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HERMITE–GALERKIN AND FIBONACCI-COLLOCATION METHODS FOR SOLVING TWO-DIMENSIONAL NONLINEAR FREDHOLM INTEGRAL EQUATION

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Abstract. This paper presents two interesting computational methods, the Hermite–Galerkin and the Fibonacci collocation methods. These methods are applied to reduce the two-dimensional nonlinear Fredholm integral equation to a nonlinear system of algebraic equations. The existence and uniqueness of the solution are discussed. We use Maple18 to get the solution of that nonlinear system. Finally, some numerical examples show the implementation and accuracy of the proposed methods.

1. Introduction

Integral equations play an important role in several problems in engineering, mechanics, physical chemistry, electrostatics and physics which can be expressed by using two-dimensional nonlinear Fredholm integral equations. For instance, electrical engineering [7], telegraph equations [16], plasma physics [6] and electromagnetic scattering [11]. Owing to the increasing of integral equations applications, many numerical methods are developed and analyzed for solving different types of integral equations by various methods such as radial basis functions [23], barycentric interpolation collocation methods [14], triangular orthogonal functions [22], iterative methods [12], Gauss product quadrature rule [4], meshless methods [2], the Nyström type methods [9], Galerkin methods [5, 10], Chebyshev polynomials [28], wavelet method [3], differential transform method [29] and many others.

Two-dimensional nonlinear integral equations are usually difficult to be solved analytically. Consequently, in most cases, it is required to approximate solutions. Although there are a lot of numerical methods, the Galerkin and collocation methods are the two commonly used for solving the integral equations. Therefore, in this paper, we focused on proposing a numerical solution for the two-dimensional nonlinear Fredholm integral equations (2D-NFIE) depending on the Hermite–Galerkin and Fibonacci-collocation methods. M. Rahman, [27] and Doaa Sh. Mohamed et al [24] worked on the solution of one-dimensional and two-dimensional linear integral equations, respectively, using the Hermite–Galerkin method. Mostefa N. [25] and Mohammad A. et al [1] studied Fibonacci polynomials to solve integral equations in one-dimensional case. Farshid Mirzaee and Seyede Hoseini in [18] have used the Fibonacci-collocation method for solving a class of Fredholm-Volterra integral equations in two-dimensional spaces. Also, there are many different applications of Fibonacci-collocation method (see [17, 19–21]). One can observe that the Hermite–Galerkin and Fibonacci collocation methods are uncommon in solving the two-dimensional nonlinear Fredholm integral equations. The advantage of the proposed methods is that the problem under consideration is transformed into a system of algebraic equations which can be solved via a suitable numerical method.

Consider the following two-dimensional nonlinear Fredholm integral equation:

$$u(x,y) = f(x,y) + \int_0^1 \int_0^1 K(x,y,t,s,u(t,s))\, dt\, ds, \tag{1}$$

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where $u(x,y)$ is an unknown function defined on $[0,1]\times[0,1]$ and $f(x,y)$, $K(x,y,t,s,u(t,s))$ are analytical functions.

In this paper, we assume

$$K(x,y,t,s,u(t,s)) = K(x,y,t,s)(u(t,s))^p, \tag{2}$$

where p is a positive integer.

The paper is organized as follows: In Section 2, we study the existence and uniqueness of the solution of equation (1). In Section 3, we describe the Hermite and Fibonacci polynomials. In Section 4, we apply Hermite–Galerkin and Fibonacci collocation methods to solve two-dimensional nonlinear Fredholm integral equation (1) with condition (2). Some numerical examples applied to obtain the efficiency of the methods are presented in Section 5. Finally, conclusions of the work are given in Section 6.

2. On the Existence of the Solution of the Two-dimensional Nonlinear Fredholm Integral Equations

In this section, we prove an existence and uniqueness theorem for two-dimensional nonlinear Fredholm integral equations. Consider Equation (1) on the complete metric space of complex valued continuous functions as follows:

$$X = (C(d,S)), d(g,w) = \sup\{|g(x,y)-w(x,y)|,\ (x,y)\in S\},\ S=[0,1]\times[0,1].$$

Theorem 1. *Let f and K be continuous functions on S and $S\times S\times\mathscr{C}$, respectively, and let there exist a nonnegative constant $L\leq 1$ such that*

$$|\ K(x,y,t,s,u(t,s)) - K(x,y,t,s,v(t,s))\ | \leq\ L\ \ |\ u(t,s)-v(t,s)\ |\ .$$

Then Equation (1) *has only one solution u on S.*

Proof. Consider the iterative scheme

$$u_{n+1}(x,y) = f(x,y) + \int_0^1\int_0^1 K(x,y,t,s,u_n(t,s))\ dt\ ds,\ n=1,2,\ldots. \tag{3}$$

We have

$$\begin{aligned}|\ u_{n+1}(x,y)-u_n(x,y)\ | &= \left|\int_0^1\int_0^1 [K(x,y,t,s,u_n(t,s)) - K(x,y,t,s,u_{n-1}(t,s))]\ dt\ ds\right| \\ &\leq L\int_0^1\int_0^1 |u_n(t,s)-u_{n-1}(t,s)|\ dt\ ds \leq Ld(u_n,u_{n-1}) \\ &\Longrightarrow d(u_{n+1},u_n)\leq Ld(u_n,u_{n-1}).\end{aligned}$$

Hence

$$\begin{gathered} d(u_{n+1},u_n) \leq L^{n-1} d(u_2,u_1) \\ \Longrightarrow |\ u_{n+1}(x,y)-u_n(x,y)\ | \leq L^{n-1}d(u_2,u_1).\end{gathered}$$

Since X is a complete metric space, and $0\leq L\leq 1$, we conclude by using the Weierstrass M-test that

$$\sum_{n=1}^{\infty}(u_{n+1}(x,y)-u_n(x,y))$$

is absolutely and uniformly convergent on S. Due to the fact that $u_n(x,y)$ can be written as

$$u_n(x,y) = u_1(x,y) + \sum_{k=1}^{n-1}(u_{k+1}(x,y)-u_k(x,y)),$$

so there exists a unique solution $u \in X$ such that $\lim_{n\longrightarrow\infty} u_n = u$. Taking the limit of both sides of Equation (3), we obtain

$$\begin{aligned} u(x,y) = \lim_{n\longrightarrow\infty} u_{n+1}(x,y) &= \lim_{n\longrightarrow\infty} \left(f(x,y) + \int_0^1\int_0^1 K(x,y,t,s,u_n(t,s))\ dt\ ds \right) \\ &= f(x,y) + \int_0^1\int_0^1 K(x,y,t,s,\lim_{n\longrightarrow\infty} u_n(t,s))\ dt\ ds \\ &= f(x,y) + \int_0^1\int_0^1 K(x,y,t,s,u(t,s))\ dt\ ds. \end{aligned}$$

Therefore the limit function u is the unique solution $u \in X$ such that

$$u(x,y) = f(x,y) + \int_0^1\int_0^1 K(x,y,t,s,u(t,s))\ dt\ ds. \qquad \square$$

3. Preliminaries

In this section, we review some basics that will be applied in this paper.

3.1. Hermite Polynomials.

The differential equation $y'' - 2xy' + 2\lambda y = 0$ has polynomial solutions called Hermite polynomials [26] which were introduced for the first time by Pierre-Simon Laplace in 1810. Pafnuty Chebyshev studied them in detail in 1859, but his work was ignored in 1864. Charles Hermite defined the multidimensional polynomials in 1865. Hermite polynomials are a mutually orthogonal function with weight functions, which can be determined easily by using Rodrigues formula in the form

$$H_n(x) = (-1)^n\ e^{x^2}\ \frac{d^n}{dx^n}(e^{-x^2})\ ,\ n = 0,1,2,\ldots . \tag{4}$$

The first few Hermite polynomials are

$$H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2,\ H_3(x) = 8x^3 - 12x,\ H_4(x) = 16x^4 - 48x^2 + 12,$$
$$H_5(x) = 32x^5 - 160x^3 + 120x,\ H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120.$$

Hermite polynomials have a generating function

$$w(x,t) = e^{2xt - x^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n,\ |t| < \infty.$$

3.2. Fibonacci Polynomials.

Leonardo of Pisa or, Fibonacci, was an Italian mathematician of the 13th century [8, 13]. In 1202, Fibonacci posed and solved a problem relating to the growth of a population of rabbits, generation by generation, based on idealized expectations. The solution was a sequence of numbers known as Fibonacci numbers, denoted by F_n. Each number in this sequence is the sum of the previous two numbers, starting with 0 and 1, in the form $0, 1, 1, 2, 3, 5, 8, \ldots$.

For any positive real number k, the k-Fibonacci sequence, say $\{F_{k,n}\}_{n\in N}$, is defined recurrently by

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1}\ ,\ n \geq 1,$$

with the initial conditions

$$F_{k,0} = 0,\ F_{k,1} = 1.$$

If k is a real variable x, then $F_{k,n} = F_{x,n}$ and they correspond to the Fibonacci polynomials defined by

$$F_{n+1}(x) = \begin{cases} 1, & n = 0 \\ x, & n = 1 \\ xF_n(x) + F_{n-1}(x), & n > 1. \end{cases} \tag{5}$$

The first few Fibonacci polynomials are

$$F_1(x) = 1,\ F_2(x) = x,\ F_3(x) = x^2 + 1,\ F_4(x) = x^3 + 2x,\ F_5(x) = x^4 + 3x^2 + 1.$$

From these expressions, the k-Fibonacci numbers can be written as

$$F_{n+1}(x) = \sum_{n=0}^{\lceil \frac{n}{2} \rceil} \binom{n-i}{i} x^{n-2i}, \quad n \geq 0, \tag{6}$$

where $\lceil \frac{n}{2} \rceil$ denotes the greatest integer $\frac{n}{2}$.

The Fibonacci polynomials have generating function

$$G(x,t) = \frac{t}{1 - t^2 - xt} = \sum_{n=1}^{\infty} F_n(x) t^n = t + xt^2 + (x^2+1)t^3 + (x^3+2x)t^4 + \cdots .$$

4. The Solution of Two-dimensional Nonlinear Integral Equations

In this section we solve two-dimensional nonlinear Fredholm integral equations of the second kind of the form (1) with Equation (2) by using the Hermite–Galerkin and Fibonacci collocation methods.

4.1. **Hermite–Galerkin Method.**

Assume that $\tilde{u}(x,y)$ is an approximate solution of equation (1) with condition (2) and apply Hermite polynomials by Galerkin's method which has the form

$$\tilde{u}(x,t) \cong \sum_{i=0}^{N} \sum_{j=0}^{N} c_{i,j}\ H_i(x)\ H_j(y), \tag{7}$$

where $H_i(x)$, $H_j(y)$ are Hermite polynomials and $c_{i,j}$ are unknown Hermite coefficients to be determined.

Substituting the right-hand side of equation (7) into equation (1) with condition (2), we get

$$\sum_{i=0}^{N} \sum_{j=0}^{N} c_{i,j}\ H_i(x)\ H_j(y) = f(x,y) + g(x,y)\ , \tag{8}$$

where

$$g(x,y) = \int_0^1 \int_0^1 K(x,y,t,s) \left[\sum_{i=0}^{N} \sum_{j=0}^{N} c_{i,j}\ H_i(t)\ H_j(s) \right]^p dtds.$$

Multiplying both sides of equation (8) by $H_q(x)\ H_r(y)$ and integrating with respect to x and y from 0 to 1 such that q and $r = 0, 1, 2, \ldots, N$, equation (8) takes the form

$$\sum_{i=0}^{N} \sum_{j=0}^{N} c_{i,j} \int_0^1 \int_0^1 H_i(x)\ H_j(y)\ H_q(x)\ H_r(y)\ dxdy = \int_0^1 \int_0^1 f(x,y)\ H_q(x)\ H_r(y)\ dxdy$$
$$+ \int_0^1 \int_0^1 g(x,y)\ H_q(x)\ H_r(y)\ dxdy. \tag{9}$$

Substituting q, $r = 0, 1, 2, \ldots, N$ into (9), we get a system of $(N+1)^2$ nonlinear algebraic equations. Solving this system, we obtain Hermite coefficients $c_{i,j}$. Consequently, the approximate solution is obtained.

4.2. **The Fibonacci-collocation Method.** This method is based on approximating the unknown function in equation (1) with condition (2) in the form

$$\overline{u}(x,y) = \sum_{v=1}^{\infty}\sum_{w=1}^{\infty} a_{v,w}\ F_v(x)\ F_w(y), \tag{10}$$

where $F_v(x)$, $F_w(y)$ are Fibonacci polynomials and $a_{v,w}$ are unknown coefficients to be determined.

Curtailing the infinite series (10), we get

$$\overline{u}(x,y) \cong \sum_{v=1}^{N+1}\sum_{w=1}^{N+1} a_{v,w}\ F_v(x)\ F_w(y). \tag{11}$$

Substituting the right-hand side of (11) into equation (1) with condition (2), we get

$$\sum_{v=1}^{N+1}\sum_{w=1}^{N+1} a_{v,w}\ F_v(x)\ F_w(y) = f(x,y) + \int_0^1\int_0^1 K(x,y,t,s)\left[\sum_{v=1}^{N+1}\sum_{w=1}^{N+1} a_{v,w}\ F_v(t)\ F_w(s)\right]^p dtds. \tag{12}$$

Using the collocation points x_L, y_R of Fibonacci polynomials [1]

$$x_L = a + \frac{b-a}{N+1}L, \quad y_R = c + \frac{d-c}{N+1}R, \tag{13}$$

for $L,\ R = 1, 2, \ldots, N+1$, $x_L \in [a,b]$ and $y_R \in [c,d]$, equation (12) can be written as

$$\sum_{v=1}^{N+1}\sum_{w=1}^{N+1} a_{v,w}\ F_v(x_L)\ F_w(y_R) = f(x_L,y_R) + \int_0^1\int_0^1 K(x_L,y_R,t,s)\left[\sum_{v=1}^{N+1}\sum_{w=1}^{N+1} a_{v,w}\ F_v(t)\ F_w(s)\right]^p dtds. \tag{14}$$

Thus we have obtained a nonlinear system of equations which contains $(N+1)^2$ unknown coefficients. Solving this system to obtain the values of these coefficients, we arrive at the approximate solution $\overline{u}(x,y)$.

5. Numerical Examples

Some numerical examples of (2D-NFIE) are presented to illustrate the previous methods. The results are obtained by Maple18.

Table 1. Absolute Error of Example 1 by HG, FC and DE methods for $N = 2, 4$.

(x,y)	$n=2$			$n=4$		
$(2^{-l},2^{-l})$	DE method	HG method	FC method	DE method	HG method	FC method
$l=1$	6.4729×10^{-8}	2.87×10^{-9}	1.0×10^{-9}	2.1740×10^{-7}	6.3734×10^{-7}	1.3279×10^{-9}
$l=2$	2.2477×10^{-8}	0	2.0×10^{-9}	4.7907×10^{-8}	2.4844×10^{-7}	4.9599×10^{-10}
$l=3$	1.0452×10^{-8}	4.6×10^{-9}	5.0×10^{-9}	1.1171×10^{-8}	5.9911×10^{-7}	1.0595×10^{-9}
$l=4$	3.2170×10^{-9}	1.10×10^{-8}	1.0×10^{-9}	2.6921×10^{-9}	4.1209×10^{-8}	1.7469×10^{-10}
$l=5$	8.7976×10^{-10}	1.60×10^{-8}	1.2×10^{-10}	6.6045×10^{-10}	2.8366×10^{-7}	1.5643×10^{-10}
$l=6$	2.2938×10^{-10}	1.89×10^{-8}	1.4×10^{-10}	1.6354×10^{-10}	1.0137×10^{-6}	3.9380×10^{-10}

TABLE 2. Absolute Error of Example 2 by the HG and FC methods for $N = 2, 4$.

(x,y)	$n=2$		$n=4$	
	HG method	FC method	HG method	FC method
$(0,0)$	1.67×10^{-7}	6.96×10^{-10}	7.41×10^{-4}	2.47×10^{-10}
$(0.1,0.1)$	4.52×10^{-8}	3.88×10^{-10}	6.51×10^{-6}	1.52×10^{-10}
$(0.2,0.2)$	3.30×10^{-9}	1.86×10^{-10}	5.38×10^{-5}	3.08×10^{-13}
$(0.3,0.3)$	3.19×10^{-9}	6.24×10^{-11}	1.15×10^{-5}	1.64×10^{-11}
$(0.4,0.4)$	1.68×10^{-8}	7.58×10^{-12}	1.21×10^{-5}	2.91×10^{-11}
$(0.5,0.5)$	2.63×10^{-8}	4.22×10^{-11}	3.15×10^{-5}	9.55×10^{-12}
$(0.6,0.6)$	2.36×10^{-8}	5.56×10^{-11}	5.36×10^{-6}	8.38×10^{-12}
$(0.7,0.7)$	1.09×10^{-8}	5.71×10^{-11}	1.21×10^{-5}	2.18×10^{-11}
$(0.8,0.8)$	6.23×10^{-11}	5.18×10^{-11}	7.34×10^{-5}	1.62×10^{-11}
$(0.9,0.9)$	1.32×10^{-8}	3.96×10^{-11}	4.94×10^{-6}	3.69×10^{-11}
$(1,1)$	8.26×10^{-8}	1.61×10^{-11}	1.82×10^{-4}	2.16×10^{-11}

Example 1. Consider the following 2D-NFIE [15]:

$$u(x,y) = x^2y^2 - 0.01\Big(\frac{x^2y}{30} + \frac{x^2}{42}\Big) + 0.01\int_0^1\int_0^1 x^2t(y+s^2)\ (u(t,s))^2\ dtds, \tag{15}$$

with the exact solution $u(x,y) = x^2y^2$.

Table 1 gives the absolute error of equation (15) by the Hermite–Galerkin (HG) and Fibonacci-collocation (FC) methods for different values of x and y according to Section 4.

Figure 1 expresses the exact solution of equation (15). Figures 2, 3 and 4 clarify the approximate solution and absolute error by the HG method. Also, Figures 5, 6 and 7 show the approximate solution and absolute error by the FC method. Moreover, these methods are compared to double-exponential (DE) Sinc Nyström methods [15] that given for $N = 2, 4$.

Example 2. Consider the following 2D-NFIE:

$$u(x,y) = xy - 0.01\Big(\frac{y^3}{3}\Big)\ (x^2\sin x + 2x\cos x - 2\sin x) + 0.01\int_0^1\int_0^1 x^3y^2\cos(xt)\ (u(t,s))^2\ dtds, \tag{16}$$

with the exact solution $u(x,y) = xy$.

The absolute error of equation (16) for different values of x and y by the HG and FC methods are obtained in Table 2. Figure 8 expresses the exact solution of Equation (16). Figures 9, 10 and 11 clarify the approximate solution and absolute error by HG method. Also, Figures 12, 13 and 14 show the approximate solution and absolute error by FC method.

6. Conclusion

In this paper, two numerical methods are presented to solve the two-dimensional nonlinear Fredholm integral equations by converting equation (1) with condition (2) to a system of nonlinear algebraic equations. After comparing the results of the given examples, we can establish the following deductions.

1. The Fibonacci-collocation method is more effective than the Hermite–Galerkin method.
2. The error of the Fibonacci-collocation method decreases when the value of N increases.
3. The error of the Hermite–Galerkin method increases when the value of N increases.
4. We have compared our methods with the DE Sinc Nyström method [15] and We deduce that the Fibonacci-collocation method is more accurate.

We can develop these methods for solving three-dimensional linear and nonlinear Fredholm and Volterra integral equations by some modifications.

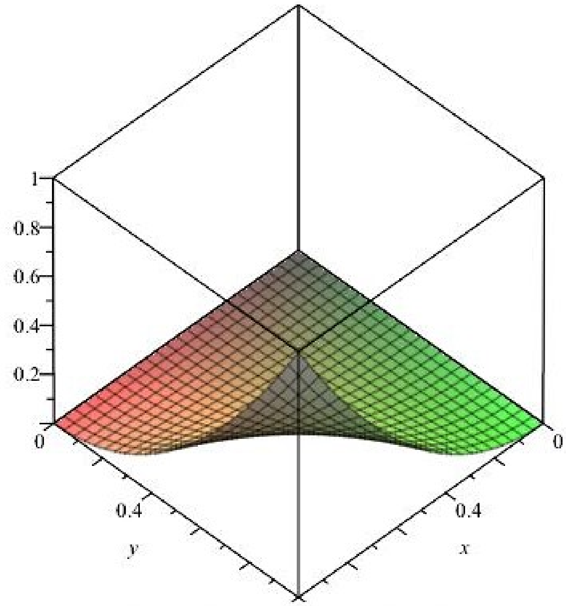

FIGURE 1. The exact solution Example 1.

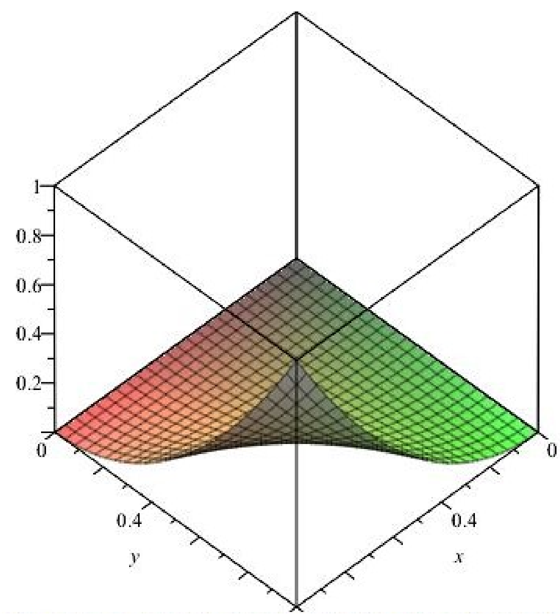

FIGURE 2. Approximate solution of Example 1 by HG metod, $N = 4$.

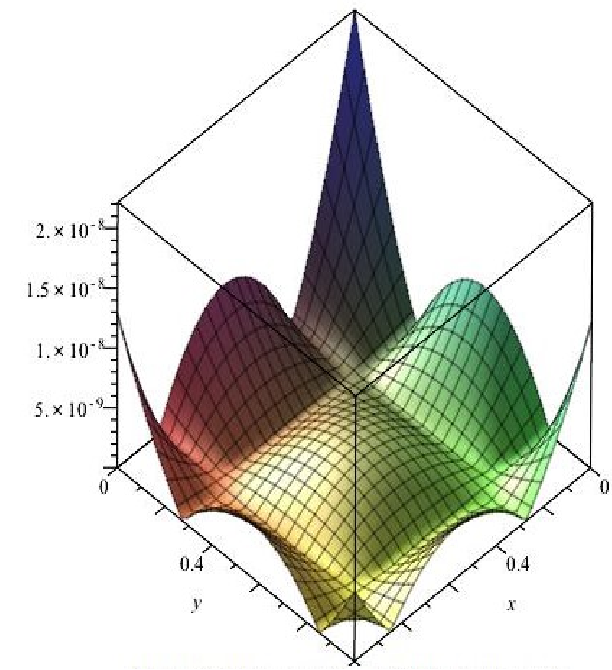

FIGURE 3. Absolute error of Example 1 by HG method, $N = 2$.

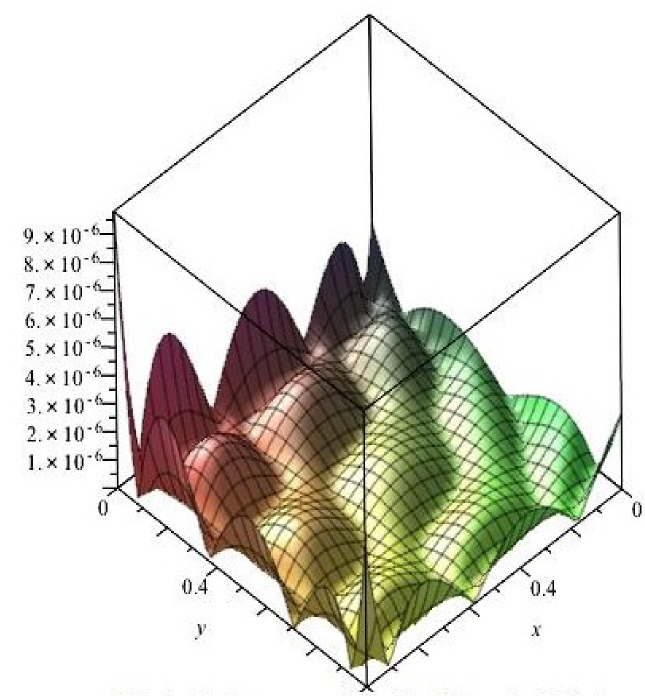

FIGURE 4. Absolute error of Example 1 by HG method, $N = 4$.

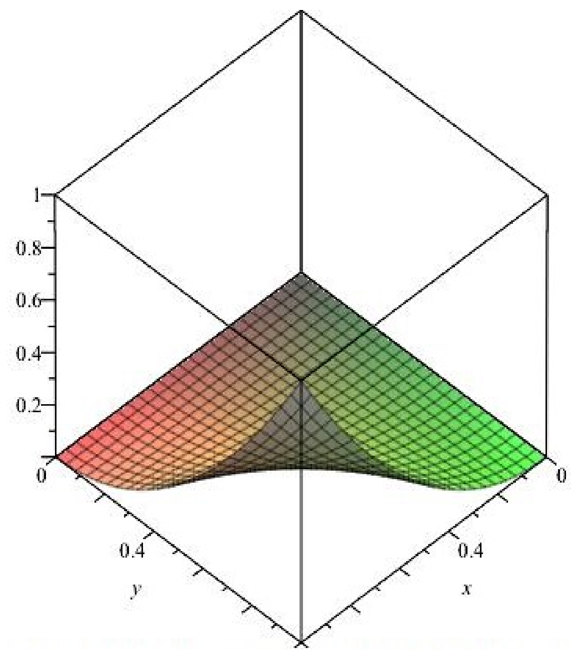

FIGURE 5. Approximate solution of Example 1 by FC method, $N = 4$.

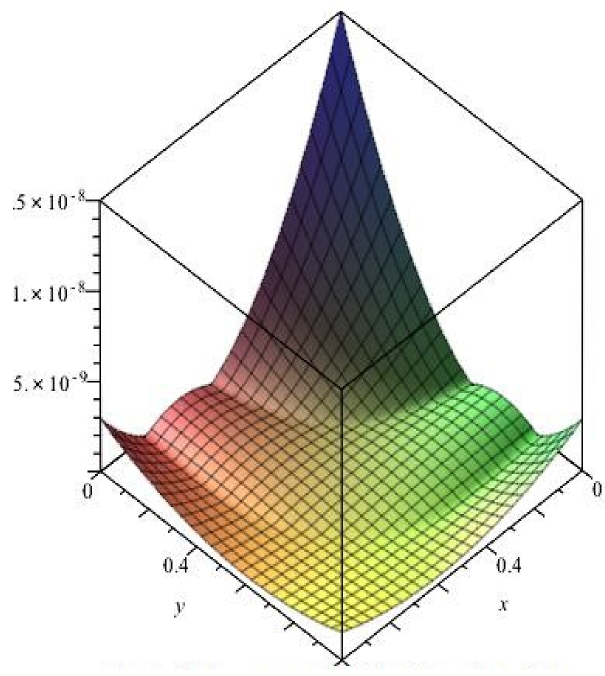

FIGURE 6. Absolute error of Example 1 by FC method, $N = 2$.

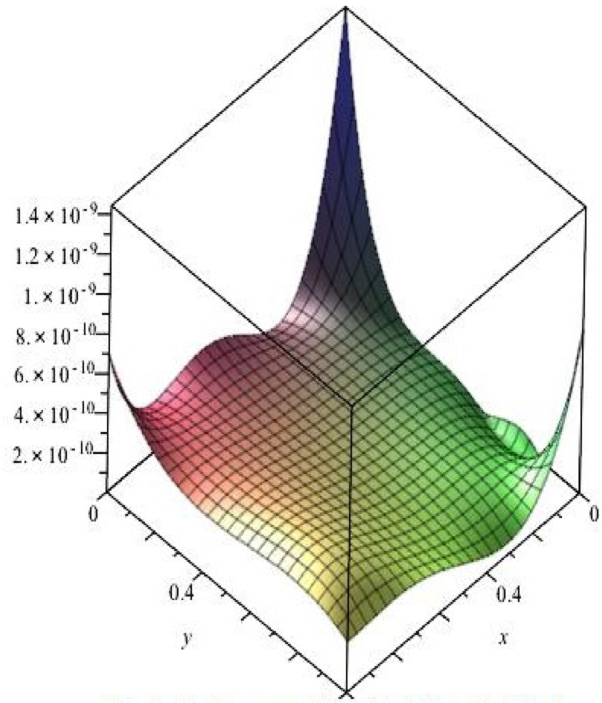

FIGURE 7. Absolute error of Example 1 by FC method, $N = 4$.

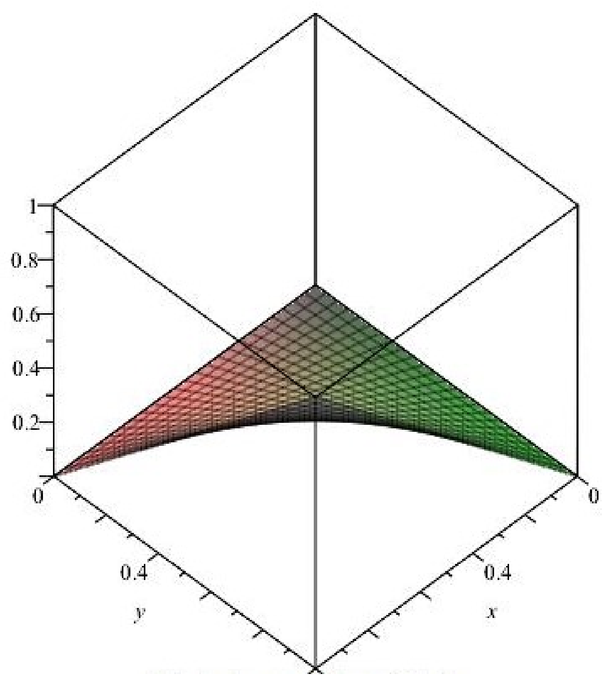

FIGURE 8. Exact solution Example 2.

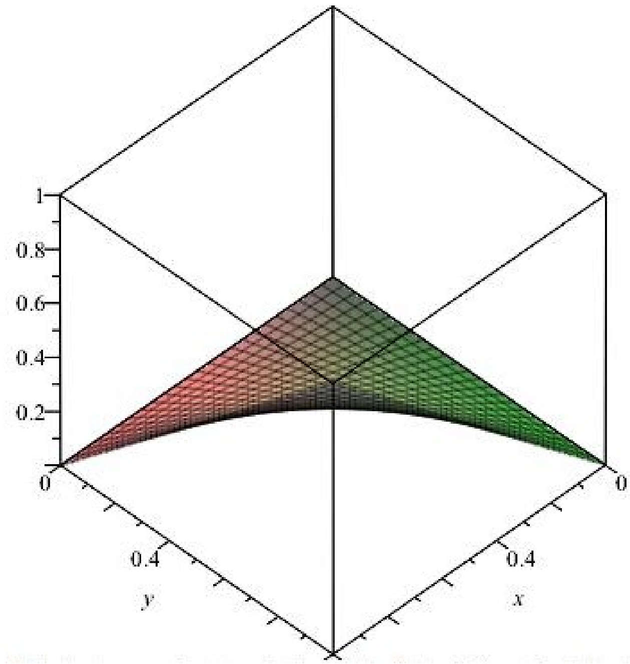

FIGURE 9. Approximate solution of Example 2 by HG method, $N = 4$.

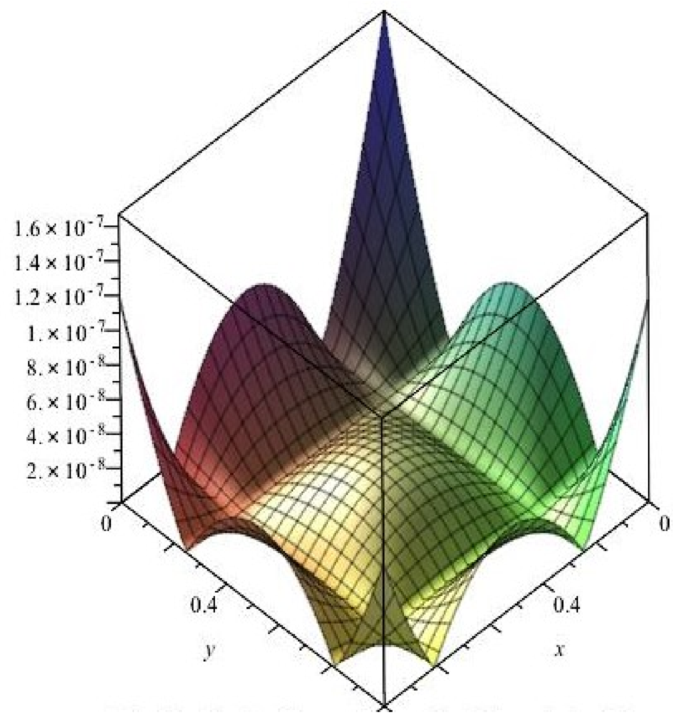

FIGURE 10. Absolute Error of Example 2 by HG method, $N = 2$.

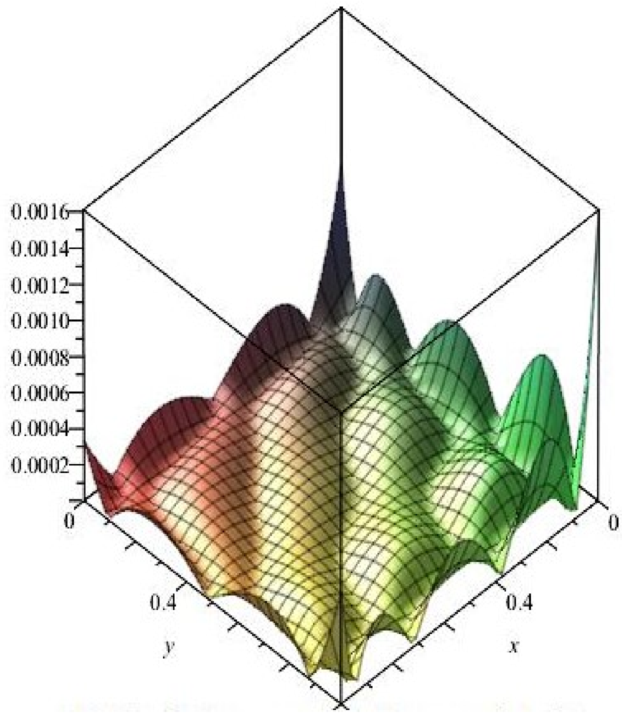

FIGURE 11. Absolute Error of Example 2 by HG method, $N = 4$.

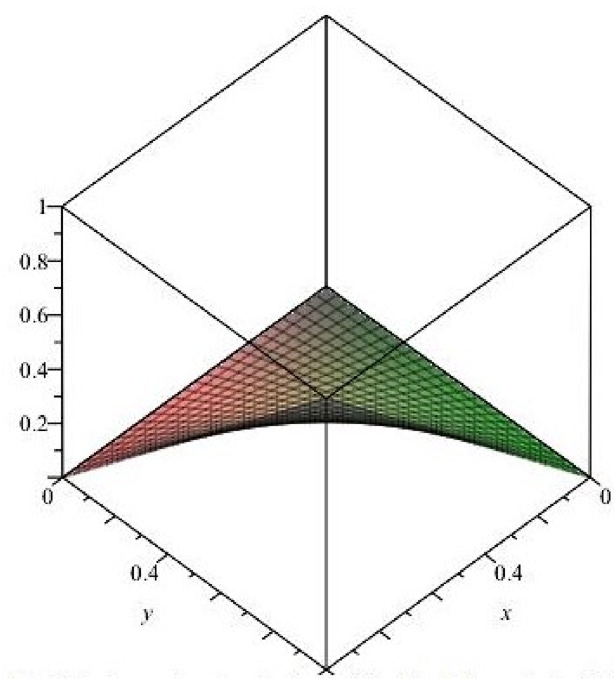

FIGURE 12. Approximate solution of Example 2 by FC method, $N = 4$.

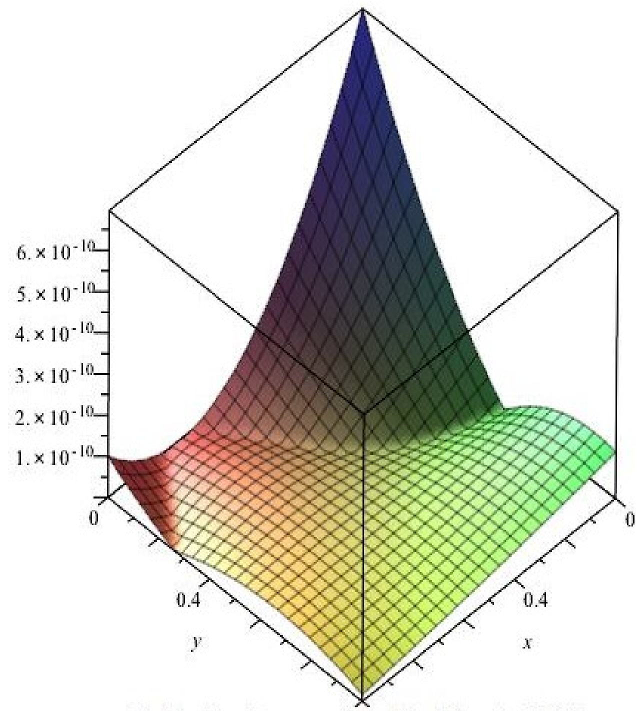

FIGURE 13. Absolute Error of Example 2 by FC method, $N = 2$.

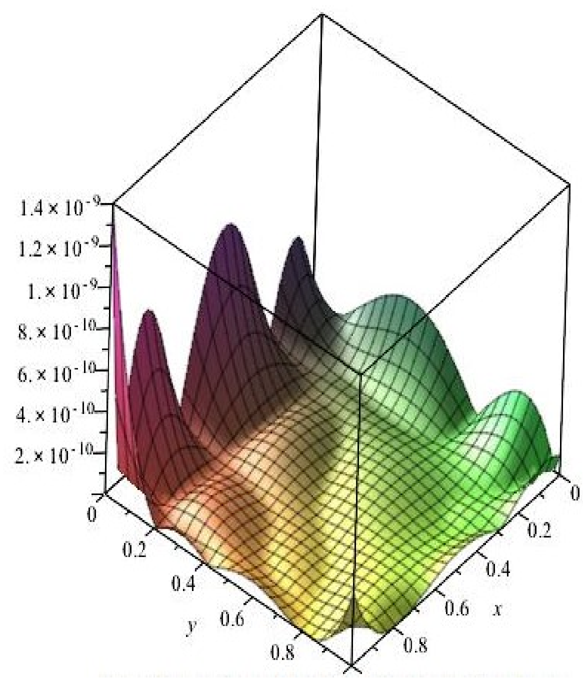

FIGURE 14. Absolute Error of Example 2 by FC method, $N = 4$.

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