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A REMARK ON CONSTRUCTION OF J-UNITARY MATRIX POLYNOMIALS

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Dedicated to the memory of Edem Lagvilava

Abstract. A certain algorithm for construction of J-unitary matrices has been proposed in L. Ephremidze, A Saatashvili, I. Spitkovsky, On J-unitary matrix polynomials. *J. Math. Sci.*, 2022. `https://link.springer.com/article/10.1007/s10958-022-05878-w`. In this note, we provide an example which shows that the algorithm does not work in all situations when the problem has a solution.

1. Introduction

Let J be a diagonal matrix

$$J = \operatorname{diag}(j_1, j_2, \dots, j_{m-1}, 1), \tag{1}$$

where each j_k is either positive or negative 1, $j_k = \pm 1$. Without lose of generality, we assume that $j_m = 1$. A matrix $U \in \mathbb{C}^{m\times m}$ is called J-unitary if $UJU^* = J$, where $*$ denotes conjugate transpose. A matrix function $U(t)$, where $\mathbb{T} := \{t \in \mathbb{C} : |t| = 1\}$, is called J-unitary if

$$U(t)JU^*(t) = J, \quad t \in \mathbb{T}.$$

If $u(t) = \sum\limits_{k=0}^{N} c_k t^k$, $c_k \in \mathbb{C}$, is a polynomial, $u \in \mathcal{P}_N^+$, let $\tilde{u}(t) = \sum\limits_{k=0}^{N} \overline{c_k} t^{-k}$. Note that $\widetilde{u}(t) = \overline{u(t)}$ for $t \in \mathbb{T}$.

J-unitary matrix polynomials of the special structure

$$U(t) = \begin{pmatrix} u_{11}(t) & u_{12}(t) & \cdots & u_{1m}(t) \\ u_{21}(t) & u_{22}(t) & \cdots & u_{2m}(t) \\ \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}(t) & u_{m-1,2}(t) & \cdots & u_{m-1,m}(t) \\ \widetilde{u_{m1}}(t) & \widetilde{u_{m2}}(t) & \cdots & \widetilde{u_{mm}}(t) \end{pmatrix}, \quad u_{ij} \in \mathcal{P}_N^+, \tag{2}$$

with the property

$$\det U(t) = 1, \quad \text{for} \quad t \in \mathbb{T}, \tag{3}$$

play a crucial role in the generalization of Janashia–Lagvilava method [2,5] for J-spectral factorization [3]. Particularly, it can be proved (see [1, Theorem 4.1]) that for a matrix function F of the form

$$F(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1(t) & \zeta_2(t) & \zeta_3(t) & \cdots & \zeta_{m-1}(t) & 1 \end{pmatrix}, \quad \zeta_k \in \mathcal{P}_N^-, \tag{4}$$

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where $\mathcal{P}_N^- := \left\{ \sum_{k=1}^{N} \alpha_k t^{-k} : \alpha_k \in \mathbb{C} \right\}$, there exists a J-unitary matrix polynomial (2) satisfying (3) such that

$$FU \in (\mathcal{P}_N^+)^{m\times m}, \tag{5}$$

if and only if the matrix function $F(t)JF^*(t)$ possesses the left J-spectral factorization. The latter condition means that the representation

$$F(t)\,JF^*(t) = \Phi_+(t)\,J\,\Phi_+^*(t) \tag{6}$$

is valid, where $\Phi \in (\mathcal{P}_N^+)^{m\times m}$ and $\det\Phi(t) \neq 0$ for $|t| \leq 1$. This is also equivalent to the condition that the left partial indices of (6) are equal to zero [6].

In [1], an algorithm is proposed for construction of J-unitary matrix polynomials of the aforementioned structure. The algorithm is a generalization of the Janashia–Lagvilava method for matrices J with indefinite structure, i.e., for matrices (1) with some j_k equal to -1. Similarly to this method, for a matrix function (4), the algorithm explicitly constructs J-unitary matrix polynomial (2) satisfying (3) such that (5) holds. It is demonstrated in [1] that the algorithm works well for every matrix function (4) except for some isolated singular cases. These exceptional situations are not surprising since unlike the classical case, where $J = I$ and the Janashia–Lagvilava method works well, $F(t)\,JF^*(t)$ might not have zero left partial indices and the corresponding matrix polynomial (2) would not exist at all. However, the question naturally arises (see [1, Remark 5.2]) if the algorithm works in every situation where the left partial indices are equal to 0, i.e., the representation (6) holds. In the present paper, we answer negatively to this question. Particularly, an example of the matrix function (4) is constructed for which the corresponding J-unitary matrix polynomial exists, however, it cannot be determined by the algorithm proposed in [1]. This example indicates that further refinement of the algorithm is desirable for the singular case.

2. The Algorithm for Constructing J-Unitary Matrix Polynomials

In this section, we describe the algorithm for constructing J-unitary matrix function (2) for a given matrix function (4) presented in [1]. A careful examination reveals that this algorithm does not work in all situations when the problem has a solution. However, it works generically, except for some isolated singular cases.

Consider the following system of conditions:

$$\begin{cases} \zeta_1 x_m - j_1\widetilde{x_1} \in \mathcal{P}^+, \\ \zeta_2 x_m - j_2\widetilde{x_2} \in \mathcal{P}^+, \\ \vdots \\ \zeta_{m-1} x_m - j_{m-1}\widetilde{x_{m-1}} \in \mathcal{P}^+, \\ \zeta_1 x_1 + \zeta_2 x_2 + \cdots + \zeta_{m-1}x_{m-1} + \widetilde{x_m} \in \mathcal{P}^+, \end{cases} \tag{7}$$

where $\zeta_i \in \mathcal{P}_N^-$, $i = 1, 2, \ldots, m-1$, are the entries of F in (4), and $\mathcal{P}^+ = \cup_{N\geq 1}\mathcal{P}_N^+$ is the set of all polynomials.

A vector function $\mathbf{u} = \big(u_1, u_2, \ldots, u_{m-1}, \widetilde{u_m}\big)^T$, where $u_i \in \mathcal{P}_N^+$ for each $i = 1, 2, \ldots, m$, is called a solution of (7) if all the conditions in (7) are satisfied whenever $x_i = u_i$, $i = 1, 2, \ldots, m$, and it is proved in [1, Lemma 5.1] that if

$$\mathbf{u} = \big(u_1, u_2, \ldots, \widetilde{u_m}\big)^T \quad\text{and}\quad \mathbf{v} = \big(v_1, v_2, \ldots, \widetilde{v_m}\big)^T$$

are two (possibly identical) solutions of system (7), then

$$\sum_{k=1}^{m-1} j_k u_k \widetilde{v_k} + \widetilde{u_m} v_m = \text{const}\,. \tag{8}$$

Therefore, the goal is to construct m linearly independent solutions of (7). To this end, (7) is rewritten in equivalent form of a linear system of equations. Namely, equating all the coefficients of the non-positive powers of t of the functions in the left-hand side of (7) to zero, except for the free term of the qth function which is a set equal to 1, one arrives at the following system of algebraic equations in the block matrix form, which is denoted by $\mathbb{S}_q$:

$$\mathbb{S}_q := \begin{cases} \Gamma_1 X_m - j_1 \overline{X_1} = \mathbf{0}, \\ \Gamma_2 X_m - j_2 \overline{X_2} = \mathbf{0}, \\ \Gamma_q X_m - j_q \overline{X_q} = \mathbf{1}, \\ \Gamma_{m-1} X_m - j_{m-1} \overline{X_{m-1}} = \mathbf{0}, \\ \Gamma_1 X_1 + \cdots + \Gamma_{m-1} X_{m-1} + \overline{X_m} = \mathbf{0}. \end{cases} \tag{9}$$

Here, we use the following notation:

$$\Gamma_i = \begin{pmatrix} 0 & \gamma_{i1} & \gamma_{i2} & \cdots & \gamma_{i,N-1} & \gamma_{iN} \\ \gamma_{i1} & \gamma_{i2} & \gamma_{i3} & \cdots & \gamma_{iN} & 0 \\ \gamma_{i2} & \gamma_{i3} & \gamma_{i4} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \gamma_{iN} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad i = 1, 2, \ldots, m-1, \tag{10}$$

$$X_i = (a_{i0}, a_{i1}, \ldots, a_{iN})^T, \quad i = 1, 2, \ldots, m,$$

where

$$\zeta_i(t) = \sum_{n=1}^{N} \gamma_{in} t^{-n}, \quad i = 1, 2, \ldots, m-1, \quad x_i(t) = \sum_{n=0}^{N} a_{in} t^n, \quad i = 1, 2, \ldots, m,$$

and

$$\mathbf{0} = (0, 0, \ldots, 0)^T \in \mathbb{C}^{(N+1)\times 1}, \quad \mathbf{1} = (1, 0, 0, \ldots, 0)^T \in \mathbb{C}^{(N+1)\times 1}.$$

Determining X_i, $i = 1, 2, \ldots, m-1$, from the first $m-1$ equations of (9),

$$X_i = j_i \left(\overline{\Gamma_i}\, \overline{X_m} - \delta_{iq}\, \mathbf{1} \right), \tag{11}$$

$i = 1, 2, \ldots, m-1$, and then substituting them in the last equation of (9), one gets

$$j_1 \Gamma_1\ \overline{\Gamma_1}\, \overline{X_m} + j_2 \Gamma_2\ \overline{\Gamma_2}\, \overline{X_m} + \cdots + j_{m-1} \Gamma_{m-1}\ \overline{\Gamma_{m-1}}\, \overline{X_m} + \overline{X_m} = j_q \Gamma_q\ \mathbf{1} \tag{12}$$

(it is assumed that the right-hand side is equal to $\mathbf{1}$ when $q = m$) or, equivalently,

$$\Delta\, \overline{X_m} = j_q\, \Gamma_q\, \mathbf{1}, \tag{13}$$

where

$$\Delta = \sum_{k=1}^{m-1} j_k \Gamma_k \Gamma_k^* + I_{N+1} \tag{14}$$

(Γ^* is used in place of $\overline{\Gamma}$ because $\Gamma^T = \Gamma$). The algorithm is continued under the additional restriction that

$$\det \Delta \neq 0. \tag{15}$$

Therefore (13) has a unique solution for every right-hand side.

Finding $\overline{X_m}$ from (13) and then determining $X_1, X_2, \ldots, X_{m-1}$ from (11), one gets the unique solution of $\mathbb{S}_q$ denoted by $(X_1^q, X_2^q, \ldots, X_{m-1}^q, X_m^q)$. Suppose

$$X_i^q := (a_{i0}^q, a_{i1}^q, \ldots, a_{iN}^q)^T, \quad i = 1, 2, \ldots, m, \tag{16}$$

and let

$$V = \begin{pmatrix} v_{11} & v_{12} & \cdots & v_{1m} \\ v_{21} & v_{22} & \cdots & v_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ v_{m-1,1} & v_{m-1,2} & \cdots & v_{m-1,m} \\ \widetilde{v_{m1}} & \widetilde{v_{m2}} & \cdots & \widetilde{v_{mm}} \end{pmatrix}, \tag{17}$$

where

$$v_{ij}(z) = \sum_{n=0}^{N} a_{in}^{j} z^n, \quad 1 \le i,\ j \le m, \tag{18}$$

Then the columns of (17) are the solutions of the system (7) and it is proved in [1] that if $\det \Delta_0 \neq 0$, where Δ_0 is the $N \times N$ submatrix of Δ obtained by deleting its first row and column, then $V(1)$ is invertible and

$$\mathbf{U}(z) = V(z)\big(V(1)\big)^{-1} \tag{19}$$

is the desired J-unitary matrix polynomial.

Theorem 5.1 in [1] proves that if $\det \Delta \neq 0$ and $\det \Delta_0 = 0$, then the desired J-unitary matrix polynomial does not exist. Namely, $\det V(t) \equiv 0$ in this situation and the algorithm cannot be applied as $\big(V(1)\big)^{-1}$ does not exist in (19). However, the following specific question was left unanswered in [1]: If we know that $\det \Delta = 0$, can we again claim that the desired J-unitary matrix polynomial does not exist? An example constructed in the following section provides the negative answer to this question.

3. The Specific Example

In this section, we provide an example of the matrix function (4) such that the left partial indices of $F(t)\,JF^*(t)$ are equal to 0, i.e., the representation (6) holds, and (15) does not hold. As it was mentioned above, for such F, there exists a J-unitary matrix polynomial (2) satisfying (3) such that (5) holds, however, this U cannot be constructed by the algorithm proposed in [1].

The example is similar to the one given in [1]. Namely, let $m = 2$, $J = \mathrm{diag}(-1, 1)$, and

$$F(t) = \begin{pmatrix} 1 & 0 \\ \sqrt{\alpha}(t^{-1} + t^{-2}) & 1 \end{pmatrix}, \tag{20}$$

where the positive constant α is specified later.

We used the symbolic computations of MATLAB to obtain some of the following relations. The equations where the variable t is involved are assumed to hold for $t \in \mathbb{T}$.

The corresponding to (20) matrix Γ is

$$\Gamma = \begin{pmatrix} 0 & \sqrt{\alpha} & \sqrt{\alpha} \\ \sqrt{\alpha} & \sqrt{\alpha} & 0 \\ \sqrt{\alpha} & 0 & 0 \end{pmatrix}$$

and, consequently, (see (15))

$$\Delta = -\Gamma\Gamma^* + I_3 = -\begin{pmatrix} 2\alpha - 1 & \alpha & 0 \\ \alpha & 2\alpha - 1 & \alpha \\ 0 & \alpha & \alpha - 1 \end{pmatrix}. \tag{21}$$

The determinant of (21) is $-(\alpha^3 - 6\alpha^2 + 5\alpha - 1)$ which has the roots $\alpha = 5.0489\ldots, 0.6431\ldots, 0.3080\ldots$.

Also, for $\alpha \neq 0$, we have

$$F(t)\,J\,F^*(t) = F_1(t)F_2(t)F_3(t), \tag{22}$$

where

$$F_1(t) = \begin{pmatrix} -\sqrt{\alpha}t + \sqrt{\alpha} - 1/\sqrt{\alpha} & t - 1 \\ -\alpha & \sqrt{\alpha} \end{pmatrix}, \quad F_2(t) = \begin{pmatrix} t & 0 \\ (\alpha^2 - 3\alpha + 1)/\sqrt{\alpha} & t^{-1} \end{pmatrix},$$

and $F_3(t) =$

$$\begin{pmatrix} \sqrt{\alpha}t^{-3} & \alpha t^{-2} + (\alpha - 1)t^{-1} + 1 \\ -(\alpha^2 - 3\alpha + 1)t^{-2} + (\alpha - 1)t^{-1} - 1 & -(\alpha^2 - 3\alpha + 1)\sqrt{\alpha}t^{-1} - (\alpha^3 - 5\alpha^2 + 5\alpha)/\sqrt{\alpha} \end{pmatrix}.$$

One can check that $\det F_1(t) = -1$ and $\det F_3(t) = 1$. Hence the left partial indices of (22) coincide with those of F_2 for each $\alpha \neq 0$.

It is well known (see, e.g., formula (1.23) in [4]) that the left partial indices of $\begin{pmatrix} t & 0 \\ \varepsilon & t^{-1} \end{pmatrix}$ are $(1,-1)$ if $\varepsilon = 0$ and $(0,0)$ if $\varepsilon \neq 0$, since

$$\begin{pmatrix} t & 0 \\ \varepsilon & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 0 & -1/\varepsilon \\ 1 & (\varepsilon t)^{-1} \end{pmatrix}.$$

Therefore, (cf., [1, Example]), partial indices of (22) are nonzero if $\alpha^2 - 3\alpha + 1 = 0$ (that is $\alpha = 2.6180\ldots, 0.3820\ldots$) and they are zero otherwise. Hence the left partial indices are equal to 0 and, correspondingly, the left J-spectral factorization of (22) exists even for values of α such that $\alpha^3 - 6\alpha^2 + 5\alpha - 1 = 0$, i.e., when the determinant of (21) is equal to 0. According to the above-mentioned Theorem 4.1 in [1], the desired unitary matrix function $\mathbf{U}$ exists in such situations.

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