Transactions of A. Razmadze
Mathematical Institute
Vol. **176** (2022), issue 3, 451–455

LACUNARY RELATIVE UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS

BINOD CHANDRA TRIPATHY

Abstract. In this article, we introduce the notions of lacunary convergence and strong Cesàro summability relatively uniform convergence of sequences of operators defined on a compact subset of real numbers. We establish some of their properties and relationship between these two notions.

1. Introduction and Preliminaries

The notion of lacunary sequence is found in the article by Freedman et al. [5]. The studies of the $|(C,1)|$ of strongly Cesàro summable sequences with general lacunary sequence θ result in a larger class N_θ, a $BK-$space (Banach space of sequences for which the co-ordinate linear functionals are continuous). Thereafter these notions were investigated and linked with the summability theory due to Tripathy and Baruah [7], Tripathy and Dutta [8,9], Tripathy et al. [10], Tripathy and Mahanta [11] and others.

Our main objective in this article is to study the classes of lacunary convergence and strongly Cesàro summable sequences relatively uniform convergence of sequences of operators defined on a compact subset of real numbers.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \to \infty$, as $n \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$, $h_r = k_r - k_{r-1}$ for $r = 1, 2, 3, \dots$.

Let (x_k) be a sequence of real or complex numbers, then the sums of the form $\sum\limits_{i=k_{r-1}+1}^{k_r} |x_i| = \sum\limits_{i\in I_r} |x_i|$ will often be written for convenience as $\sum\limits_{I_r} |x_i|$.

A sequence (x_n) of real or complex terms is said to be converge lacunarily to L, if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| = 0.$$

The class of all lacunary convergent sequences, denoted by N_θ of real or complex terms is a Banach space with respect to the following norm:

$$||(x_k)|| = \sup_r \frac{1}{h_r} \sum_{k\in I_r} |x_k|.$$

A sequence (x_n) of real or complex terms is said to be strongly Cesàro summable to L, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L| = 0.$$

The class of all strongly Cesàro summable sequences of real or complex terms denoted by $|C,1|$ is a Banach space with respect to the norm

$$||(x_k)|| = \sup_n \frac{1}{n} \sum_{k=1}^{n} |x_k|.$$

2020 *Mathematics Subject Classification.* 40H05, 46A11, 46B28, 46B45, 46B50, 47A58, 47B65, 47L05.

Key words and phrases. Lacunary convergence; Relative uniform convergence; Positive linear operator; Strongly Cesàro summbility; Scale function.

Remark 1. It is clear from the examples discussed by Freedman et al. [5, p. 511] that the lacunary limit of a sequence (x_n) is dependent on the lacunary sequence under consideration. Hence the lacunary limit in the case of relative uniform convergence with respect to the scale function is also dependent on the lacunary sequence.

2. Definitions and Background

Throughout the article, we consider the domain of the operators D, a compact subset of the set of real numbers.

E. H. Moore [6] introduced the notion of uniform convergence of a sequence of functions relative to a scale function. This was improved by Chittenden [1] as follows.

Definition 1. A sequence (f_n) of functions defined on a compact subset D of the real space is said to converge relatively uniformly to a limit function f, if there exists a function $\sigma(x)$ called the scale function such that for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that for every $n > n_0$ the inequality

$$|f_k(x) - f(x)| < \varepsilon|\sigma(x)|$$

holds uniformly for $x \in D$.

Example 1. Consider the sequence of functions (f_k), $f_k : (0, 1] \to R$, defined by $f_k(x) = k^{-2}x^{-1}$, for all $x \in (0, 1]$ and all $k \in N$.

This sequence of functions does not converge uniformly to 0 on (0,1]. It converges to 0 uniformly with respect to the scale function $\sigma(x)$ defined by $\sigma(x) = x^{-1}$, for all $x \in (0, 1]$.

Definition 2. A sequence (f_n) of functions defined on a compact subset D of the real space is said to converge lacunary relatively uniformly to a limit function f, if there exists a function $\sigma(x)$ called the scale function such that for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for every $r > n_0$ the inequality

$$\tau_r(x) = \frac{1}{h_r}\sum_{k\in I_r}|f_k(x) - f(x)| < \varepsilon|\sigma(x)|$$

holds uniformly for $x \in D$.

The class of all lacunary relatively uniformly convergent sequence of functions (f_n) with respect to a scale function $\sigma(x)$ is denoted by N_θ^u.

Here, we define the space $|C, 1|$ of a strongly Cesàro summable sequence by

$$|C, 1| = \left\{(f_k) : \text{ there exists } L \text{ such that } \frac{1}{n}\sum_{k=1}^{n}|f_k - f| \to 0, \quad \text{as } n \to \infty\right\}.$$

A sequence of functions (f_n) is said to be relatively uniformly strongly Cesàro summable to f, if for each $\varepsilon > 0$, there exists a scale function $\sigma(x)$ such that

$$\frac{1}{n}\sum_{k=1}^{n}|f_k(x) - f(x)| < \varepsilon|\sigma(x)|.$$

The class of all relatively uniformly strongly Cesàro summable sequences is denoted by $|C, 1|^u$. The class of relatively uniformly Cesàro summable to the null operators is denoted by $|C, 1|_0^u$.

3. Main Results

In this section, we establish the results of this article.

Theorem 1. *If θ is a lacunary sequence, then for $|C, 1|^u \subseteq N_\theta^u$, it is necessary and sufficient that* $\liminf_{r\to\infty} q_r > 1$.

Proof. For the sufficiency we assume $\liminf_{r\to\infty} q_r > 1$, then there exist $\delta > 0$ and $M(\delta) > 0$ such that $1 + M(\delta) \le q_r$ for all $r \ge 1$.

Now, for $(f_k) \in |C,1|_0^u$ we have

$$\begin{aligned}\tau_r =& \frac{1}{h_r}\sum_{i=1}^{k_r}|f_i| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}}|f_i| \\ =& \frac{k_r}{h_r}(\frac{1}{k_r}\sum_{i=1}^{k_r}|f_i|) - \frac{k_{r-1}}{h_r}(\frac{1}{k_{r-1}}\sum_{i=1}^{k_r}|f_i|).\end{aligned}$$

Since $h_r = k_r - k_{r-1}$, we have
$\frac{k_r}{h_r} \le \frac{1+M(\delta)}{M(\delta)}$ and $\frac{k_{r-1}}{h_r} \le \frac{1}{M(\delta)}$, as $M(\delta) > 0$ and $q_r = \frac{k_r}{k_{r-1}}$.

The terms $\frac{1}{k_r}\sum_{i=1}^{k_r}|f_i|$ and $\frac{1}{k_{r-1}}\sum_{i=1}^{k_r}|f_i|$ both converge to 0 relatively uniformly with respect to the scale function $\sigma(x)$.

Hence τ_r converges to 0 relatively uniformly with respect to the scale function $\sigma(x)$, that is, $f_i \in N_\theta^{0,u}$.

Therefore $|C,1|^u \subseteq N_\theta^u$.

For the sufficiency, we assume $\liminf_{r\to\infty} = 1$.

Since θ is lacunary, we can find a subsequence (k_{r_j}) of θ satisfying $\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j}$ and $\frac{k_{r_j-1}}{k_{r_{j-1}}} > j$, where $r_j \ge r_{j-1} + 2$.

Define the sequence of functions (f_i) by

$$f_i(x) = \begin{cases} 1, & \text{if } \ i \in I_{r_j}, \ \text{ for some } \ j = 1, 2, \dots \\ 0, & \text{otherwise,} \end{cases}$$

for all $x \in D$, where D is the compact domain on which the sequence of functions are defined.

Then for any real number L,

$$\frac{1}{h_{r_j}}\sum_{I_{r_j}}|f_i - L| = |1 - L|; \quad j = 1, 2, \dots$$

and

$$\frac{1}{h_r}\sum_{I_r}|f_i - L| = |L| \quad \text{for } \ r \ne r_j.$$

It follows that $(f_i) \in N_\theta^u$.

But (f_i) is strongly summable if we consider t as sufficiently large, then there exists a unique j for which $k_{r_j-1} < t \le k_{r_{j+1}-1}$ and we write

$$\frac{1}{t}\sum_{i=1}^{t}|f_i| \le \frac{k_{r_{j-1}} + h_{r_j}}{k_{r_j} - 1} \le \frac{1}{j} + \frac{1}{j} = \frac{2}{j}.$$

Now, if $t \to \infty$, it follows that $j \to \infty$. Hence $(f_i) \in |C,1|_0^u$. □

Theorem 2. *If θ is a lacunary sequence, then for $N_\theta^u \subseteq |C,1|^u$, it is necessary and sufficient that $\liminf_{r\to\infty} q_r < \infty$.*

Proof. For the sufficiency, we consider $\limsup_{r\to\infty} q_r < \infty$, there exist H and $M(H) > 0$ such that $q_r < M(H)$ for all $r \ge 1$. Considering $(f_i) \in N_\theta^{0,u}$, relative to the scale function $\sigma(x)$ and $\varepsilon > 0$, we can find $R > 0$ and $K > 0$ such that $\tau_i < K$ for all $i = 1, 2, \dots$.

Then if t is any integer with $k_{r-1} < t \le k_r$, where $r > R$, we can write

$$\begin{aligned}\frac{1}{t}\sum_{i=1}^{t}|f_i| \le &\frac{1}{k_{r-1}}\sum_{i=1}^{t}|f_i|\\ =&\frac{1}{k_{r-1}}\left(\sum_{I_1}|f_i| + \dots + \sum_{I_r}|f_i|\right)\\ =&\frac{1}{k_{r-1}}\tau_1 + \frac{k_2-k_1}{k_{r-1}}\tau_2 + \dots + \frac{k_R-k_{R-1}}{k_{r-1}}\tau_r\\ +&\frac{k_{R+}-k_R}{k_{r-1}}\tau_{R+1} + \dots + \frac{k_r-k_{r-1}}{k_{r-1}}\tau_r\\ \le&\Big(\sup_{i\ge 1}\tau_i\Big)\frac{k_R}{k_{r-1}} + \Big(\sup_{i\ge R}\tau_i\Big)\frac{k_r-k_R}{k_{r-1}}\\ =&\Big\{k.\frac{k_R}{k_{r-1}} + \varepsilon.M(H)\Big\}\sigma(x).\end{aligned}$$

Since $k_{r-1} \to \infty$ as $t \to \infty$, it follows that $\frac{1}{t.\sigma(x)}\sum_{i=1}^{t}|f_i| \to 0$ and, consequently, $(f_k) \in |C,1|_0^u$.

For the necessary part, we consider $\limsup_{r\to\infty} q_r = \infty$ and construct a sequence in N_θ^u that is not strongly Cesàro summable.

We select a subsequence (k_{r_j}) of θ so that $q_{r_j} > j$ and then define (f_k) by

$$f_i(x) = \begin{cases}1, & \text{if } k_{r_{j-1}} < i \le 2k_{r_j-1}, \text{ for some } j = 1,2,\dots\\ 0, & \text{otherwise,}\end{cases}$$

for all $x \in D$.

Then $\tau_{r_j} = \frac{k_{r_j-1}}{k_{r_j}-k_{r_j-1}} < \frac{1}{j-1}$ if $r = r_j$, $\tau_r = 0$. Thus $(f_i) \in N_\theta^{0,u}$.

Any sequence in $|\sigma_1^u|$ consisting of only $\bar{\theta}$'s and I's has a strong limit f, where $M(l) = 1$ or $M(l) = \bar{\theta}$.

For the sequence (f_i) and $i = 1,2,\dots,k_{r_j}$,

$$\begin{aligned}\frac{1}{k_{r_j}}\sum|f_i - f| \ge &\frac{1}{k_{r_j}}(k_{r_j} - 2k_{r_j-1})\\ =&1 - \frac{2k_{r_j}-1}{k_r}\\ >&1 - \frac{2}{j},\end{aligned}$$

which converges to I, and for $i = 1,2,\dots,2k_{r_j}-1$,

$$\frac{1}{2k_{r_j}-1}\sum_i|f_i| \ge \frac{k_{r_j-1}}{2k_{r_j}-1} = \frac{1}{2}.$$

Thus $(f_i) \in |\sigma_1^u|$. □

The following result is a consequence of the above two results.

Proposition 3. *Let θ be a lacunary sequence. Then $N_\theta^u = |C,1|^u$, if and only if*

$$1 < \liminf_{r\to\infty} q_r \le \limsup_{r\to\infty} q_r < \infty.$$

References

1. E. W. Chittenden, Relatively uniform convergence of sequences of functions. *Trans. Amer. Math. Soc.* **15** (1914), no. 2, 197–201.
2. E. W. Chittenden, On the limit functions of sequences of continuous functions converging relatively uniformly. *Trans. Amer. Math. Soc.* **20** (1919), no. 2, 179–184.

3. E. W. Chittenden, Relatively uniform convergence and the classification of functions. *Trans. Amer. Math. Soc.* **23** (1922), no. 1, 1–15.
4. K. Demirci, S. Orhan, Statistically relatively uniform convergence of positive linear operators. *Results Math.* **69** (2016), no. 3-4, 359–367.
5. A. R. Freedman, J. J. Sember, M. Raphael, Some Cesàro-type summability spaces. *Proc. London Math. Soc. (3)* **37** (1978), no. 3, 508–520.
6. E. H. Moore, *An Introduction to a Form of General Analysis.* The New Haven Mathematical Colloquium, Amer. Math. Soc. Colloquium, 5th, New Haven, 1910.
7. B. C. Tripathy, A. Baruah, Lacunary statically convergent and lacunary strongly convergent generalized difference sequences of fuzzy real numbers. *Kyungpook Math. J.* **50** (2010), no. 4, 565–574.
8. B. C. Tripathy, A. J. Dutta, Lacunary bounded variation sequence of fuzzy real numbers. *J. Intell. Fuzzy Systems* **24** (2013), no. 1, 185–189.
9. B. C. Tripathy, A. J. Dutta, Lacunary I-convergent sequences of fuzzy real numbers. *Proyecciones* **34** (2015), no. 3, 205–218.
10. B. C. Tripathy, B. Hazarika, B. Choudhary, Lacunary I-convergent sequences. *Kyungpook Math. J.* **52** (2012), no. 4, 473–482.
11. B. C. Tripathy, S. Mahanta, On a class of generalized lacunary difference sequence spaces defined by Orlicz functions. *Acta Math. Appl. Sin. Engl. Ser.* **20** (2004), no. 2, 231–238.

(Received 05.05.2021)

Department of Mathematics; Tripura University, Agartala – 799022, Tripura, INDIA
Email address: tripathybc@rediffmail.com; tripathybc@yahoo.com