Transactions of A. Razmadze
Mathematical Institute
Vol. **176** (2022), issue 3, 313–321

NORMALIZATION OF WIENER–HOPF FACTORIZATION FOR MATRIX FUNCTIONS WITH DISTINCT PARTIAL INDICES

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Dedicated to the memory of Edem Lagvilava

Abstract. For a matrix function $A(t)$ with different partial indices we study the normalization of the Wiener–Hopf factorization $A(t) = A_-(t)D(t)A_+(t)$ that guarantees the uniqueness of the factors $A_\pm(t)$. Previously, this problem was fully investigated for the cases of the stable factorization and the factorization of second-order matrix functions. A notion of a P-normalized factorization is introduced. The definition of this concept uses the Birkhoff factorization of matrix functions. It is shown that in the case of different partial indices, the matrix function $A(t)$ admits P-normalization if and only if the matrix $A_-(\infty)$ admits PLU-factorization. P-normalization allows us to find the Birkhoff factorization of $A(t)$ and to obtain explicit estimates of absolute errors for the factors of an approximate factorization.

1. Introduction

In mathematics and its applications, various factorization problems for matrix functions often arise. The *spectral factorization* of positive definite matrix functions plays an important role in Linear System and Control Theory (see the review [14]). In these branches of applied mathematics, the *fractional factorization* of rational matrix functions [12] is also often used. One of the main problems of linear finite-dimensional dynamical systems (the minimal realization problem) is solved in terms of fractional factorization [12]. In the theory of linear differential equations, the problem of *Birkhoff's factorization* [5] has arisen. Finally, the problem of the *Wiener–Hopf factorization* of matrix functions [9,13] is of fundamental importance in mathematical physics, in the theory of differential equations and in complex analysis. It is the main tool for solving the systems of singular integral equations with the Cauchy kernel and the systems of Wiener–Hopf equations and is an essential step in solving nonlinear evolutionary equations by the inverse scattering method. Also, the Wiener–Hopf factorization is widely used in applications to mechanics.

Let us recall the definitions of these factorization problems. We will study the problems on the unit circle $\mathbb{T}$. The Wiener–Hopf and Birkhoff factorizations are considered in the matrix Wiener algebra $W^{p\times p}(\mathbb{T})$ [9,13].

Spectral factorization. Let $S(z)$ be a positive-definite (a.e.) $p \times p$ matrix function with integrable on $\mathbb{T}$ elements. We suppose that the Paley–Wiener condition, $\log\det S(t) \in L_1(\mathbb{T})$, is satisfied. Then $S(z)$ admits the following (left) spectral factorization:

$$S(z) = S^+(z)(S^+(z))^*.$$

Here, the elements of the factor $S^+(z)$ belong to the Hardy space H_2, $\det S^+(z)$ is an outer function and $(S^+(z))^* = (\overline{S^+(1/\overline{z})})^T$.

Right fractional factorization. This is a generalization to the matrix case of a representation of a proper rational fraction as a ratio of two polynomials. Let $R(z)$ be a regular rational $p \times p$ matrix function. The right coprime fractional factorization of $R(z)$ is its representation as

$$R(z) = N_r(z)D_r^{-1}(z),$$

2020 *Mathematics Subject Classification.* 47A68, 30E25.
Key words and phrases. Wiener–Hopf factorization; Partial indices; Normalization of factorization.

where $N_r(z)$, $D_r(z)$ are the right coprime matrix polynomials and $\det D_r(z) \not\equiv 0$. Also, we suppose that $D_r(z)$ is column proper. This implies that the matrix composed of the coefficients at the highest powers of the columns is an invertible matrix.

Right Wiener–Hopf factorization. Let $A(t)$ be a matrix function from the matrix Wiener algebra $W^{p\times p}(\mathbb{T})$ that is invertible on the unit circle $\mathbb{T}$. Then it can be represented in the following form:

$$A(t) = A_-(t)D(t)A_+(t), \quad t \in \mathbb{T}. \tag{1}$$

Here, $A_\pm(t)$ belong to the group $GW_\pm^{p\times p}(\mathbb{T})$ invertible elements of the subalgebra $W_\pm^{p\times p}(\mathbb{T})$ consisting of absolutely convergent matrix Fourier series for which the Fourier coefficients with negative/positive indices are equal to zero. The middle factor $D(t)$ is the diagonal matrix $D(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}]$, where integers $\rho_1 \le \dots \le \rho_p$ are the right partial indices of $A(t)$. The relation $\rho_1 + \dots + \rho_p = \varkappa = \operatorname{ind}_{\mathbb{T}} \det A(z)$ is valid. This factorization is called the right Wiener–Hopf factorization. Similarly, (by rearranging the factors $A_\pm$), the left Wiener–Hopf factorization is defined.

Right Birkhoff factorization. This factorization was introduced by G. Birkhoff [5] in connection with some problems for the ordinary differential equations. The right Birkhoff factorization $A(t)$ is its representation in the following form:

$$A(t) = D_b(t)B_-(t)B_+(t), \quad t \in \mathbb{T}, \tag{2}$$

where $B_\pm(t) \in GW_\pm^{p\times p}(\mathbb{T})$ and $D_b(t) = \operatorname{diag}\left[t^{\beta_1}, \dots, t^{\beta_p}\right]$, $\beta_1, \dots, \beta_p$ are the right Birkhoff indices of $A(t)$. In contrast to partial indices, the Birkhoff indices are not uniquely determined by the matrix function $A(t)$.

It turns out that these factorization problems are closely related. The relationship between the spectral factorization problem and the Wiener–Hopf canonical factorization problem is well known (see, e.g., [13, Ch.7]). If a matrix function $S(z) \in W^{p\times p}(\mathbb{T})$ is positive definite on $\mathbb{T}$, then its spectral factorization is a right Wiener–Hopf factorization and any right Wiener–Hopf factorization $S(z) = S^+(z)S^-(z)$ can be reduced to a spectral factorization after an appropriate normalization of the factor $S^+(z)$.

I. C. Gohberg and M. A. Kaashuk developed the space state method for constructing the Wiener–Hopf factorization of a rational matrix function using a minimal realization (see, for example, the review [10]). Since the minimal realization of the linear system can be constructed in terms of the fractional factorization of its transfer function, they actually obtained a connection between these factorizations. The explicit connection between the fractional factorization of rational matrix functions and the Wiener–Hopf factorization of meromorphic in the disc $|z| < 1$ matrix functions was established in [1].

The connection between the Wiener–Hopf factorization and the Birkhoff factorization was found in [6]. It turned out that among all possible sets of Birkhoff indices there always exists a set obtained by some permutation of the right partial indices. This means that if $A(t) = A_-(t)D(t)A_+(t)$ is an arbitrary Wiener–Hopf factorization of $A(t)$, then one of the Birkhoff factorizations can be written in the form

$$A(t) = PD(t)P^{-1}B_-(t)B_+(t), \quad t \in \mathbb{T},$$

where P is some permutation matrix. Here the matrix function $B(t) = B_-(t)B_+(t)$ admits a right canonical factorization. Thus, the Wiener–Hopf factorization $A(t)$ with non-zero partial indices can be reduced to the canonical factorization of $B(t)$. This important fact was firstly discovered by I. S. Čebotaru [6] in developing the projection methods for solving systems of discrete Wiener–Hopf equations with non-zero partial indices.

For the practical applications of these factorization problems, it is necessary to develop algorithms for their approximate solutions. However, this is complicated due the non-uniqueness of solutions of the factorization problems.

In the spectral factorization $S(z) = S^+(z)(S^+(z))^*$ the factor $S^+(z)$ is unique up to a constant right unitary factor U [7]. The choice of U allows us to perform the desired normalization of the spectral factorization. In [8], a new efficient method for constructing the spectral factorization is developed, which allows to find approximately the spectral factor $S^+(z)$. In this paper, the *canonical normalization* of the spectral factorization is used. By an arbitrary spectral factor $S^+(z)$, the canonically normalized factor $S_c^+(z) = S^+(z)U$ is constructed by using $U = (S^+(0))^{-1}\sqrt{S^+(0)(S^+(0))^{-1}}$. The canonically normalized spectral factor is uniquely determined by the condition that the matrix $S_c^+(0)$ is positive definite.

For the Wiener–Hopf factorization, the situation is more complicated. According to the theorem on the general form of the factorization, the factor $A_-(z)$ is found up to the right factor $Q_-(z)$, which is an upper block triangular matrix function with entries that are polynomial in z^{-1} (see, e.g., [9, Chapter VIII, Theorem 1.2]). The problem of normalization of the Wiener–Hopf factorization (except for the trivial case of a matrix function with equal partial indices) has not been studied. The well-known theorem of M. A. Shubin (see [13, Theorem 6.15]) on the continuity of factorization factors is incomplete, since the factorization is not unique. Hence this theorem cannot be applied to constructing an approximate factorization.

In this paper, we want to fill a gap in the theory of Wiener–Hopf factorization associated with a normalization. We restrict ourselves to the case of matrix functions with different partial indices. We will describe the method for a canonical normalization that guarantees the uniqueness of the factorization. This method essentially uses the Birkhoff factorization. Since the normalization of the Wiener–Hopf factorization for a matrix function with equal partial indices is not difficult, the obtained results allow us to completely solve the normalization problem for the second-order matrix functions. For this class, this was previously done in [3] which containes a detailed analysis of stable cases of normalizations and proposes a supplement to the Shubin theorem. Moreover, it is shown that utilization of the normalized factorization allows to obtain an error estimate for the factors of the approximate factorization. In [4], the normalization problem was studied for a matrix function with a stable system of partial indices.

The results of this work were announced in [2].

2. P-normalization and Uniqueness of the Wiener–Hopf Factorization

The main tool in the normalization theory is the Gohberg–Krein theorem on the general form of the factorization factors $A_\pm(t)$ [9, Ch. VIII, Theorem 1.2]. Let us formulate it in a form that is convenient for us.

Let $\rho_1, \dots, \rho_p$ be an arbitrary set of integers, sorted in ascending order: $\rho_1 \le \dots \le \rho_p$. We assume that this set contains s different numbers $\varkappa_1 < \dots < \varkappa_s$ of multiplicity $k_1, \dots, k_s$, respectively. Let $\mathcal{Q}_-(\rho_1, \dots, \rho_p)$ be the set of all block triangular matrix functions of the form

$$Q_-(t) = \begin{pmatrix} Q_{11} & Q_{12} & \dots & Q_{1s} \\ 0 & Q_{22} & \dots & Q_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{ss} \end{pmatrix}. \tag{3}$$

Here, the block Q_{ij} has dimension $k_i \times k_j$, the diagonal blocks Q_{ii} are the constant invertible $k_i \times k_i$ matrices, and the off-diagonal blocks $Q_{ij}(t)$ are matrix polynomials in the variable t^{-1} of degree at most $\varkappa_j - \varkappa_i$. The set $\mathcal{Q}_-(\rho_1, \dots, \rho_p)$ is a subgroup of the group $GW_-^{p\times p}(\mathbb{T})$.

Let $D(t) = \operatorname{diag}[t^{\rho_1}, \dots, t^{\rho_p}]$. Define a matrix function

$$Q_+(t) = D^{-1}(t)Q_-^{-1}(t)D(t).$$

$Q_+(t)$ has the same form as (3), and only in this case, $Q_{ij}(t)$ are the matrix polynomials in t of degree at most $\varkappa_j - \varkappa_i$. Thus, $Q_+(z) \in GW_+^{p\times p}(\mathbb{T})$.

The Gohberg–Krein theorem on the general form of the factorization states that if (1) is a Wiener–Hopf factorization of the matrix function $A(t)$ with partial indices $\rho_1 \le \dots \le \rho_p$, then the representation

$$A(t) = G_-(t)D(t)G_+(t), \tag{4}$$

where $G_-(t) = A_-(t)Q_-(t)$, $G_+(t) = Q_+(t)A_+(t)$, is also a Wiener–Hopf factorization of $A(t)$ for any $Q_-(t) \in \mathcal{Q}_-(\rho_1, \dots, \rho_p)$. Moreover, any factorization $A(t)$ can be obtained from the original factorization (1) in a similar way with an appropriate choice of $Q_-(t)$.

Definition 1. The transition from the original factorization (1) to the factorization (4) by using any matrix function $Q_-(t) \in \mathcal{Q}_-(\rho_1, \dots, \rho_p)$ will be called **the normalization of the factorization** (1) **at infinity**. The matrix function $Q_-(t)$ is called the **normalization matrix**.

Thus, the normalization of the factorization at infinity is determined by the choice of $Q_-(t) \in \mathcal{Q}_-(\rho_1, \dots, \rho_p)$. Since the $\mathcal{Q}_-(\rho_1, \dots, \rho_p)$ is a group, we can produce the normalization in several steps choosing the normalization matrix arbitrarily at each step.

Our task is to produce (in some sense) a canonical choice of $Q_-(t)$. The main condition that determines the choice of a canonical representative $Q_-(t) \in \mathcal{Q}_-(\rho_1, \dots, \rho_p)$ will be related to the Birkhoff factorization of the matrix function of $A(t)$.

Definition 2. Let P be a permutation matrix of order p. The Wiener–Hopf factorization of the matrix-function $A(t)$:

$$A(t) = C_-(t)D(t)C_+(t)$$

is called P-**normalized** if the following conditions are fulfilled:

(1) The matrix function $B_-(t) = PD^{-1}(t)P^{-1}C_-(t)D(t)$ belongs to the algebra $W_-^{p\times p}(\mathbb{T})$;
(2) $B_-(\infty) = P$.

We consider the P-**normalization** as canonical normalization of the factorization.

It turns out that if the P-normalized factorization $A(t)$ exists, then it is unique and generates a Birkhoff factorization $A(t)$.

Theorem 1. *Suppose a matrix function $A(t) \in GW^{p\times p}(\mathbb{T})$ has a P-normalized Wiener–Hopf factorization*

$$A(t) = C_-(t)D(t)C_+(t).$$

Then

(1) *this Wiener–Hopf factorization generates a Birkhoff factorization by the formula*

$$A(t) = PD(t)P^{-1}B_-(t)B_+(t),$$

where $B_-(t) = PD^{-1}(t)P^{-1}C_-(t)D(t)$, $B_+(t) = C_+(t)$;

(2) *the given P-normalized Wiener–Hopf factorization is unique.*

Proof. Condition (1) of Definition 2 is equivalent to the statement that $B_-(t) \in GW_-^{p\times p}(\mathbb{T})$. The existence of the above Birkhoff factorization is verified directly.

Let us prove the uniqueness of the P-normalized Wiener–Hopf factorization. Assume that $A(t) = \widetilde{C}_-(t)D(t)\widetilde{C}_+(t)$ is another P-normalized factorization of $A(t)$ and

$$A(t) = PD(t)P^{-1}\widetilde{B}_-(t)\widetilde{B}_+(t)$$

is the corresponding Birkhoff factorization. Then $\widetilde{B}_-^{-1}(t)B_-(t) = \widetilde{B}_+(t)B_+^{-1}(t)$, and, therefore, by Liouville's theorem, this matrix function is a constant invertible matrix. Hence $\widetilde{B}_-^{-1}(t)B_-(t) = \widetilde{B}_-^{-1}(\infty)B_-(\infty) = I_p$, due to condition (2) of Definitions 2. Thus $\widetilde{B}_-(t) = B_-(t)$ and $\widetilde{C}_-(t) = C_-(t)$, $\widetilde{C}_+(t) = C_+(t)$. □

Remark 1. The condition $B_-(\infty) = P$, which ensures the uniqueness of the P-normalized factorization, can be replaced by $B_-(\infty) = A_0$, where A_0 is any invertible matrix. The condition $B_-(\infty) = P$ allows us to obtain a simplest form of the factors $C_-(t)$, $B_-(t)$ in P-normalized factorizations.

3. Existence of a P-normalization for Matrix Functions with Different Systems of Partial Indices

In this section, we prove that the P-normalized factorization always exists for a matrix function having the different right partial indices $\rho_1 < \cdots < \rho_p$. The explicit form of the factor $C_-(t)$ for such a factorization and the form of $B_-(t)$ in the corresponding Birkhoff factorization will also be obrtained. It turns out that the existence of the P-normalization is equivalent to the existence of the so-called PLU-factorization of the invertible numerical matrix $A_-(\infty)$.

Recall (see, e.g., [11]) the definition of an *LU-factorization* of a numerical invertible matrix A_0. If A_0 is a product of lower and upper triangular matrices L and U, $A_0 = LU$, then A_0 is said to be admitting an LU-factorization. A necessary and sufficient condition for the existence of an LU-factorization of the matrix A_0 is that all leading minors of this matrix are nonzeros. If we fix the diagonal elements of the matrix L, then the LU-factorization will be unique. We assume that all diagonal elements of L are equal to 1.

In general, by permutation of rows of A_0, it is always possible to make all its leading minors non-zero. It means that there exists a permutation matrix P^{-1} such that $P^{-1}A_0$ admits an LU-factorization, i.e., A_0 is represented as $A_0 = PLU$. This is the *PLU–factorization* A_0. In general, the permutation P is non-unique in this representation of A_0.

Since the matrix $A_-(\infty)$ is invertible, there always exists its PLU–factorization. Note that if there exists a Wiener–Hopf factorization $A(t) = A_-(t)D(t)A_+(t)$ for which $A_-(\infty)$ admits the PLU–factorization with the given P, then it follows from the theorem on the general form of factorization that any Wiener–Hopf factorization $A(t)$ has this property.

Theorem 2. *Suppose a matrix function $A(t) \in GW^{p\times p}(\mathbb{T})$ has different right partial indices $\rho_1 < \rho_2 < \cdots < \rho_p$, and let $\rho_{ji} = \rho_j - \rho_i$ for $i < j$. The matrix function $A(t)$ admits the P-normalized factorization if and only if for some factorization $A(t) = A_-(t)D(t)A_+(t)$ the numerical matrix $A_-(\infty)$ admits PLU-factorization.*

If this condition is fulfilled, then the P-normalized Wiener–Hopf factorization and the corresponding Birkhoff factorization have the form

$$A(t) = C_-(t)D(t)C_+(t), \quad A(t) = PD(t)P^{-1}B_-(t)B_+(t),$$

where

$$C_-(t) = P\begin{pmatrix} 1+t^{-1}c_{11}^- & t^{-\rho_{21}-1}c_{12}^- & \cdots & t^{-\rho_{p1}-1}c_{1p}^- \\ c_{21}^- & 1+t^{-1}c_{22}^- & \cdots & t^{-\rho_{p2}-1}c_{2p}^- \\ \cdots & \cdots & \cdots & \cdots \\ c_{p-1,1}^- & c_{p-1,2}^- & \cdots & t^{-\rho_{p,p-1}-1}c_{p-1,p}^- \\ c_{p,1}^- & c_{p,2}^- & \cdots & 1+t^{-1}c_{pp}^- \end{pmatrix}, \tag{5}$$

$$B_-(t) = P\begin{pmatrix} 1+t^{-1}c_{11}^- & t^{-1}c_{12}^- & \cdots & t^{-1}c_{1p}^- \\ t^{-\rho_{21}}c_{21}^- & 1+t^{-1}c_{22}^- & \cdots & t^{-1}c_{2p}^- \\ \cdots & \cdots & \cdots & \cdots \\ t^{-\rho_{p-1,1}}c_{p-1,1}^- & t^{-\rho_{p-1,2}}c_{p-1,2}^- & \cdots & t^{-1}c_{p-1,p}^- \\ t^{-\rho_{p,1}}c_{p,1}^- & t^{-\rho_{p,2}}c_{p,2}^- & \cdots & 1+t^{-1}c_{pp}^- \end{pmatrix}, \tag{6}$$

Here, $c_{ij}^-(t) \in W_-(\mathbb{T})$.

Proof. For convenience, we use the notation $\mathcal{O}(t^{-\ell})$ for a function of the form $t^{-\ell}c_-(t)$, where $c_-(t)$ is a function that is analytic in the neighborhood of infinity, $\ell \geq 0$.

First, we prove the theorem for the case of $P = I$, i.e., when there exists a Wiener–Hopf factorization $A(t) = A_-(t)D(t)A_+(t)$ for which $A_-(\infty)$ admits an LU-factorization: $A_-(\infty) = LU$.

Let us show that a factorization of the form (5) always exists. We normalize the original factorization by taking $Q_0 = U^{-1} \in \mathcal{Q}_-(\rho_1,\dots,\rho_p)$ as the normalizing matrix. Then $A(t)$ admits the factorization $A(t) = A_-^{(0)}(t)D(t)A_+^{(0)}(t)$, $A_-^{(0)}(t) = A_-(t)Q_0$, for which

$$A_-^{(0)}(t) = \begin{pmatrix} 1+t^{-1}a_{11}^{(0)} & t^{-1}a_{12}^{(0)} & \cdots & t^{-1}a_{1,p-1}^{(0)} & t^{-1}a_{1p}^{(0)} \\ a_{21}^{(0)}(t) & 1+t^{-1}a_{22}^{(0)} & \cdots & t^{-1}a_{2,p-1}^{(0)} & t^{-1}a_{2p}^{(0)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p-1,1}^{(0)} & a_{p-1,2}^{(0)} & \cdots & 1+t^{-1}a_{p-1,p-1}^{(0)} & t^{-1}a_{p-1,p}^{(0)} \\ a_{p,1}^{(0)} & a_{p,2}^{(0)} & \cdots & a_{p,p-1}^{(0)} & 1+t^{-1}a_{p,p}^{(0)} \end{pmatrix}, \tag{7}$$

where $a_{ij}^{(0)}(t) \in W_-(\mathbb{T})$.

Let us prove by induction on the number of rows s that there always exists a normalized factorization $A(t)=A_-^{(s)}(t)D(t)A_+^{(s)}(t)$ for which the matrix function $A_-^{(s)}(t)$ has the first s rows as in (5), and the remaining $p-s$ rows as in matrix (7).

Let $s=1$. Take as a normalizing matrix

$$Q_1(t)=\begin{pmatrix}1 & t^{-1}q_{12} & \cdots & t^{-1}q_{1p}\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\end{pmatrix}\in\mathcal{Q}_-(\rho_1,\dots,\rho_p),$$

where $q_{ij}(t)$ are arbitrary polynomials in t^{-1} of degree at most $\rho_{ji}-1$. Now,

$$A(t)=A_-^{(1)}(t)D(t)A_+^{(1)}(t),\ A_-^{(1)}(t)=A_-^{(0)}(t)Q_1(t),$$

is a new normalized Wiener–Hopf factorization. The first row of the factor $A_-^{(1)}(t)$ has the form

$$\begin{pmatrix}1+t^{-1}a_{11}^{(0)} & t^{-1}(1+t^{-1}a_{11}^{(0)})q_{12}+t^{-1}a_{12}^{(0)} & \cdots & t^{-1}(1+t^{-1}a_{11}^{(0)})q_{1p}+t^{-1}a_{1p}^{(0)}\end{pmatrix}.$$

For any choice of polynomials $q_{12}(t),\dots,q_{1p}(t)$, the rows of $A_-^{(1)}(t)$ with indices $2,\dots,p$ have the same form as the corresponding rows of the matrix $A_-^{(0)}(t)$ (see formula (7)). Let us represent the element $t^{-1}(1+t^{-1}a_{11}^{(0)})q_{1j}+t^{-1}a_{1j}^{(0)}$ of the first row in the form

$$t^{-1}(1+t^{-1}a_{11}^{(0)})\big[q_{1j}+\frac{a_{1j}^{(0)}}{1+t^{-1}a_{11}^{(0)}}\big].$$

The function $\frac{a_{1j}^{(0)}(t)}{1+t^{-1}a_{11}^{(0)}(t)}$ is analytic in a neighborhood of infinity. Let $\frac{a_{1j}^{(0)}(t)}{1+t^{-1}a_{11}^{(0)}(t)}=\sum\limits_{k=0}^{\infty}q_k^{(1j)}t^{-k}$ be its expansion into the Laurent series. Set $q_{1j}(t)=-\sum\limits_{k=0}^{\rho_{j1}-1}q_k^{(1j)}t^{-k}$. Then

$$t^{-1}(1+t^{-1}a_{11}^{(0)})q_{1j}+t^{-1}a_{1j}^{(0)}=\mathcal{O}(t^{-\rho_{j1}-1}),\ \ j=2,\dots,p.$$

It easily follows that $\mathcal{O}(t^{-\rho_{j1}-1})=t^{-\rho_{j1}-1}a_{1j}^{(1)}(t)$ for some $a_{1j}^{(1)}(t)\in W_-(\mathbb{T})$. Hence the statement is true for $s=1$.

Let us assume that the Wiener–Hopf factorization $A(t)=A_-^{(s-1)}(t)D(t)A_+^{(s-1)}$ has already been obtained at the $(s-1)$-th normalization step, where the factor $A_-^{(s-1)}(t)$ has the desired structure. Let us partitione $A_-^{(s-1)}(t)$ into the blocks

$$A_-^{(s-1)}(t)=\left(\begin{array}{c|c}A_{11}^{(s-1)} & A_{12}^{(s-1)}\\ \hline A_{21}^{(s-1)} & A_{22}^{(s-1)}\end{array}\right),$$

where $A_{11}^{(s-1)}$ is the $s\times s$ block. By virtue of the prescribed structure of $A_-^{(s-1)}(t)$, these blocks have the following form:

$$A_{11}^{(s-1)}=\begin{pmatrix}1+t^{-1}a_{11}^{(s-1)} & t^{-\rho_{21}-1}a_{12}^{(s-1)} & \dots & t^{-\rho_{s1}-1}a_{1s}^{(s-1)}\\ a_{21}^{(s-1)} & 1+t^{-1}a_{22}^{(s-1)} & \dots & t^{-\rho_{s2}-1}a_{2s}^{(s-1)}\\ \vdots & \vdots & \ddots & \vdots\\ a_{s1}^{(s-1)} & a_{s2}^{(s-1)} & \dots & 1+t^{-1}a_{ss}^{(s-1)}\end{pmatrix};$$

$$A_{12}^{(s-1)}=\begin{pmatrix}t^{-\rho_{s+1,1}-1}a_{1,s+1}^{(s-1)} & t^{-\rho_{s+2,1}-1}a_{1,s+2}^{(s-1)} & \dots & t^{-\rho_{p,1}-1}a_{1,p}^{(s-1)}\\ \vdots & \vdots & & \vdots\\ t^{-\rho_{s+1,s-1}-1}a_{s-1,s+1}^{(s-1)} & t^{-\rho_{s+2,s-1}-1}a_{s-1,s+2}^{(s-1)} & \dots & t^{-\rho_{p,s-1}-1}a_{s-1,p}^{(s-1)}\\ t^{-1}a_{s,s+1}^{(s-1)} & t^{-1}a_{s,s+2}^{(s-1)} & \dots & t^{-1}a_{sp}^{(s-1)}\end{pmatrix};$$

$$A_{21}^{(s-1)} = \begin{pmatrix} a_{s+1,1}^{(s-1)} & a_{s+1,2}^{(s-1)} & \dots & a_{s+1,s}^{(s-1)} \\ \vdots & \vdots & & \vdots \\ a_{p,1}^{(s-1)} & a_{p,2}^{(s-1)} & \dots & a_{p,s}^{(s-1)} \end{pmatrix};$$

$$A_{22}^{(s-1)} = \begin{pmatrix} 1+t^{-1}a_{s+1,s+1}^{(s-1)} & t^{-1}a_{s+1,s+2}^{(s-1)} & \dots & t^{-1}a_{s+1,p}^{(s-1)} \\ a_{s+2,s+1}^{(s-1)} & 1+t^{-1}a_{s+2,s+2}^{(s-1)} & \dots & t^{-1}a_{s+2,p}^{(s-1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p,s+1}^{(s-1)} & a_{p,s+2}^{(s-1)} & \dots & 1+t^{-1}a_{pp}^{(s-1)} \end{pmatrix}.$$

Here, all functions $a_{ij}^{(s-1)}(t)$ belong to the algebra $W_-(\mathbb{T})$.

Apply to the factor $A_-^{(s-1)}(t)$ the normalization matrix $Q_s(t) \in \mathcal{Q}_-(\rho_1,\dots,\rho_p)$ of the following block structure

$$Q_s(t) = \left(\begin{array}{c|c} I_s & Q_{12} \\ \hline 0 & I_{p-s} \end{array}\right),$$

where

$$Q_{12} = \begin{pmatrix} t^{-1}q_{1,s+1} & \cdots & t^{-1}q_{1,p} \\ \vdots & & \vdots \\ t^{-1}q_{s,s+1} & \cdots & t^{-1}q_{s,p} \end{pmatrix}.$$

Here, $q_{ij}(t)$ are polynomials in t^{-1} of degree at most $\rho_{ji}-1$.

Then at the s normalization step we get the following factorization factor

$$A_-^{(s)}(t) = A_-^{(s-1)}(t)Q_s(t) = \left(\begin{array}{c|c} A_{11}^{(s-1)} & A_{11}^{(s-1)}Q_{12} + A_{12}^{(s-1)} \\ \hline A_{21}^{(s-1)} & A_{21}^{(s-1)}Q_{12} + A_{22}^{(s-1)} \end{array}\right).$$

The blocks $A_{11}^{(s-1)}$, $A_{21}^{(s-1)}$ have already the required form. It is easy to see that for any choice of polynomials $q_{ij}(t)$, the block $A_{21}^{(s-1)}Q_{12} + A_{22}^{(s-1)}$ can be written as

$$\begin{pmatrix} 1+t^{-1}a_{s+1,s+1}^{(s)} & t^{-1}a_{s+1,s+2}^{(s)} & \dots & t^{-1}a_{s+1,p}^{(s)} \\ a_{s+2,s+1}^{(s)} & 1+t^{-1}a_{s+2,s+2}^{(s)} & \dots & t^{-1}a_{s+2,p}^{(s)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p,s+1}^{(s)} & a_{p,s+2}^{(s)} & \dots & 1+t^{-1}a_{pp}^{(s)} \end{pmatrix}, \quad a_{i,j}^{(s)} \in W_-(\mathbb{T}).$$

It remains to choose the polynomials $q_{ij}(t)$, $\deg q_{ij}(t) \le \rho_{ji}-1$ such that the block $A_{11}^{(s-1)}Q_{12} + A_{12}^{(s-1)}$ has the desired structure. Let us extract the j-th column of this block

$$A_{11}^{(s-1)} \begin{pmatrix} t^{-1}q_{1j} \\ t^{-1}q_{2j} \\ \vdots \\ t^{-1}q_{sj} \end{pmatrix} + \begin{pmatrix} t^{-\rho_{j,1}-1}a_{1,j}^{(s-1)} \\ \vdots \\ t^{-\rho_{j,s-1}-1}a_{s-1,j}^{(s-1)} \\ t^{-1}a_{s,j}^{(s-1)} \end{pmatrix}, \quad s+1 \le j \le p.$$

Now, we consider $p-s$ systems of equations of the form

$$A_{11}^{(s-1)} \begin{pmatrix} t^{-1}\widetilde{q}_{1j} \\ t^{-1}\widetilde{q}_{2j} \\ \vdots \\ t^{-1}\widetilde{q}_{sj} \end{pmatrix} + \begin{pmatrix} t^{-\rho_{j,1}-1}a_{1,j}^{(s-1)} \\ \vdots \\ t^{-\rho_{j,s-1}-1}a_{s-1,j}^{(s-1)} \\ t^{-1}a_{s,j}^{(s-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad s+1 \le j \le p. \tag{8}$$

The elements of the matrix of this system are the functions from $W_-(\mathbb{T})$ and therefore are analytic in the neighborhood of infinity. Obviously, $\det A_{11}^{(s-1)}(\infty) = 1$. Thus, the matrix function $A_{11}^{(s-1)}(t)$ is invertible in the class of matrix functions that are analytic in the neighborhood of infinity. Therefore, each of the systems of equations (8) has a unique solution $(t^{-1}\widetilde{q}_{1j}, t^{-1}\widetilde{q}_{2j}, \dots, t^{-1}\widetilde{q}_{sj})^T$, where the functions $\widetilde{q}_{ij}(t)$ are analytic in the neighborhood of infinity. Let us expand the function $\widetilde{q}_{ij}(t)$ into a Laurent series in the vicinity of infinity as follows:

$$\widetilde{q}_{ij}(t) = \sum_{k=0}^{\infty} q_k^{(ij)} t^{-k}, \quad 1 \le i \le s, \quad s+1 \le j \le p,$$

and define the polynomials $q_{ij}(t) = \sum\limits_{k=0}^{\rho_{ji}-1} q_k^{(ij)} t^{-k}$. Let us represent $q_{ij}(t)$ as $q_{ij}(t) = \widetilde{q}_{ij}(t) + \mathcal{O}(t^{-\rho_{ji}})$. Taking into account the structure of the matrix function $A_{11}^{(s-1)}(t)$, we get

$$A_{11}^{(s-1)} \begin{pmatrix} t^{-1}q_{1j} \\ t^{-1}q_{2j} \\ \vdots \\ t^{-1}q_{sj} \end{pmatrix} + \begin{pmatrix} t^{-\rho_{j,1}-1} a_{1,j}^{(s-1)} \\ \vdots \\ t^{-\rho_{j,s-1}-1} a_{s-1,j}^{(s-1)} \\ t^{-1} a_{s,j}^{(s-1)} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(t^{-\rho_{j1}-1}) \\ \mathcal{O}(t^{-\rho_{j2}-1}) \\ \vdots \\ \mathcal{O}(t^{-\rho_{js}-1}) \end{pmatrix}, \quad s+1 \le j \le p.$$

Since the elements of the column on the left-hand side of this relation belong to the algebra $W_-(\mathbb{T})$, the right-hand side can actually be represented as

$$\begin{pmatrix} \mathcal{O}(t^{-\rho_{j1}-1}) \\ \mathcal{O}(t^{-\rho_{j2}-1}) \\ \vdots \\ \mathcal{O}(t^{-\rho_{js}-1}) \end{pmatrix} = \begin{pmatrix} t^{-\rho_{j1}-1} a_{1j}^{(s)} \\ t^{-\rho_{j2}-1} a_{2j}^{(s)} \\ \vdots \\ t^{-\rho_{js}-1} a_{sj}^{(s)} \end{pmatrix}$$

for some functions $a_{ij}^{(s)}(t)$ from the algebra $W_-(\mathbb{T})$. This means that the block $A_{11}^{(s-1)} Q_{12} + A_{12}^{(s-1)}$ has the required form. We have completed the s-th normalization step. After p normalization steps we find that $A_-^{(p)}(t)$ has the form (5) for $P = I$.

Thus the existence of a normalized factorization of the form (5) for $P = I$ is proved. Let us now verify that this factorization is I-normalized. In order to do this, we find the matrix function $B_-(t) = D^{-1}(t)C_-(t)D(t)$. It is easy to see that it has the form (6) for $P = I$. Since all elements of $B_-(t)$ belong to the algebra $W_-(\mathbb{T})$ and $B_-(\infty) = I$, the conditions of Definition 2 are satisfied. Therefore, by Theorem 1, the normalized factorization of the form (5) is unique. For the case $P = I$, the theorem is proved.

Now, we suppose that for $A(t)$ there exists a factorization $A(t) = A_-(t)D(t)A_+(t)$ for which $A_-(\infty)$ admits the PLU -factorization: $A_-(\infty) = PLU$. Note that in this case, any factorization of $A(t)$ has this property.

Let us define the matrix function $F(t) = P^{-1}A(t)$. Then $F(t) = (P^{-1}A_-(t))D(t)A_+(t)$ is its Wiener–Hopf factorization, and the condition $F_-(\infty) = LU$ is satisfied for $F_-(t) = P^{-1}A_-(t)$. By the proved part of the theorem, $F(t)$ admits the I-normalized Wiener–Hopf factorization $F(t) = K_-(t)D(t)F_+(t)$. Then $A(t) = (PK_-(t))D(t)F_+(t)$ is the P-normalized factorization of $A(t)$. The theorem is completely proved. □

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(Received 26.06.2022)

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