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ON SOME EXTRAPOLATION IN GENERALIZED GRAND MORREY SPACES AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. Weighted extrapolation in generalized grand Morrey spaces is investigated. The obtained results are applied to derive one-weight estimates for some operators of Harmonic Analysis and to study regularity properties of solutions of second order partial differential equations with discontinuous coefficients in the frame of generalized grand Morrey spaces under the Muckenhoupt condition on weights.

1. Introduction

Let (X, d, μ) be a quasi-metric measure space ($QMMS$, briefly) with a quasi-metric d and measure μ. We say that a measure μ satisfies doubling condition if there is a positive constant C_{dc} such that for all $x \in X$ and $r > 0$, $\mu B(x, 2r) \le C_{dc} \mu B(x, r)$. We will deal with the $QMMS$ with doubling measure. Such a $QMMS$ is called space of homogeneous type (SHT, briefly).

There are many important examples of an SHT: (a) Carleson (regular) curves on $\mathbb{C}$ with arc-length measure $d\nu$ and Euclidean distance on $\mathbb{C}$; (b) nilpotent Lie groups with Haar measure and homogeneous norm (homogeneous groups); (c) bounded domain Ω in $\mathbb{R}^n$ together with induced Lebesgue measure satisfying the so-called $\mathcal{A}$ condition, i.e., there is a positive constant C such that for all $x \in \overline{\Omega}$ and $\rho \in (0, \ell)$,

$$\mu(\widetilde{B}(x, \rho)) \ge C \rho^n, \tag{1}$$

where ℓ is a diameter of Ω and $\widetilde{B}(x, \rho) := \Omega \cap B(x, \rho)$.

Morrey spaces describe regularity problems for solutions of elliptic PDEs more precisely than Lebesgue spaces. Morrey spaces were introduced by C. B. Morrey [23] in 1938.

Let w be a weight function on X, i.e., w is a μ- a.e. positive integrable function on X. Let $L^{p,r}_w(X)$ be the weighted Morrey space defined with respect to the norm (cf., [19])

$$\|f\|_{L^{p,r}_w(X)} := \sup_B \frac{1}{\big(w(B)\big)^{\frac{1}{p}+r}} \|f\|_{L^p_w(B)},$$

where $1 < p < \infty$, $-1/p \le r < 0$. If $-1/p = r$, then we have a weighted Lebesgue space $L^p_w(X)$.

In 1992, T. Iwaniec and C. Sbordone [10], in their studies related to the integrability properties of the Jacobian in a bounded open set Ω, introduced a new type of function spaces $L^{p)}(\Omega)$, called *grand Lebesgue spaces*. Their generalized version, $L^{p),\theta}(\Omega)$, appeared in L. Greco, T. Iwaniec and C. Sbordone [9] in 1997 when studying the existence and uniqueness of the solution of certain non-linear PDEs.

Harmonic Analysis related to these spaces and their associate spaces (called *small Lebesgue spaces*), was intensively studied during the last years due to various applications (see also monograph [17] and references cited therein).

Denote by Φ_p the class of non-decreasing functions $\varphi(\cdot)$ on $(0, p-1)$ such that $\lim\limits_{\varepsilon \to 0} \varphi(\varepsilon) = 0$.

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We are interested in the weighted grand Morrey space $L_w^{p),r,\varphi(\cdot)}(X)$ with a weight function w defined by the norm

$$\|f\|_{L_w^{p),r,\varphi(\cdot)}(X)} := \sup_{0<\varepsilon<p-1} \sup_B \frac{\varphi(\varepsilon)}{\big(w(B)\big)^{\frac{1}{p-\varepsilon}+r}} \|f\|_{L_w^{p-\varepsilon}(B)} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)\|f\|_{L_w^{p-\varepsilon,r}(X)},$$

where $1<p<\infty$, $-1/p \le r < 0$ and $\varphi(\cdot) \in \Phi_p$.

We are stimulated to investigate the extrapolation problem in such a type of grand Morrey space because of the papers [7, 18], where the same problem was studied in the classical weighted Morrey spaces $L_w^{p,\lambda}(\mathbb{R}^n)$ and weighted grand Morrey spaces $L_w^{p),\lambda,\theta}$, respectively. The study of the one-weight problem for integral operators in weighted classical Morrey spaces with Muckenhoupt weights defined on $\mathbb{R}^n$ was initiated in paper [19]. Similar problem for sublinear operators involving maximal, Calderón-Zygmund and fractional integrals in the classical weighted Morrey spaces with A_p weights was investigated, for example, in [7, 13, 24, 25, 27, 28, 30].

It should be emphasize that the one-weight boundedness problem for sublinear operators involving their commutators in grand Morrey spaces was explored in [16] and [15]. Weighted extrapolation in grand Lebesgue spaces was established in [12].

Unweighted grand Morrey spaces were introduced and studied in [20]. Later, they were generalized in [26] by introducing grand Morrey spaces defined by the norm including the "grandification" taken not only with respect to p but also for another parameter λ.

We say that a weight function w belongs to the Muckenhoupt class $A_s(X)$ (or A_s) $1<s<\infty$, if

$$\big[w\big]_{A_s} := \sup_B \left(\frac{1}{\mu(B)}\int_B w(x)d\mu(x)\right)\left(\frac{1}{\mu(B)}\int_B w^{1-s'}(x)d\mu(x)\right)^{s-1} < \infty,$$

where the supremum is taken over all balls $B \subset X$. The symbol $\big[w\big]_{A_s}$ is called the characteristic of w. Further, a weight w belongs to $A_1(X)$ if $Mw(x) \le Cw(x)$ a.e., where

$$Mw(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B w(y)d\mu(y).$$

The characteristic $[w]_{A_1(X)}$ is defined as the essential supremum of Mw/w.

Further, the following monotonicity property holds for the Muckenhoupt classes

$$A_r(X) \subset A_s(X), \quad 1 \le r < s < \infty.$$

2. Weighted Extrapolation

The main result regarding the extrapolation reads as follows:

Theorem 2.1. *Let $1 \le p_0 < \infty$ and let $\mathcal{F}(X)$ be a collection of non-negative measurable pairs of functions defined on X. Suppose that for all $(f,g) \in \mathcal{F}(X)$ and for all $w \in A_{p_0}(X)$, the inequality*

$$\|g\|_{L_w^{p_0}(X)} \le CN\big([w]_{A_{p_0}(X)}\big)\|f\|_{L_w^{p_0}(X)} \tag{2}$$

holds, where $N\big([w]_{A_{p_0}(X)}\big)$ is the positive constant depending only on the characteristic $[w]_{A_{p_0}(X)}$ such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing, and the constant C does not depend on (f,g) and w. Then for every $1<p<\infty$, $-1/p \le r < 0$, $\varphi(\cdot) \in \Phi_p$ and $w \in A_p(X)$, we have

$$\|g\|_{L_w^{p),r,\varphi(\cdot)}(X)} \le C\overline{C}\|f\|_{L_w^{p),r,\varphi(\cdot)}(X)}, \quad (f,g) \in \mathcal{F}(X),$$

where C is the constant from (2), and the constant $\overline{C}$ is independent of (f,g).

Extrapolation statement regarding the A_∞ class of weights reads as follows:

Theorem 2.2. *Let $\mathcal{F}(X)$ be a family of pairs of functions (f,g), where f and g are defined on X. Suppose that for some $p_0 \in (0,\infty)$ and for all $w \in A_\infty$, we have*

$$\|g\|_{L^{p_0}_w(X)} \le CN([w]_{A_l(X)})\|f\|_{L^{p_0}_w(X)}, \quad (f,g) \in \mathcal{F}, \tag{3}$$

for some $l \ge 1$, where $N\big([w]_{A_{p_0}(X)}\big)$ is the positive constant depending only on the characteristic $[w]_{A_{p_0}(X)}$ such that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing and the constant C does not depend on w and (f,g). Then for every $1<p<\infty$, $-1/p \le r < 0$, $\varphi \in \Phi_p$ and $w \in A_\infty(X)$, we have

$$\|g\|_{L^{p),r,\varphi(\cdot)}_w(X)} \le C\overline{C}\|f\|_{L^{p),r,\varphi(\cdot)}_w(X)}, \quad (f,g) \in \mathcal{F}, \tag{4}$$

where C is the same constant as in (3) *and $\overline{C}$ is independent of (f,g).*

These statements for $\varphi(t) = t^\theta$, $t>0$ were proved in [18].

Remark 2.1. From the extrapolation results and the fact that the Muckenhoupt condition $w \in A_{p_0}(X)$ on the weights guarantees a one–weight inequality in the classical Lebesgue spaces $L^{p_0}_w(X)$, we have the one-weight norm estimates for operators of Harmonic Analysis such as Calderón–Zygmund singular integrals, commutators of singular integrals, fractional integrals and commutators of fractional integrals in grand Morrey spaces defined on an SHT (cf., [18]).

3. Applications to PDEs

In the last thirty years a number of papers have been devoted to the study of local and global regularity properties of strong solutions to elliptic equations with discontinuous coefficients. To be more precise, let us consider the second order equation

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} a_{ij}(x) D_{x_i x_j} u = f(x) \quad \text{for almost all } x \in \Omega, \tag{5}$$

where $\mathcal{L}$ is a uniformly elliptic operator over the bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$.

We assume that a domain Ω satisfies $\mathcal{A}$ condition (see (1)). In this case, Ω with the induced Lebesgue measure and Euclidean metrics is an SHT. Hence the previous statements are valid for such domains.

Regularizing properties of $\mathcal{L}$ in Hölder spaces (i.e., $\mathcal{L}u \in C^\alpha(\bar\Omega)$ imply that $u \in C^{2+\alpha}(\bar\Omega)$) have been well studied in the case of Hölder continuous coefficients $a_{ij}(x)$. Also, a unique classical solvability of the Dirichlet problem for (5) has been derived in this case (we refer to [8] and references therein). In the case of uniformly continuous coefficients a_{ij}, an L^p-Schauder theory has been elaborated for the operator $\mathcal{L}$ (see [1,8]). In particular, $\mathcal{L}u \in L^p(\Omega)$ always implies that the strong solution to (5) belongs to the Sobolev space $W^{2,p}(\Omega)$ for each $p \in (1,\infty)$. However, the situation becomes rather difficult if one tries to allow discontinuity at the principal coefficients of $\mathcal{L}$. In general, it is well-known (cf. [21]) that an arbitrary discontinuity of a_{ij} implies that the L^p-theory of $\mathcal{L}$ and the strong solvability of the Dirichlet problem for (5) break down. A notable exception of that rule is the two-dimensional case ($\Omega \subset \mathbb{R}^2$). It was shown by G. Talenti that the solely condition on the measurability and boundedness of the a_{ij}'s ensures isomorphic properties of $\mathcal{L}$ considered as a mapping from $W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega)$ into $L^2(\Omega)$. To handle with the multidimensional case ($n \ge 3$), it is necessary to add additional properties on $a_{ij}(x)$ to the uniform ellipticity in order to guarantee for $\mathcal{L}$ to possess the regularizing property in Sobolev functional scales. In particular, if $a_{ij}(x) \in W^{1,n}(\Omega)$ (cf., [22]), or if the difference between the largest and the smallest eigenvalues of $\{a_{ij}(x)\}$ is small enough (the Cordes condition), then $\mathcal{L}u \in L^2(\Omega)$ yields that $u \in W^{2,2}(\Omega)$ and these results can be extended to $W^{2,p}(\Omega)$ for $p \in (2-\varepsilon, 2+\varepsilon)$ with sufficiently small ε.

Later, the Sarason class VMO of functions with a vanishing mean oscillation was used in the study of local and global Sobolev regularity of strong solutions to (5).

Next, we define the space BMO, then the smallest VMO class, where we consider the coefficients a_{ij} and further the class in which we consider the known term f.

In the sequel, let Ω be an open bounded set in $\mathbb{R}^n$.

Definition 3.1. Let $f \in L^1_{\text{loc}}(\Omega)$. We define the integral mean $f_{x,R}$ by

$$f_{x,R} := \frac{1}{|\Omega \cap B(x,R)|} \int\limits_{\Omega \cap B(x,R)} f(y)dy,$$

where $B(x,R)$ ranges in the class of balls centered in x with radius R and $|\Omega \cap B(x,R)|$ is the Lebesgue measure of $\Omega \cap B(x,R)$. If we are not interested in specifying which the center is, we just use the notation f_R.

We now give the definition of Bounded Mean Oscillation functions (BMO) that appeared first in the note by F. John and L. Nirenberg [11].

Definition 3.2. Let $f \in L^1_{\text{loc}}(\Omega)$. We say that f belongs to $BMO(\Omega)$ if the seminorm $\|f\|_*$ is finite, where

$$\|f\|_* := \sup_{B(x,R)} \frac{1}{|B(x,R)|} \int\limits_{B(x,R)} |f(y) - f_{x,R}|dy.$$

Next, we consider the definition of the space of Vanishing Mean Oscillation functions given first by D. Sarason [29].

Definition 3.3. Let $f \in BMO(\Omega)$ and

$$\eta(f,R) := \sup_{\rho \le R} \frac{1}{|B_\rho|} \int\limits_{B_\rho} |f(y) - f_\rho|dy,$$

where $B\rho$ ranges over the class of balls of $\mathbb{R}^n$ of radius ρ. Further, a function $f \in VMO(\Omega)$ if $\lim_{R \to 0} \eta(f,R) = 0$.

The Sarason class is then expressed as the subspace of the functions in the John-Nirenberg class whose BMO norm over a ball vanishes as the radius of the balls tends to zero. This property implies a number of good features of VMO functions not shared by general BMO functions; in particular, they can be approximated by smooth functions.

This class of functions was considered by many others. First, we recall the paper by F. Chiarenza, M. Frasca and P. Longo [6], where the authors answer a question raised thirty years before by C. Miranda [22]. In his paper, the author considers a linear elliptic equation where the coefficients a_{ij} of the higher order derivatives are in the class $W^{1,n}(\Omega)$ and asks whether the gradient of the solution is bounded if $p > n$. In [6] the authors suppose that $a_{ij} \in VMO$ and prove that Du is Hölder continuous for all $p \in]1, +\infty[$.

Also, it is possible to check that the bounded uniformly functions are in VMO, as well as the functions of fractional Sobolev spaces $W^{\theta, \frac{n}{\theta}}$, $\theta \in]0,1[$.

The study of Sobolev regularity of strong solutions of (5) was initiated in 1991 with the pioneering work by F. Chiarenza, M. Frasca and P. Longo [5]. It was found that if $a_{ij}(x) \in VMO \cap L^\infty(\Omega)$ and $\mathcal{L}u \in L^p(\Omega)$, then $u \in W^{2,p}(\Omega)$ for *each* value of p in the range $(1,\infty)$. Moreover, the well-posedness of the Dirichlet problem for (5) in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ was proved. As a consequence, Hölder's continuity of the strong solution or of its gradient follows if the exponent p is sufficiently large.

Thanks to the fundamental accessibility of these two papers [4, 6], many other authors have used VMO class to obtain the regularity results for PDEs and for systems with discontinuous coefficients.

Continuing the study of regularity of PDEs, we see that Hölder's continuity can be inferred for small p if one has more information on $\mathcal{L}u$, such as its belonging to a suitable Morrey class $L^{p,\lambda}(\Omega)$.

Let us denote by $M^{p,\lambda}(\Omega)$ the Morrey space defined on a domain $\Omega \subset \mathbb{R}^n$ which is defined with respect to the norm

$$\|f\|_{p,\lambda} := \sup_{\substack{x \in \Omega \\ 0 < \rho < diam\, \Omega}} \left(\frac{1}{\rho^\lambda} \int\limits_{\Omega \cap B_\rho(x)} |f(y)|^p \, dy \right)^{1/p}.$$

The exponent λ may take the values that do not belong to $]0,n[$ but the unique cases of real interest are those for which $\lambda \in]0,n[$. Indeed, from the definition, we immediately see that $M^{p,\lambda}(\Omega) = L^p(\Omega)$,

if $\lambda \le 0$. Sometimes later we will explicitly use the fact that $M^{p,0}(\Omega) = L^p(\Omega)$.
Moreover, if $\lambda = n$, by applying the Lebesgue differentiation theorem, we find that

$$\lim_{\rho\to 0^+} \frac{1}{\rho^n} \int\limits_{\Omega\cap B_\rho(x)} |f(y)|^p\, dy = \lim_{\rho\to 0^+} \frac{1}{\rho^n} \int\limits_{B_\rho(x)} |f(y)|^p dy = C|f(x)|^p$$

for every Lebesgue point or, equivalently, almost everywhere in Ω. Then in order that $f(x) \in M^{p,n}(\Omega)$, it is necessary and sufficient that f is bounded. It means that $M^{p,n}(\Omega) = L^\infty(\Omega)$.
If $\lambda > n$, then $M^{p,\lambda}(\Omega) = \{0\}$.

Using the spaces defined above, there arises a natural problem, namely, to study the regularizing properties of the operator $\mathcal{L}$ in Morrey spaces in the case of VMO principal coefficients. In [2], it is proved that each $W^{2,p}$-viscosity solution to (5) lies in $C^{1+\alpha}(\Omega)$ if $f(x)$ belongs to the Morrey space $M^{n,n\alpha}(\Omega)$ with $\alpha \in (0,1)$.

One of the main results of this note is to obtain the local regularity, in the grand Morrey Spaces, for the highest order derivatives of solutions of elliptic non-divergence form with coefficients that may be discontinuous.

We recall that in the case of continuous coefficients of the above kind equation, the results are obtained by S. Agmon, A. Douglis and L. Nirenberg [1]. Later, discontinuous coefficients were considered by S. Campanato [3].

Then this paper can be regarded as a continuation of the study of L^p regularity of solutions of second order elliptic PDEs for the maximum order derivatives of the solutions to a certain class of linear elliptic equations in nondivergence form with discontinuous coefficients (see also [18] for related topics).

Let us consider the second order differential operator

$$\mathcal{L} \equiv a_{ij}(x) D_{ij}, \qquad D_{ij} \equiv \frac{\partial^2}{\partial x_i \partial x_j}.$$

Here we have adopted the usual summation convention on repeated indices.

In the sequel, we need the following regularity and ellipticity assumptions on the coefficients of $\mathcal{L}$, $\forall i,j = 1,\dots,n$:

$$\begin{cases} a_{ij}(x) \in L^\infty(\Omega) \cap VMO, \\ a_{ij}(x) = a_{ji}(x), \quad \text{a.a. } x \in \Omega \\ \exists\ \kappa > 0 :\ \kappa^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \kappa|\xi|^2, \quad \forall \xi \in \mathbb{R}^n,\ \text{a.a. } x \in \Omega. \end{cases} \tag{6}$$

Set η_{ij} for the VMO-modulus of the function $a_{ij}(x)$ and let $\eta(r) = \Big(\sum\limits_{i,j=1}^{n} \eta_{ij}^2\Big)^{1/2}$. We denote by $\Gamma(x,t)$ the normalized fundamental solution of $\mathcal{L}$, i.e.,

$$\Gamma(x,\xi) = \frac{1}{n(2-n)\omega_n \sqrt{\det\{a_{ij}(x)\}}} \Big(\sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j\Big)^{(2-n)/2}$$

for a.a. x and all $\xi \in \mathbb{R}^n \setminus \{0\}$, where $A_{ij}(x)$ stand for the entries of the inverse matrix of the matrix $\{a_{ij}(x)\}_{i,j=1,\dots,n}$, and ω_n is the measure of the unit ball in $\mathbb{R}^n$. We set also

$$\Gamma_i(x,\xi) = \frac{\partial}{\partial \xi_i}\Gamma(x,\xi), \quad \Gamma_{ij}(x,\xi) = \frac{\partial}{\partial \xi_i \partial \xi_j}\Gamma(x,\xi),$$
$$M = \max_{i,j=1,\dots,n} \max_{|\alpha|\le 2n} \left\| \frac{\partial^\alpha \Gamma_{ij}(x,\xi)}{\partial \xi^\alpha} \right\|_{L^\infty(\Omega\times\Sigma)}.$$

It is well known that $\Gamma_{ij}(x,\xi)$ are the Calderón–Zygmund kernels in the ξ variable.

Theorem 3.1. *Let* (6) *be true,* $1 < p < \infty$, $-1/p \le r < 0$, $\varphi(\cdot) \in \Phi_p$. *Let* Ω *be a domain satisfying* $\mathcal{A}$ *condition* (*see* (1)) *and let* w *be a weight on* Ω *such that* $w \in A_p(\Omega)$. *Then there exist positive* *constants* $c = c(n, \kappa, p, r, \varphi(\cdot), M, w)$ *and* $\rho_0 = \rho_0(C, n)$ *such that for every ball* $B_\rho \subset\subset \Omega$, $\rho < \rho_0$ *and every* $u \in W_0^{2,p}(B_\rho)$ *such that* $D_{ij}u \in L_w^{p),r,\varphi(\cdot)}(B_\rho)$, *for* $w \in A_p(\Omega)$, *we have*

$$\|D_{ij}u\|_{L_w^{p),r,\varphi(\cdot)}(B_\rho)} \le c\|\mathcal{L}u\|_{L_w^{p),r,\varphi(\cdot)}(B_\rho)}, \quad \forall i, j = 1, \dots, n. \tag{7}$$

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