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# A NOTE ON CANCELLATION LAW FOR SUBSETS OF TOPOLGICAL GROUPS

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**Abstract.** In this note, we generalize a result of R. Urbański from paper [3] which states that for subsets $A, B, C$ of topological vector space $X$ the implication

$$A + B \subset B + C \Longrightarrow A \subset C$$

holds, provided that $B$ is bounded and $C$ is closed and convex. In Theorem 2.1, we generalize this result to dyadic convex subsets of a topological group.

## 1. Introduction

Let $(X+, \tau)$ be a topological group, i.e., an algebraical commutative group $(X, +)$ equipped with topology $\tau$ such that the addition $+$ is a continuous mapping. Assume that:

(a) the equation $x + x = 0$ has only one solution $x = 0$,

(b) for every $x \in X$ there exists $v \in X$ such that $x = v + v$.

Obviously, for every $x \in X$, the element $v$ which existence is guaranteed by $(b)$ is unique by $(a)$ and we denote it by $\frac{1}{2}x$.

Analogously, we can define $\frac{1}{2^n}x$ as $\frac{1}{2}\left(\frac{1}{2^{n-1}}x\right)$ for $n \in \mathbb{N}$.

Since for every integer $m \in \mathbb{Z}$ the operation $mx$ is defined as

$$mx = \underbrace{x + x + \cdots + x}_{m-times}$$

if $m > 0$ and

$$mx = \underbrace{(-x) + (-x) + \cdots + (-x)}_{-m-times}$$

if $m < 0$, therefore we have just defined an element $rx$ for $x \in X$ and $r \in D_2$, where $D_2 = \{2^{-n}m : n \in \mathbb{N}, m \in \mathbb{Z}\}$.

**Definition 1.1.** Let $A \subset X$. We say that $A$ is a *dyadic convex set* if $pA + qA \subset A$ for every $0 < p$, $q \in D_2$ such that $p + q = 1$.

**Definition 1.2.** Let $A \subset X$ and $k \in \mathbb{N}$. We say that $A$ is a *k-dyadic convex set* if $pA + qA \subset A$ for every $0 < p$, $q \in D_2$ such that $p^{\frac{1}{k}} + q^{\frac{1}{k}} = 1$.

**Definition 1.3.** Let $A \subset X$. We say that $A$ is a *bounded set* if for every open subset $V \subset X$ such that $0 \in V$ there exists $j \in \mathbb{N}$ such that $2^{-j}A \subset V$.

**Example.** The cartesian product $D_2^n = D_2 \times \cdots \times D_2$ is a dyadic convex set and intersection of a dyadic convex set with a convex set is a dyadic convex set.

**Lemma 1.4.** *Let $X$ be a topological group and let $\mathfrak{B}$ be any base of neighbourhoods of $0$ for the topology in $X$. Then*

$$\overline{A} = \bigcap_{V \in \mathfrak{B}} (A + V).$$

*Proof.* The proof is easy and we omit it. □

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## 2. The Main Result

In this section, we prove the following theorem which is the main result of the present paper.

**Theorem 2.1.** *Let $X$ be a topological group such that for every $x \in X$ there exists a unique element $v \in X$ such that $x = v + v$. Assume that $A, B, C \subset X$ are subsets such that $B$ is bounded and $C$ is dyadic convex and closed. Then the following implication*

$$A + B \subset B + C \Longrightarrow A \subset C$$

*holds true.*

*Proof.* Without loss of generality, we may assume that $0 \in B$. Let us first observe that

$$2A + B \subset A + (A + B) \subset A + (B + C) = (A + B) + C \subset (B + C) + C = B + 2C.$$

From the above equality we can easily obtain that

$$2^j A + B \subset B + 2^j C,$$

for every $j \in \mathbb{N}$. Dividing the last inclusion by $2^j$, we obtain

$$A + 2^{-j} B \subset 2^{-j} B + C.$$

Now, let $\mathfrak{B}$ be any basis of neighbourhoods of 0 in X and let $V \in \mathfrak{B}$. Since $B$ is bounded, there exists $j \in \mathbb{N}$ such that $2^{-j} B \subset V$. Therefore we get

$$A \subset A + 2^{-j} B \subset 2^{-j} B + C \subset V + C.$$

Thus

$$A \subset \bigcap_{V \in \mathfrak{B}} (V + C) = \overline{C} = C. \qquad \square$$

**Theorem 2.2.** *Let $X$ be a topological group such that for every $x \in X$ there exists a unique element $v \in X$ such that $x = v + v$. Assume that $A, B, C \subset X$ are subsets such that $B$ is bounded and $C$ is $k$-dyadic convex, closed and satisfying the following condition:*

$$2^{(k-1)n} C \subset C,$$

*for every $n \in \mathbb{N}$. Then the following implication*

$$A + B \subset B + C \Longrightarrow A \subset C$$

*holds true.*

*Proof.* Without loss of generality, we may assume that $0 \in B$. Similarly, as in the proof of Theorem 2.1 we can prove the following inclusion:

$$2^j A + B \subset B + 2^{kj} C,$$

for every $j \in \mathbb{N}$.

Hence

$$A + 2^{-j} B \subset 2^{-j} B + 2^{(k-1)j} C.$$

Now let $\mathfrak{B}$ be any base of neighbourhoods of 0 in X and let $V \in \mathfrak{B}$. Since $B$ is bounded, there exists $j \in \mathbb{N}$ such that $2^{-j} B \subset V$. Therefore we get

$$A \subset A + 2^{-j} B \subset 2^{-j} B + 2^{(k-1)j} C \subset V + 2^{(k-1)j} C \subset V + C.$$

Thus

$$A \subset \bigcap_{V \in \mathfrak{B}} (V + C) = \overline{C} = C. \qquad \square$$

**Remark 2.3.** Every dyadic convex set is 1-dyadic convex set, thus Theorem 2.2 generalizes Theorem 2.1.

**Remark 2.4.** Every convex subset of topological vector space is, obviously, a dyadic convex subset of the additive group of this space. Thus, Theorem 2.2 generalizes the result obtained by Urbański in [3].

**Remark 2.5.** From Theorem 2.1, it follows that for every topological group $X$ the family $\mathfrak{U}(X)$ of dyadic convex closed and bounded subsets of X forms a semigroup with the law of cancellation and therefore can be embedded algebraically into a group.

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