \subseteq W, T^{n_0}(X) \subseteq W, \text{for some } n_0 \geq 1\}.$$

Let $T \in \mathcal{P}(X, W)$, T_W denotes the restriction of T over the T-invariant subspace W of X. Thus, T is not surjective. On the other hand, if $q(T) = \infty$, or $p(T) = \infty$, then by [12, Theorem 4.1], we have

$$\sigma(T) = \sigma(T_W),\ \sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T_W),\ \sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T_W) \text{ and } \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(T_W).$$

So, applying Theorems 1 and 2, we obtain

T verifies the property (az) if and only if T_W verifies the property (az).

Note that by Theorem 7 the same result is obtained, if $0 \in \sigma_{uw}(T_W)$. Truly, if $0 \in \sigma_{uw}(T_W)$ so, $0 \in \sigma_{uw}(T)$, whereby, if $\lambda \notin \sigma_{uw}(T_W)$ or $\lambda \notin \sigma_{uw}(T)$, then $\lambda \neq 0$, and by [12, Lemma 3.2], if $\lambda \neq 0$, we obtain

$$R(\lambda I - T_W) = R(\lambda I - T) \cap W,\ \alpha(\lambda I - T) = \alpha(\lambda I - T_W),\ \beta(\lambda I - T) = \beta(\lambda I - T_W).$$

In this way, the result is: $\sigma(T) = \sigma(T_W)$, $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T_W)$, whereby

$$\operatorname{int}(\Delta^+(T)) = \emptyset \text{ if and only if } \operatorname{int}(\Delta^+(T_W)) = \emptyset.$$

Thus by Theorem 7, T verifies the property (az) if and only if T_W verifies the property (az).

v) Let $T \in L(X)$ checking property (az) be such that $\sigma_{uw}(T) = \sigma_{uw}(T + K)$, where $K \in L(X)$ is algebraic that commutes with T and $\sigma_{uw}(T) \bigcap \sigma(K) = \emptyset$. Then $T + K$ verifies the property (az). Since T has the property (az), by Theorem 2, we have $\sigma_{uw}(T) = \sigma_b(T)$. Now, if $\lambda \notin \sigma_{uw}(T)$, then T^* has the SVEP at λ. Also, $\sigma(K) = \sigma(K^*)$ and K^* is algebraic. By [5, Theorem 2.3], we find that $T^* + K^*$ has the SVEP at $\lambda \notin \sigma_{uw}(T) = \sigma_{uw}(T + K)$. Hence by Theorem 3, $T + K$ verifies the property (az).

Dually, if K is an algebraic operator that commutes with T, with $\sigma_{lw}(T) = \sigma_{lw}(T + K)$ and $\sigma_{lw}(T) \bigcap \sigma(K) = \emptyset$, then by Theorem 3, we have that $T^* + K^*$ verifies the property (az).

6. Conclusions

In this article, we have studied the property (az), equivalent to the property (gaz). Therefore we were able to add important results to these two properties which have found some applications. It should be noted that the obtained in this paper results make it possible to establish that the set of operators that verify the property (gaz) or, equivalently, the property (az), is closed in $L(X)$. Other results are obtained in a simplified way, for example, those of Section 4.

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