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ON 3-DIMENSIONAL QUASI-PARA-SASAKIAN MANIFOLDS AND RICCI SOLITONS

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Abstract. The purpose of this paper is to study 3-dimensional quasi-para-Sasakian manifolds and Ricci solitons. First, we prove that a 3-dimensional non-paracosymplectic quasi-para-Sasakian manifold is an η-Einstein manifold if and only if the structure function β is constant. Further, it is shown that a Ricci soliton on a 3-dimensional quasi-para-Sasakian manifold with β=constant is expanding. Moreover, we show that if a 3-dimensional quasi-para-Sasakian manifold admits a Ricci soliton, then the flow vector field V is Killing, and the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure. Besides, we study gradient Ricci solitons and prove that if a 3-dimensional non-paracosymplectic quasi-para-Sasakian manifold with β=constant admits a gradient Ricci soliton, then the manifold is an Einstein one. Also, a suitable example of a 3-dimensional quasi-para-Sasakian manifold is constructed to verify our results.

1. Introduction

In recent years, the theory of almost contact and almost paracontact geometry is an active branch of research. Many authors have studied almost contact manifolds and pointed out its importance in many applied areas such as geometric optics, mechanics, thermodynamics and control theory. The study of almost paracontact manifolds is also of interest from the standpoint of pseudo-Riemannian geometry and mathematical physics. S. Kaneyuki and S. F. Williams [21] inititaed the study of almost paracontact geometry. A systematic study of almost paracontact metric manifolds was carried out by S. Zamkovoy in paper [39]. Since then, several authors studied these manifolds by emphasizing the similarities and differences with respect to the most well-known almost contact case. Comparing with the huge literature in almost contact geometry, it seems that there are necessary new studies in almost paracontact geometry. Some interesting properties of almost paracontact manifolds were studied in papers [21, 25, 32–34, 36, 37, 39] and the references therein.

The notion of normal almost contact metric manifolds of dimension 3 was studied by O. Olszak in [23]. He derived certain necessary and sufficient conditions for an almost contact metric structure on a manifold to be normal. Recently, J. Welyczko studied curvature and torsion of Frenet-Legendre curves in 3-dimensional normal almost paracontact metric manifolds. C. L. Bejan and M. Crasmareann [1] considered second order parallel tensors and Ricci solitons in a 3-dimensional normal paracontact geometry. Further, I. K. Erken [12] studied some classes of 3-dimensional normal almost paracontact metric manifolds.

A quasi-Sasakian manifold, introduced by D. E. Blair [3], is a normal almost contact metric manifold whose fundamental 2-form Φ is closed. Quasi-Sasakian manifolds unifies Sasakian and cosymplectic manifolds, and also can be viewed as an odd-dimensional counterpart of Kaehler structures.

An almost paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called quasi-para-Sasakian if the structure is normal and its fundamental 2-form Φ is closed. These manifolds are analogues to the quasi-Sasakian manifolds and they belong of the class $\mathbb{G}_5$ of the classification given in [40]. Basic properties of quasi-para-Sasakian manifolds and their general curvature identities are investigated systematically

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in papers [13, 14] and [15]. A quasi-para-Sasakian manifold is a paracontact manifold with pseudo-Riemannian metric, whereas a quasi-Sasakian manifold is a contact manifold with Riemannian metric. Therefore these two notions are completely different.

On the other hand, the study of Ricci soliton on a pseudo-Riemannian manifold is an interesting topic of research in modern differential geometry. A Ricci soliton is a natural generalization of an Einstein metric and this notion was introduced by Hamilton [19]. A pseudo-Riemannian metric g on a smooth manifold M^n is said to be a Ricci soliton if there exists a real number λ such that its Ricci tensor S satisfies

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

where V is a vector field on M (called the potential vector field) and $\pounds_V$ denotes the Lie differentiation along the vector field V. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive respectively. Also, a Ricci soliton with V zero or Killing is an Einstein metric. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton. The Ricci flow [18] is an evolution equation for the metrics on a pseudo-Riemannian manifold defined by $\frac{\partial}{\partial t} g_{ij}(t) = -2S_{ij}$. Ricci solitons are self-similar solutions of the Ricci flow, and play an important role in understanding its singularities. The theoretical physicists have also been looking into the equation of a Ricci soliton in relation with the string theory. In 1985, Friedan [16] made an effort in this direction and discussed some aspects of it. Later, the study of Ricci solitons in the context of contact geometry has initiated by R. Sharma [28] and then continued by several authors in papers [10, 17, 27, 28, 30, 31, 35]. The problem of studying Ricci solitons in the context of paracontact metric geometry was initiated by G. Calvaruso and D. Perrone [5]. The case of Ricci solitons in a 3-dimensional paracontact geometry was studied in papers [1] and [6]. Some properties of Ricci solitons on almost paracontact metric manifolds have been studied in papers [2, 4, 8, 9, 20, 22, 24, 26] and the references therein.

Motivated by these circumstances, in this paper we focus our study to quasi-para-Sasakian manifolds of dimension three. The present paper is organized as follows: In Section 2, we give a brief description about almost paracontact structures. In Section 3, we study 3-dimensional quasi-para-Sasakian manifolds. In the next section, we study 3-dimensional quasi-para-Sasakian manifolds with the structure function β is constant admitting a Ricci soliton. Also, we justify our result by providing a suitable example. The final section is devoted to study the gradient almost Ricci solitons on quasi-para-Sasakian 3-manifolds. Throughout the paper, several interesting results and their consequences are discussed.

2. Almost Paracontact Structures

If on a $(2n+1)$-dimensional smooth manifold M^{2n+1} there exist a $(1,1)$-type tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \cdot \phi = 0, \tag{2}$$

where I denotes the identity endomorphism, then we say that the triple (ϕ, ξ, η) is an almost paracontact structure on M^{2n+1}. The distribution $\mathbb{D} : P \in M \to \mathbb{D}_P \subset T_P M$:

$$\mathbb{D} = \operatorname{Ker} \eta = \{X \in T_P M : \eta(X) = 0\}$$

is called paracontact distribution generated by η. The tensor field ϕ induces an almost paracomplex structure [21] on each fibre of $\mathbb{D}$, that is, the eigen distributions D^+, D^- of ϕ corresponding to the eigenvalues 1 and -1, respectively, have the same dimension n. From (2), we find that the endomorphism ϕ has rank $2n$. In general, a smooth manifold M^{2n+1} endowed with an almost paracontact structure is called an almost paracontact manifold which is denoted by $(M^{2n+1}, \phi, \xi, \eta)$. An almost paracontact manifold is called an almost paracontact metric manifold denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$, if it is additionally endowed with a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{3}$$

or equivalently,

$$g(X, \phi Y) = -g(\phi X, Y) \text{ and } g(X, \xi) = \eta(X), \tag{4}$$

for all vector fields $X, Y \in \chi(M)$. Then the metric g is said to be compatible with the almost paracontact manifold $(M^{2n+1}, \phi, \xi, \eta)$.

An almost paracontact metric manifold becomes a paracontact metric manifold if $g(X, \phi Y) = d\eta(X, Y) = \Phi(X, Y)$ for any vector fields $X, Y \in \chi(M)$. The 1-form η is then a paracontact form. An almost paracontact manifold is said to be normal if and only if the tensor field $\mathcal{N}_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically. The normality condition says that the almost paracomplex structure J defined on $M \times \mathbb{R}$ by

$$J\Big(X, \lambda \frac{d}{dt}\Big) = \Big(\phi X + \lambda\xi, \eta(X)\frac{d}{dt}\Big)$$

is integrable. On a paracontact metric manifold, we define a symmetric, trace-free $(1,1)$-type tensor field h by $h = \frac{1}{2}\pounds_\xi \phi$, where $\pounds$ is the Lie differentiation. It is known that $h\phi = -\phi h$, $h\xi = 0$, $tr\ h = tr\ h\phi = 0$ and $\nabla\xi = -\phi + \phi h$, where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold (M, g).

A paracontact structure on M^{2n+1} naturally gives rise to an almost paracomplex structure on the product $M^{2n+1} \times \mathbb{R}$. A paracontact metric manifold is called a K-paracontact manifold if its characteristic vector field ξ is Killing (or equivalently $h = 0$). A normal almost paracontact metric manifold will be called para-Sasakian if $F = d\eta$ [11] and quasi-para-Sasakian if $dF = 0$. Obviously, the class of para-Sasakian manifolds is contained in the class of quasi-para-Sasakian manifolds. The converse does not hold in general. A paracontact metric manifold will be called paracosymplectic if $dF = 0$, $d\eta = 0$ [7], more generally, α-para-Kenmotsu if $dF = 2\alpha\eta \wedge F$, $d\eta = 0$, $\alpha = \text{const}. \neq 0$.

For a three-dimensional almost paracontact metric manifold [36] M, the following three conditions are mutually equivalent:

(a) M is normal,

(b) there exist functions α, β on M such that

$$(\nabla_X \phi)Y = \beta(g(X, Y)\xi - \eta(Y)X) + \alpha(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{5}$$

(c) there exist functions α, β on M such that

$$\nabla_X \xi = \alpha(X - \eta(X)\xi) + \beta\phi X. \tag{6}$$

Here, ∇ is the Levi-Civita connection of g. The functions α, β appearing in the above equations are given by

$$2\alpha = \text{Trace}\{X \to \nabla_X \xi\}, \quad 2\beta = \text{Trace}\{X \to \phi\nabla_X \xi\}. \tag{7}$$

A three-dimensional normal almost para contact metric manifold is said to be

- paracosymplectic if $\alpha = \beta = 0$ [7],
- quasi-para Sasakian if and only if $\alpha = 0$ and $\beta \neq 0$ [11,36],
- β-para-Sasakian if and only if $\alpha = 0$ and β is constant, in particular, para-Sasakian if $\beta = -1$ [36,39],
- α-para-Kenmotsu if α is a non-zero constant and $\beta = 0$ [29].

3. Three-dimensional Quasi-para-Sasakian Manifolds

An almost paracontact metric manifold (M, ϕ, ξ, η, g) is called a 3-dimensional quasi-para-Sasakian [36] if and only if there exists a certain function β on M such that

$$(\nabla_X \phi)Y = \beta(g(X, Y)\xi - \eta(Y)X) \tag{8}$$

for all vector fields $X, Y \in \chi(M)$.

From (8), we have

$$\nabla_X \xi = \beta\phi X, \tag{9}$$

or equivalently,

$$(\nabla_X \eta)(Y) = \beta g(\phi X, Y) \tag{10}$$

for any vector fields $X, Y \in \chi(M)$.

A quasi-para-Sasakian manifold is paracosympletic if and only if $\beta = 0$. If $\beta = \text{const}$ and $\beta \neq 0$, then the manifold reduces to a β-para-Sasakian manifold and if, in particular, $\beta = -1$, the manifold becomes a para-Sasakian manifold. Here, we have the following

Proposition 3.1. *In a 3-dimensional non-paracosymplectic quasi-para-Sasakian manifold, the structure function β satisfies the condition $\xi\beta = 0$.*

Theorem 3.2. *For any 3-dimensional quasi-para-Sasakian manifold (M, ϕ, ξ, η, g), we have*

$$dr(\xi) = 0.$$

Proof. The Ricci operator Q of a 3-dimensional quasi-para-Sasakian manifold (M, ϕ, ξ, η, g) is known as the following

$$QY = \Big(\frac{r}{2} + \beta^2\Big) Y - \Big(\frac{r}{2} + 3\beta^2\Big) \eta(Y)\xi - \eta(Y)\phi \operatorname{grad} \beta + d\beta(\phi Y)\xi, \tag{11}$$

where r is the scalar curvature of the manifold M.
Differentiating (11) covariantly with respect to X and using (8) and (9), we get

$$\begin{aligned}(\nabla_X Q)Y =&\Big(\frac{dr(X)}{2} + 2\beta d\beta(X)\Big)Y - \Big(8\beta d\beta(X) + \frac{dr(X)}{2}\Big)\eta(Y)\xi\\ &-\beta\Big(3\beta^2 + \frac{r}{2}\Big)g(\phi X, Y)\xi - \beta g(\phi X, Y)\phi \operatorname{grad} \beta\\ &-\eta(Y)\phi\nabla_X \operatorname{grad} \beta + g(\nabla_X \operatorname{grad} \beta, \phi Y)\xi.\end{aligned} \tag{12}$$

For any point $p \in U \subset M$, there exists a local orthonormal ϕ-basis $\{e_1 = \phi e_2,\ e_2 = \phi e_1,\ e_3 = \xi\}$, where $g(e_1, e_1) = -g(e_2, e_2) = g(e_3, e_3) = 1$. Replacing X by e_i in the equation (12) and taking summation over i, and then using the following well-known formula for pseudo-Riemannian manifolds

$$\text{Trace}\{X \to (\nabla_X Q)Y\} = \frac{1}{2}dr(Y),$$

we obtain

$$\begin{aligned}\frac{1}{2}dr(Y) =&g((\nabla_{e_1} Q)Y, e_1) + g((\nabla_{e_2} Q)Y, e_2) + g((\nabla_{e_3} Q)Y, e_3)\\ =&\Big(\frac{dr(e_1)}{2} + 2\beta d\beta(e_1)\Big)g(e_1, Y) - \beta g(\phi e_1, Y)g(\phi \operatorname{grad} \beta, e_1)\\ &-g(\phi\nabla_{e_1} \operatorname{grad} \beta, e_1)\eta(Y) + \Big(\frac{dr(e_2)}{2} + 2\beta d\beta(e_2)\Big)g(e_2, Y)\\ &-\beta g(\phi e_2, Y)g(\phi \operatorname{grad} \beta, e_2) - g(\phi\nabla_{e_2} \operatorname{grad} \beta, e_2)\eta(Y)\\ &-g(\nabla_\xi\phi \operatorname{grad} \beta, \xi)\eta(Y) + g(\nabla_\xi \operatorname{grad} \beta, \phi Y).\end{aligned}$$

Putting $Y = \xi$ in the above equation, we get $dr(\xi) = 0$. This completes the proof of our theorem. □

Next, we prove the following

Theorem 3.3. *A 3-dimensional non-paracosymplectic quasi-para-Sasakian manifold is an η-Einstein manifold if and only if the structure function β is constant.*

Proof. From (11), we write

$$\begin{aligned}S(X, Y) =&g(QX, Y)\\ =&-\Big(\frac{r}{2} + \beta^2\Big) g(\phi X, \phi Y) + \phi X(\beta)\eta(Y) + \phi Y(\beta)\eta(X) - 2\beta^2\eta(X)\eta(Y).\end{aligned} \tag{13}$$

A 3-dimensional quasi-para-Sasakian manifold (M, ϕ, ξ, η, g) is called η-Einstein, if the Ricci tensor S satisfies

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{14}$$

for all vector fields $X, Y \in \chi(M)$; here, a, b are smooth scalar functions on M.
Then from (13) and (14), we have

$$\eta(X)d\beta(\phi Y) + \eta(Y)d\beta(\phi X) = (a - \frac{r}{2} - \beta^2)g(X, Y) + (b + \frac{r}{2} + 3\beta^2)\eta(X)\eta(Y). \tag{15}$$

Taking $Y = \xi$ in the last equation, we get

$$d\beta(\phi X) = (a + b + 2\beta^2)\eta(X). \tag{16}$$

Now, taking ϕX instead of X in the above equation and using Proposition 3.1, we obtain β is const.

Conversely, from (13), it is clear that if $\beta = \text{const}.$, then we have

$$S(X,Y) = \Big(\frac{r}{2} + \beta^2\Big) g(X,Y) - \Big(\frac{r}{2} + 3\beta^2\Big)\eta(X)\eta(Y). \tag{17}$$

That is, M is an η-Einstein manifold. This proves our result. □

The Riemannian curvature tensor of a 3-dimensional quasi-para-Sasakian manifold with $\beta = \text{const}$ is given by

$$\begin{aligned} R(X,Y)Z =& \Big(\frac{r}{2} + 2\beta^2\Big)(g(Y,Z)X - g(X,Z)Y) - \Big(\frac{r}{2} + 3\beta^2\Big)(g(Y,Z)\eta(X)\xi \\ & - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y). \end{aligned} \tag{18}$$

We recall the following result for later use.

Theorem 3.4 ([13, Theorem 1, I. K. Erken])**.** *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a quasi-para-Sasakian manifold of constant curvature K. Then $K \le 0$. Furthermore,*

- *If $K = 0$, the manifold is paracosymplectic.*
- *If $K < 0$, the structure (ϕ, ξ, η, g) is obtained by a homothetic deformation of a para-Sasakian structure M^{2n+1}.*

4. Ricci Solitons on 3-dimensional Quasi-para-Sasakian Manifolds with $\beta = \text{const}$

Theorem 4.1. *A Ricci soliton on a 3-dimensional quasi-para-Sasakian manifold with $\beta = \text{const}$ is expanding.*

Proof. Let us consider a Ricci soliton on a 3-dimensional quasi-para-Sasakian manifold M with the constant structure functin β. Then from (1), we have

$$(\pounds_V g)(X,Y) = -2S(X,Y) - 2\lambda g(X,Y) \tag{19}$$

for any vector fields $X, Y \in \chi(M)$. Using (17) in (19), we deduce that

$$(\pounds_V g)(X,Y) = -\{r + 2(\lambda + \beta^2)\} g(X,Y) + (r + 6\beta^2)\eta(X)\eta(Y) \tag{20}$$

for any vector fields $X, Y \in \chi(M)$. Taking the covariant differentiation of (20) with respect to Z and using (9), we obtain

$$\begin{aligned} (\nabla_Z \pounds_V g)(X,Y) =& - dr(Z)\{g(X,Y) - \eta(X)\eta(Y)\} \\ & - (r+6)\beta\{g(X,\phi Y)\eta(Z) + g(X,\phi Z)\eta(Y)\} \end{aligned} \tag{21}$$

for any vector fields $X, Y, Z \in \chi(M)$. By Yano [38], we have the following commutation formula:

$$\begin{aligned} &(\pounds_V \nabla_X g - \pounds_X \nabla_V g - \nabla_{[V,X]} g)(Y,Z) \\ &= -g((\pounds_V \nabla)(X,Y), Z) - g((\pounds_V \nabla)(X,Z), Y) \end{aligned}$$

for any vector fields $X, Y, Z \in \chi(M)$. Since the pseudo-Riemannian metric g is parallel with respect to the Levi-Civita connection ∇, the above relation becomes

$$(\nabla_X \pounds_V g)(Y,Z) = g((\pounds_V \nabla)(X,Y), Z) + g((\pounds_V \nabla)(X,Z), Y)$$

for any vector fields $X, Y, Z \in \chi(M)$. As $(\pounds_V \nabla)(X,Y) = (\pounds_V \nabla)(Y,X)$, it follows from the above relation that

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y), Z) =& (\nabla_X \pounds_V g)(Y,Z) + (\nabla_Y \pounds_V g)(Z,X) \\ & - (\nabla_Z \pounds_V g)(X,Y) \end{aligned} \tag{22}$$

for any vector fields $X, Y, Z \in \chi(M)$. Using (21) in (22), we have

$$\begin{aligned} 2g((\pounds_V \nabla)(X,Y),Z) =& dr(Z)[g(X,Y)-\eta(X)\eta(Y)] \\ &+(r+6)\beta[g(\phi X,Z)\eta(Y)+g(\phi Y,Z)\eta(X)] \\ &-dr(Y)[g(X,Z)-\eta(X)\eta(Z)] \\ &-(r+6)\beta[g(Y,\phi Z)\eta(X)+g(Y,\phi X)\eta(Z)] \\ &-dr(X)[g(Y,Z)-\eta(Y)\eta(Z)] \\ &-(r+6)\beta[g(X,\phi Y)\eta(Z)+g(X,\phi Z)\eta(Y)]. \end{aligned} \tag{23}$$

Making use of the skew-symmetric property of ϕ in (23) and then removing Z, it follows that

$$\begin{aligned} (\pounds_V \nabla)(X,Y) =& \frac{1}{2}\operatorname{grad} r[g(X,Y)-\eta(X)\eta(Y)] - \frac{1}{2}dr(Y)[X-\eta(X)\xi] \\ &-\frac{1}{2}dr(X)[Y-\eta(Y)\xi] + (r+6)\beta[\eta(Y)\phi X+\eta(X)\phi Y], \end{aligned} \tag{24}$$

where $dr(Z) = g(\operatorname{grad} r, Z)$. Substituting $Y = \xi$ in (24) and making use of Theorem 3.2, we have

$$(\pounds_V \nabla)(X,\xi) = (r+6)\beta\phi X \tag{25}$$

for any vector field $X \in \chi(M)$. Now, taking the covariant differentiation of (25) along an arbitrary vector field and then making use of the relation (8), we get

$$(\nabla_X \pounds_V \nabla)(Y,\xi) = dr(X)\beta\phi Y + (r+6)\beta^2\{g(X,Y)\xi-\eta(Y)X\} \tag{26}$$

for any vector fields $X, Y, Z \in \chi(M)$. Using (26) in the following identity (see Yano [38]):

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z),$$

one obtains

$$(\pounds_V R)(X,Y)\xi = dr(X)\beta\phi Y - dr(Y)\beta\phi X + (r+6)\beta^2\{\eta(X)Y-\eta(Y)X\}. \tag{27}$$

This yields

$$(\pounds_V R)(X,\xi)\xi = -(r+6)\beta^2[X-\eta(X)\xi] \tag{28}$$

for any vector field $X \in \chi(M)$. Setting $Y = \xi$ in (20), it follows that

$$(\pounds_V g)(X,\xi) = -2(\lambda-2\beta^2)\eta(X). \tag{29}$$

Lie-differentiating $g(X,\xi) = \eta(X)$ along V and then using (29), we obtain

$$(\pounds_V \eta)(X) - g(\pounds_V \xi, X) - 2(\lambda-2\beta^2)\eta(X) = 0 \tag{30}$$

for any vector field $X \in \chi(M)$. Further, taking the Lie-differentiation of $\eta(\xi) = 1$ along V and then using (30), one can obtain

$$\eta(\pounds_V \xi) = 0 \text{ and } (\pounds_V \eta)(\xi) = 2(\lambda-2\beta^2). \tag{31}$$

Next, Lie-differentiating the equation $R(X,\xi)\xi = \beta^2\{\eta(X)\xi - X\}$ along V and taking into account (31), we obtain

$$(\pounds_V R)(X,\xi)\xi = 2(\lambda-2\beta^2)\eta(X)\xi \tag{32}$$

for any vector field $X \in \chi(M)$. Comparing this equation with (28), we get

$$-2(\lambda-2\beta^2)\eta(X)\xi = (r+6\beta^2)[X-\eta(X)\xi]. \tag{33}$$

Putting $X = \xi$ in the above equation, we have

$$\lambda = 2\beta^2. \tag{34}$$

This means that the Ricci soliton is expanding. This proves our theorem. □

Theorem 4.2. *If a 3-dimensional quasi-para-Sasakian manifold with* $\beta = \text{const}$ *admits a Ricci soliton, then the flow vector field* V *is Killing, and the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure.*

Proof. Taking into account (34) in relation (33), we obtain

$$r = -6\beta^2, \tag{35}$$

that is, the scalar curvature of the manifold is constant. Moreover, using $\lambda = 2\beta^2$ and $r = -6\beta^2$ in (20) yields $(\pounds_V g)(X,Y) = 0$. Thus, the potential vector field V is Killing. Next, using (35) in (17), we obtain $S = -2\beta^2 g$. Making use of this relation in equation (18), we conclude that

$$R(X,Y)Z = -\beta^2\{g(Y,Z)X - g(X,Z)Y\} \tag{36}$$

for any vector fields $X, Y, Z \in \chi(M)$. This means that M is of constant negative curvature $-\beta^2$. Hence, using Theorem 3.4, we can say that the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure. This completes the proof. □

It is known that a 3-dimensional non-paracosymplectic quasi-para-Sasakian manifold with $\beta =$ const is locally ϕ-symmetric if and only if the scalar curvature is constant. So, by the above discussion, we can also state the following

Corollary 4.3. *A 3-dimensional non-paracosymplectic quasi-para-Sasakian manifold with $\beta = \mathrm{const}$. admitting a Ricci soliton is locally ϕ-symmetric.*

Theorem 4.4. *Let (M, g) be a 3-dimensional quasi-para-Sasakian manifold with $\beta = \mathrm{constant}$. If (g, V) is a Ricci soliton such that a potential vector field V is pointwise collinear with the structure vector field ξ, then the soliton becomes trivial.*

Proof. Let V be a pointwise collinear vector field with the structure vector field ξ, that is, $V = \gamma\xi$, where γ is a smooth function on M. Then from (1), we obtain

$$g(\nabla_X \gamma\xi, Y) + g(\nabla_Y \gamma\xi, X) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{37}$$

or

$$\begin{aligned}\gamma\{g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X)\} + d\gamma(X)\eta(Y) + d\gamma(Y)\eta(X) \\ +2S(X,Y) + 2\lambda g(X,Y) = 0\end{aligned} \tag{38}$$

for any vector fields $X, Y \in \chi(M)$. Using (9) in (38), we obtain

$$d\gamma(X)\eta(Y) + d\gamma(Y)\eta(X) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{39}$$

Putting $Y = \xi$ in (39), we get

$$d\gamma(X) = [4\beta^2 - 2\lambda - (\xi\gamma)]\eta(X). \tag{40}$$

Again, putting $X = \xi$ in (40), yields $d\gamma(\xi) = 2\beta^2 - \lambda$. Putting this value in (39), we get

$$d\gamma(X) = (2\beta^2 - \lambda)\eta(X),$$

or

$$d\gamma = (2\beta^2 - \lambda)\eta. \tag{41}$$

Applying d on both sides of (41), we get

$$(2\beta^2 - \lambda)d\eta = 0.$$

Since $d\eta \neq 0$, we have $\lambda = 2\beta^2$. Using this value of λ in (41) yields γ is constant. Since ξ is Killing and γ is constant, the vector field $V(= \gamma\xi)$ is also Killing. Hence the soliton becomes trivial and this completes the proof. □

Now, we construct an example of quasi-para-Sasakian 3-manifold with constant β which admits a Ricci soliton and verify our results.

Example 4.5. Let $\mathcal{L}$ be a 3-dimensional real connected Lie group and $\mathbf{g}$ be its Lie algebra with a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = 6e_3, \quad [e_1, e_3] = 6e_2, \quad [e_2, e_3] = 6e_1.$$

We define almost paracontact structure (ϕ, ξ, η) and pseudo-Riemannian metric g on the Lie group $\mathcal{L}$ as follows:

$$\varphi(e_1) = e_2, \quad \varphi(e_2) = e_1, \quad \varphi(e_3) = 0,$$
$$\xi = e_3, \quad \eta(X) = g(X, \xi),$$
$$g(e_1, e_1) = g(e_3, e_3) = 1, \quad g(e_2, e_2) = -1,$$
$$g(e_i, e_j) = 0, \quad i \neq j \in \{1, 2, 3\}.$$

Hence $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis. From Koszul's formula, we have

$$\begin{gathered} \nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 3e_3, \quad \nabla_{e_1} e_3 = 3e_2, \\ \nabla_{e_2} e_1 = -3e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = 3e_1, \\ \nabla_{e_3} e_1 = -3e_2, \quad \nabla_{e_3} e_2 = -3e_1, \quad \nabla_{e_2} e_3 = 0, \end{gathered} \tag{42}$$

where ∇ is a Levi-Civita connection. Hence the structure is a 3-dimensional quasi-para-Sasakian structure with $\beta = 3$ is constant function. With the aid of (42), we find the following expressions:

$$\begin{gathered} R(e_1, e_2)e_1 = 9e_2, \quad R(e_1, e_2)e_2 = 9e_1, \quad R(e_1, e_2)e_3 = 0, \\ R(e_1, e_3)e_1 = 9e_3, \quad R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_3 = -9e_1, \\ R(e_2, e_3)e_1 = 0, \quad R(e_2, e_3)e_2 = -9e_3, \quad R(e_2, e_3)e_3 = -9e_2. \end{gathered} \tag{43}$$

As a result of (43), it is easy to find the following:

$$S(e_1, e_1) = -18, \quad S(e_2, e_2) = 18, \quad S(e_3, e_3) = -18.$$

Then the scalar curvature $r = \sum_{i=1}^{3} \varepsilon_i S(e_i, e_i) = -54$, where $\varepsilon_i = g(e_i, e_i)$. Thus, the Ricci tensor satisfy

$$S(X, Y) = -2\beta^2 g(X, Y), \tag{44}$$

for all $X, Y \in \mathbf{g}$. From (43), we can easily show that

$$R(X, Y)Z = -9\{g(Y, Z)X - g(X, Z)Y\}, \tag{45}$$

for any $X, Y, Z \in \mathbf{g}$. Now, consider a vector field

$$V = 3(e_1 + e_2). \tag{46}$$

In view of (42), one can easily verify that

$$(\pounds_V g)(X, Y) = 0, \tag{47}$$

for any $X, Y \in \mathbf{g}$. Unifying (47) and (44), we obtain that g is a Ricci soliton, that is, (1) holds true with V as in (46) and $\lambda = 18$. Further, the relation (45) and Theorem 3.4 shows that quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure and this verifies Theorem 4.2.

5. Gradient Almost Ricci Solitons on a 3-dimensional Quasi-para-Sasakian Manifolds with $\beta = \text{const}$

Theorem 5.1. *If 3-dimensional non-paracosymplectic quasi-para-Sasakian manifold M of constant scalar curvature with $\beta = \text{const}$. admits a gradient almost Ricci soliton, then either M is Einstein, or the soliton vector field V is pointwise collinear with the characteristic vector field ξ on an open set $\mathcal{O}$ of M.*

Proof. A Ricci soliton is called gradient almost Ricci soliton if the vector field V is the gradient of a potential function $-f$ and λ is a variable smooth function. If a pseudo-Riemannian metric g on M is a gradient almost Ricci soliton, then equation (1) assumes the form

$$\nabla\nabla f = S + \lambda g. \tag{48}$$

Equation (48) can be written as

$$\nabla_Y Df = QY + \lambda Y \tag{49}$$

for any vector field $Y \in \chi(M)$, where D is the gradient operator of g and Q is Ricci operator defined by $g(QX, Y) = S(X, Y)$. From (49), it follows that

$$R(X, Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X + X(\lambda)Y - Y(\lambda)X, \tag{50}$$

for all vector fields $X, Y \in \chi(M)$. We have

$$g(R(\xi, Y)Df, \xi) = \beta^2 g(\xi(f)\xi - Df, Y) \tag{51}$$

for any vector field $Y \in \chi(M)$. Also, in a 3-dimensional quasi-para-Sasakian manifold, it follows that

$$g((\nabla_\xi Q)Y - (\nabla_Y Q)\xi, \xi) = 0. \tag{52}$$

From (50), (51) and (52), we get

$$Y(\beta^2 f - \lambda) = \xi(\beta^2 f - \lambda)\eta(Y), \tag{53}$$

which can be written as $d(\beta^2 f - \lambda) = \xi(\beta^2 f - \lambda)\eta$. Operating it by d and applying Poincare lemma, we obtain $d(\xi(\beta^2 f - \lambda)) \wedge \eta + (\xi(\beta^2 f - \lambda))d\eta = 0$. The wedge product of this equation with η gives $(\xi(\beta^2 f - \lambda))d\eta \wedge \eta = 0$, where we have used $\eta \wedge \eta = 0$. Since $d\eta \wedge \eta \neq 0$, the last equation yields $\xi(\beta^2 f - \lambda) = 0$. Thus $\beta^2 Df = D\lambda$.
Contracting (50) and then employing $D\lambda = \beta^2 Df$, we obtain

$$QDf = -\frac{1}{2}Dr - 2\beta^2 Df.$$

Since M is of constant scalar curvature, the above equation implies

$$QDf = -2\beta^2 Df.$$

Comparing the above equation with (17) shows that

$$\Big(\frac{r}{2} + 3\beta^2\Big)\{Df - \xi(f)\xi\} = 0.$$

If $r = -6\beta^2$, then it follows from (17) that $QX = -2\beta^2 X$, that is, M is Einstein. Suppose, if $r \neq -6\beta^2$ on some open set $\mathcal{O}$ of M, then we have $Df = \xi(f)\xi$ and this completes the proof. □

Corollary 5.2. *If 3-dimensional non-paracosymplectic quasi-para-Sasakian manifold M of constant scalar curvature with $\beta = \text{const}.$ admits a gradient almost Ricci soliton, then either the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure, or the soliton vector field V is pointwise collinear with the characteristic vector field ξ on an open set $\mathcal{O}$ of M.*

Proof. From Theorem 5.1, we have M is Einstein, that is, $QX = -2\beta^2 X$. This, together with (18), show that M is of constant negative curvature $-\beta^2$. As a result of Theorem 3.4, we conclude that the quasi-para-Sasakian structure can be obtained by a homothetic deformation of a para-Sasakian structure. □

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