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TRACE INEQUALITIES FOR FRACTIONAL INTEGRALS IN CENTRAL MORREY SPACES

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Abstract. We study the trace inequality for fractional integrals K_α in central Morrey spaces. In particular, we establish necessary condition and sufficient condition governing the inequality

$$\Big\|K_\alpha f\Big\|_{L_a^{q,\lambda_2}(X,\nu)} \le C\Big\|f\Big\|_{L_a^{p,\lambda_1}(X,\mu)},$$

where (X,ρ,μ) is a space of homogeneous type, a is a point in X and ν is another measure on X. As a corollary, we have necessary and sufficient conditions on power-type weights $d\nu(x) = d(a,x)^\beta d\mu(x)$ for the trace inequality. The results are new even for the Euclidean spaces.

Preliminaries

Let (X,ρ,μ) be a quasi-metric measure space with doubling measure μ. Suppose that ν is another measure on X. In this note, we establish the necessary condition and the sufficient condition on a measure ν guaranteeing the trace inequality

$$\Big\|K_\alpha f\Big\|_{L_a^{q,\lambda_2}(X,\nu)} \le C\Big\|f\Big\|_{L_a^{p,\lambda_1}(X,\mu)}, \tag{1}$$

where $L_a^{q,\lambda_2}(X,\nu)$ and $L_a^{p,\lambda_1}(X,\mu)$ are the central Morrey spaces defined with respect to a point a and measures ν and μ, respectively, and K_α is the fractional integral operator defined on (X,ρ,μ).

As a corollary, we have the necessary and sufficient conditions on power-type weights $d\nu(x) = d(a,x)^\beta d\mu(x)$ for inequality (1).

The results are new even for fractional integral operators in Morrey spaces defined on $\mathbb{R}^n$. In particular, we have the necessary condition and the sufficient condition for a measure ν defined on $\mathbb{R}^n$ governing the inequality

$$\Big\|I_\gamma f\Big\|_{L_0^{q,\lambda_2}(\mathbb{R}^n,\nu)} \le C\Big\|f\Big\|_{L_0^{p,\lambda_1}(\mathbb{R}^n)},$$

where I_γ is the Riesz potential operator on $\mathbb{R}^n$, $0<\gamma<n$.

We assume that (X,ρ,μ) is a topological space, endowed with a complete measure μ such that the space of compactly supported continuous functions is dense in $L^1(X,\mu)$ and there exists a function (quasi-metric) $\rho : X\times X \to [0,\infty)$ satisfying the following conditions:

1) $\rho(x,x) = 0$ for $\forall x \in X$;
2) $\rho(x,y) > 0$ for $\forall x,y \in X$, where $x \ne y$;
3) $\rho(x,y) = \rho(y,x)$ for $\forall x,y \in X$;
4) there exists a positive constant such that for $\forall x,y,z \in X$,

$$\rho(x,y) \le c_1[\rho(x,z) + \rho(z,y)].$$

The triple (X,ρ,μ) is called a quasi-metric measure space.

We denote by $B(x,r)$ a ball with center x and radius r.

If μ satisfies the doubling condition

$$\mu B(x,2r) \le C\mu B(x,r),$$

where the positive constant C does not depend of x and r, then the quasi-metric measure space (X,ρ,μ) is called the space of homogeneous type (SHT briefly).

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Definition 1. We say that an SHT (X,ρ,μ) is normal (see [6]) if there exists a positive constant c such that

$$\frac{1}{c}r^N \le \mu(B(x,r)) \le cr^N,$$

where x is the center of the ball B and $r \in (0,\ell)$, where ℓ is a diameter of X. The constant N is called a dimension of (X,ρ,μ).

There are many useful for applications examples of a (normal) SHT:
(a) Rectifiable regular (Carleson) curves in $\mathbb{C}$ with Euclidean metric and arc-length measure;
(b) Nilpotent Lie groups with Haar measure;
(c) Bounded domain Ω in $\mathbb{R}^n$ together with induced Lebesgue measure satisfying the so-called $\mathcal{A}$ condition, i.e., there is a positive constant C such that for all $x \in \overline{\Omega}$ and $\delta \in (0,\ell)$,

$$\mu(\widetilde{B}(x,\delta)) \ge C\delta^n,$$

where ℓ is a diameter of Ω and $\widetilde{B}(x,\delta) := \Omega \cap B(x,\delta)$;
ect.

For the definition and some essential properties of an SHT we refer, e.g., to [3,6,7].

Let (X,ρ,μ) be a quasi-metric measure space. Suppose that ν is another measure on X, $\lambda \ge 0$ and $1 \le p < \infty$. Let $L^{p,\lambda}(X,\nu,\mu)$ be the Morrey space by means of two measures μ and ν, which is the set of all functions $f \in L^p_{\text{loc}}(X,\nu)$ such that

$$\|f\|_{L^{p,\lambda}(X,\nu)} := \sup_B \left(\frac{1}{\mu(B)^\lambda}\int\limits_B |f(y)|^p d\nu(y)\right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all balls B.

If $\nu = \mu$, then we have the Morrey space $L^{p,\lambda}(X,\mu)$ with original measure μ on X. When $\lambda = 0$, then $L^{p,\lambda}(X,\nu) = L^p(X,\nu)$ is the Lebesgue space with measure ν.

A complete characterization of the trace inequality for the fractional integral operator in $L^{p,\lambda}$ spaces defined on a quasi-metric measure space can be found in [5].

We deal with the Morrey space $L^{p,\lambda}_a(X,\nu)$, which is the set of all functions $f \in L^p_{\text{loc}}(X,\nu)$ such that

$$\|f\|_{L^{p,\lambda}_a(X,\nu)} := \sup_{r>0} \left(\frac{1}{\mu(B(a,r))^\lambda}\int\limits_{B(a,r)} |f(y)|^p d\nu(y)\right)^{\frac{1}{p}} < \infty,$$

where a is a point in X and the supremum is taken over all positive r.

In the Euclidean space $X = \mathbb{R}^n$, let us take the Lebesgue measure $d\mu = dx$ and Euclinead distance $\rho(x,y) = |x-y|$. Suppose ν is a measure on $\mathbb{R}^n$, $\lambda \ge 0$, $1 \le p < \infty$ and, finally, $a = 0$, then we have the space $L^{p,\lambda}_0(\mathbb{R}^n,\nu)$ which is the set of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{L^{p,\lambda}_0(\mathbb{R}^n,\nu)} := \sup_{r>0} \left(\frac{1}{r^{\lambda n}}\int\limits_{B(0,r)} |f(y)|^p d\nu(y)\right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over the positive radius r. If ν is the Lebesgue measure, then we have the unweighted central Morrey space $L^{p,\lambda}_0(\mathbb{R}^n)$.

Let $0 < \alpha < 1$. We consider the fractional integral operator K_α given by

$$K_\alpha(f)(x) := \int\limits_X \frac{f(y)}{\mu(B(x,\rho(x,y)))^{1-\alpha}} d\mu(y),$$

for suitable f on X.

The Riesz potential operator I_γ is a particular case of K_α; this operator is given by the formula

$$I_\gamma(f)(x) := \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy,$$

for suitable f on $\mathbb{R}^n$.

The Sobolev-type inequality for I_γ was proved by Spanne and Adams (see [2]).

The following statement is the trace inequality for the operator K_α (see [1] or [4, Ch. 6]).

Theorem A. *Let (X,ρ,μ) be an SHT. Suppose that $1<p<q<\infty$ and $0<\alpha<\frac{1}{p}$. Assume that ν is another measure on X. Then K_α is bounded from $L^p(X,\mu)$ to $L^q(X,\nu)$ if and only if*

$$\nu(B)\le c\mu(B)^{q\left(\frac{1}{p}-\alpha\right)}, \tag{2}$$

for all balls B in X.

The next statement is the boundedness characterization for the operator K_α from $L^{p,\lambda_1}(X,\mu)$ to $L^{q,\lambda_2}(X,\nu)$.

Theorem B ([5])**.** *Let (X,ρ,μ) be an SHT and let $1<p<q<\infty$. Suppose that $0<\alpha<\frac{1}{p}$, $0<\lambda_1<1-\alpha p$ and $\frac{\lambda_2}{q}=\frac{\lambda_1}{p}$. Then K_α is bounded from $L^{p,\lambda_1}(X,\mu)$ to $L^{q,\lambda_2}(X,\nu)$ if and only if condition* (2) *holds.*

Our aim is to investigate a similar problem for fractional integrals in the central Morrey spaces.

Main Results

Now we formulate the main statements of this note.

Theorem 1. *Let (X,ρ,μ) be an SHT and let $1<p<q<\infty$. Suppose that $0<\alpha<\frac{1}{p}$, $0<\lambda_1<1-\alpha p$ and $\frac{\lambda_2}{q}=\frac{\lambda_1}{p}$. Then the necessary condition for the boundedness of K_α from $L_a^{p,\lambda_1}(X,\mu)$ to $L_a^{q,\lambda_2}(X,\nu)$ is that condition* (2) *holds for all balls B centered at a. Further, the sufficient condition for this boundedness is* (2)*, but satisfied for any balls B in X.*

Now, we consider the case of power weights $d\nu(x)=d(a,x)^\beta d\mu(x)$. As a corollary, we have the following statement.

Corollary 1. *Let (X,d,μ) be a normal SHT with dimension N and let $1<p<q<\infty$. Suppose that $0<\alpha<\frac{1}{p}$, $0<\lambda_1<1-\alpha p$ and $\frac{\lambda_2}{q}=\frac{\lambda_1}{p}$. Then the inequality*

$$\|K_\alpha f\|_{L_a^{q,\lambda_2}(X,d(a,x)^\beta d\mu)}\le C\|f\|_{L_a^{q,\lambda_1}(X,\mu)}$$

holds if and only if

$$\beta=qN\Big(\frac{1}{p}-\alpha\Big)-N.$$

As a special case, it is possible to get the appropriate results for the Riesz potentials I_α. Namely, we have

Theorem 2. *Let $1<p<q<\infty$. Suppose that $0<\gamma<\frac{n}{p}$, $0<\lambda_1<n-\gamma p$ and $\frac{\lambda_2}{q}=\frac{\lambda_1}{p}$. Then the necessary condition for the boundedness of I_γ from $L_0^{p,\lambda_1}(\mathbb{R}^n)$ to $L_0^{q,\lambda_2}(\mathbb{R}^n,\nu)$ is that there is a positive constant c such that*

$$\nu(B(x,R))\le cR^{q\left(\frac{n}{p}-\gamma\right)}, \tag{3}$$

for $x=0$ and all $R>0$. The sufficient condition for this boundedness is (3)*, but satisfied for all x and all $R>0$.*

Taking now the power weight $d\nu(x)=v(x)dx=|x|^\beta dx$, we have

Corollary 2. *Let $1<p<q<\infty$. Suppose that $0<\gamma<\frac{n}{p}$, $0<\lambda_1<n-\gamma p$ and $\frac{\lambda_2}{q}=\frac{\lambda_1}{p}$. Then the trace inequality*

$$\|I_\gamma f\|_{L_0^{q,\lambda_2}(\mathbb{R}^n,|x|^\beta dx)}\le C\|f\|_{L_0^{q,\lambda_1}(\mathbb{R}^n)}$$

with the positive constant C independent of f, holds if and only if

$$\beta=q\Big(\frac{n}{p}-\gamma\Big)-n.$$

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