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VARIOUS CONVERGENCES OF MULTIFUNCTIONS

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Abstract. In the present paper, we introduce different types of convergences of nets of multifunctions from one topological space to another and compare them. Attempt has been made to formulate sufficient conditions under which these convergences preserve slight B^*-continuity of the limit multifunction.

1. Introduction

Continuity and its various weaker and stronger forms (see [5–10,13–16]) are the fundamental notions in the general topology and analysis. Consequently, different types of convergences have been studied which preserve these generalized continuities. In [12], Kupka and Toma defined the concept of strong convergence and Domnik [2] proved that strong convergence preserves upper and lower semi continuity. Ganguly and Mallick [3,4] gave the notion of e-convergence and some sufficient conditions under which this type of convergence preserves further generalized continuity, namely, e-continuity.

The present paper aims to introduce the concepts of τ_{cl}^{+} and τ_{cl}^{-}-pointwise convergence, upper and lower *cl*-convergence and nearly-strong convergence. It is proved that, in general, *cl*-convergence is stronger than pointwise convergence. Further, we formulate sufficient conditions under which the *cl*-convergence preserves upper and lower slight B^*-continuity. We establish some relationship between *cl*-convergence and nearly-strong convergence. We have proved that nearly-strong convergence preserves slight B^*-continuity.

Throughout the paper, X and Y will denote topological spaces, unless specified otherwise. By $F : X \to Y$, we shall mean that F is a multifunction with domain X and co-domain $\mathcal{S}(Y)$, the power set of Y excluding the empty set. If $F : X \to Y$ is a multifunction then for $A \subset Y$ we denote

$$F^+(A) = \{x \in X : F(x) \subset A\}$$

and

$$F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}.$$

For any open set $U \subset Y$, we denote

$$U^+ = \{B \in \mathcal{S}(Y) : B \subset U\} \text{ and } U^- = \{B \in \mathcal{S}(Y) : B \cap U \neq \emptyset\}.$$

By $\mathcal{B}^+$ and $\mathcal{P}^-$, we denote the collections of all U^+ and U^- respectively. The collection $\mathcal{B}^+$ forms a base for some topology in $\mathcal{S}(Y)$, called the upper Vietoris topology, usually denoted by τ^+. Similarly, $\mathcal{P}^-$ forms a subbase for some topology in $\mathcal{S}(Y)$, called lower Vietoris topology, τ^-. For these topologies, one may refer to [2, 15] and references therein.

A set B is said to be a B^*-set if it is not nowhere dense having the property of Baire, [5].

A multifunction $F : X \to Y$ is said to be

(a) upper slightly B^*-continuous at a point x, if for every open set $U \subset X$ containing x and for every clopen set V such that $F(x) \subset V$, there exists a B^*-set B containing x such that
$$B \subset F^+(V) \cap U.$$

(b) lower slightly B^*-continuous at a point x, if for every open set $U \subset X$ containing x and for every clopen set V such that $F(x) \cap V \neq \emptyset$, there exists a B^*-set B containing x such that
$$B \subset F^-(V) \cap U.$$

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(c) slightly B^*-continuous if it is both upper and lower slightly B^*-continuous, see [10].

A net $\{a_j : j \in \mathcal{J}\}$ of elements of Y is said to be convergent to $a \in Y$, if for each neighborhood V of a, there exists $j_0 \in \mathcal{J}$ such that $a_j \in V$ for every $j \in \mathcal{J}, j \geq j_0$, see, eg., [11].

For a space Y with topology τ, if a net $\{a_j : j \in \mathcal{J}\}$ converges to $a \in Y$, then we shall be writing $a \in \tau\text{-}\lim a_j$.

2. Pointwise, Nearly-strong and *cl*-convergence

We begin this section by defining topologies weaker than the Vietoris topologies τ^+ and τ^-. The idea is to replace open sets by clopen sets in the construction. Consider a nonempty clopen subset U of Y, and let

$$U_{cl}^+ := \{B \in \mathcal{S}(Y) : B \subset U\},$$

and

$$U_{cl}^- := \{B \in \mathcal{S}(Y) : B \cap U \neq \emptyset\}.$$

Let us denote by $\mathcal{B}_{cl}^+$ and $\mathcal{P}_{cl}^-$, the collections of all U_{cl}^+ and U_{cl}^-, respectively. It is easy to see that $\mathcal{B}_{cl}^+$ forms a base for some topology in $\mathcal{S}(Y)$, to be called upper *cl*-Vietoris topology and will be denoted by τ_{cl}^+. Similarly, $\mathcal{P}_{cl}^-$ forms a subbase for some topology in $\mathcal{S}(Y)$, to be called lower *cl*-Vietoris topology and we shall denote it by τ_{cl}^-.

Below we define some new convergences:

Definition 2.1. *A net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is said to be τ_{cl}^+-pointwise $\big($resp.τ_{cl}^--pointwise$\big)$ convergent to a multifunction $F : X \to Y$ if for every $x \in X$, $F(x) \in \tau_{cl}^+\text{-}\lim F_j(x)\big($resp. $F(x) \in \tau_{cl}^-\text{-}\lim F_j(x)\big)$ and we write $F \in \tau_{cl}^+\text{-}\lim F_j\big($resp. $F \in \tau_{cl}^-\text{-}\lim F_j\big)$.*

Definition 2.2. *A net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is said to be upper (lower) cl-convergent to a multifunction $F : X \to Y$ if for every clopen set V in Y with $F^+(V) \neq \emptyset$ $\big(F^-(V) \neq \emptyset\big)$ there exists $j_0 \in \mathcal{J}$ such that for every $j \in \mathcal{J}$ with $j > j_0$*

$$F^+(V) \subseteq F_j^+(V) \qquad \Big(F^-(V) \subseteq F_j^-(V)\Big).$$

The net $\{F_j : j \in \mathcal{J}\}$ is said to be cl-convergent to F if it is both upper and lower cl-convergent.

Definition 2.3. *A net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is said to be upper (lower) nearly-strongly convergent to a multifunction $F : X \to Y$ if for each clopen cover $\mathcal{U}$ of Y there exists $j_0 \in \mathcal{J}$ such that for every $j > j_0$ and for every $x \in X$,*

$$F_j(x) \subseteq St_{cl}(F(x), \mathcal{U}) \qquad \Big(F(x) \subseteq St_{cl}(F_j(x), \mathcal{U})\Big),$$

where for any $A \subset Y$, the set $St_{cl}(A, \mathcal{U}) = \bigcup\{B \in \mathcal{U} : B \cap A \neq \emptyset\}$ is called the cl-star of $A \subset Y$ with respect to cover $\mathcal{U}$ of Y.

The net $\{F_j : j \in \mathcal{J}\}$ is said to be nearly-strongly-convergent to F if it is both upper and lower nearly-strongly convergent.

In the next two theorems, we prove that both *cl*-convergence as well as nearly-strong convergence are stronger than the pointwise convergence.

Theorem 2.4. *If a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is upper cl-convergent converging to $F : X \to Y$ then $F \in \tau_{cl}^+\text{-}\lim F_j$.*

Proof. Let $x \in X$ and $V^+ \in \mathcal{B}_{cl}^+$ be such that $F(x) \in V^+$. Then V is clopen in (Y, τ), i.e., $x \in F^+(V)$. Since $\{F_j : j \in \mathcal{J}\}$ is upper *cl*-convergent to F and $F^+(V) \neq \emptyset$, there exists $j_0 \in \mathcal{J}$ such that for each $j \in \mathcal{J}$ with $j > j_0$, $F^+(V) \subset F_j^+(V)$. Hence $x \in F_j^+(V)$ for all $j \in \mathcal{J}$, $j > j_0$. This implies that $F \in \tau_{cl}^+\text{-}\lim F_j$. □

Theorem 2.5. *If a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is upper nearly-strongly convergent converging to $F : X \to Y$ then $F \in \tau_{cl}^+\text{-}\lim F_j$.*

Proof. Let $x \in X$ and $V^+ \in \mathcal{B}_{cl}^+$ such that $F(x) \in V^+$. Then V is clopen in (Y, τ), i.e., $x \in F^+(V)$. Since V is clopen, it follows that $\mathcal{A} = \{V, Y \setminus V\}$ forms a clopen cover of Y. Now, from the upper nearly-strong convergence of the net $\{F_j : j \in \mathcal{J}\}$, there exists $j_0 \in \mathcal{J}$ such that for each $j \in \mathcal{J}$ with $j > j_0$ and $x \in X$, we have,

$$F_j(x) \subset St_{cl}(F(x), \mathcal{A}) = V.$$

Hence $x \in F_j^+(V)$ for all $j \in \mathcal{J}$, $j > j_0$, which implies that $F \in \tau_{cl}^+\text{-}\lim F_j$. □

Results corresponding to Theorems 2.4 and 2.5 for lower convergence can be proved on the similar lines. We only state them below:

Theorem 2.6. *If a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is lower cl-convergent to $F : X \to Y$ then $F \in \tau_{cl}^-\text{-}\lim F_j$.*

Theorem 2.7. *If a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is lower nearly-strongly convergent to $F : X \to Y$ then $F \in \tau_{cl}^-\text{-}\lim F_j$.*

Comparing *cl*-convergence and nearly-strong convergence, the following can be proved:

Theorem 2.8. *Let Y be a mildly compact space. If a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ cl-converges to a multifunction $F : X \to Y$, then $\{F_j : j \in \mathcal{J}\}$ converges nearly-strongly to F.*

Proof. Let $\mathcal{A}$ be a clopen cover of Y. Since Y is mildly compact, $\mathcal{A}$ admits a finite subcover, say, $\mathcal{U}$. Since $\{F_j : j \in \mathcal{J}\}$ is lower *cl*-convergent to F, there exists $j_0 \in \mathcal{J}$ such that for all $j > j_0$ and $U \in \mathcal{U}$,

$$F^-(U) \subseteq F_j^-(U).$$

Let $x \in X$, $y \in F(x)$ and $j \in \mathcal{J}$ with $j > j_0$. Then there exists $U \in \mathcal{U}$ with $y \in U$, so that $F(x) \cap U \neq \emptyset$. This gives that

$$x \in F^-(U) \subset F_j^-(U),$$

i.e., for all $j > j_0$,

$$F_j(x) \cap U \neq \emptyset,$$

which further gives that

$$U \subset St_{cl}(F_j(x), \mathcal{A}).$$

Thus $y \in St_{cl}(F_j(x), \mathcal{A})$ for every $x \in X$ and for every $j \in \mathcal{J}$ with $j > j_0$, i.e., $F(x) \subset St_{cl}(F_j(x), \mathcal{A})$ for all $j > j_0$.

We now prove that $F_j(x) \subset St_{cl}(F(x), \mathcal{A})$ for every $x \in X$ and every $j \in \mathcal{J}$ with $j > j_0$. On the contrary, let $F_{j'}(x_0) \nsubseteq St_{cl}(F(x_0), \mathcal{A})$ for some $x_0 \in X$ and for some $j' > j_0$. Take

$$z \in F_{j'}(x_0) \qquad \text{such that} \qquad z \notin St_{cl}(F(x_0), \mathcal{A}).$$

Then $z \notin St_{cl}(F(x_0), \mathcal{U})$ for some finite subcover $\mathcal{U}$ of $\mathcal{A}$. Let $\mathcal{U}' = St_{cl}(F(x_0), \mathcal{U})$. Then

$$F^+(U) \subseteq F_{j'}^+(U) \qquad \text{for all} \qquad U \in \mathcal{U}'$$

and $x_o \notin F_{j'}^+(U')$ for all $U \in \mathcal{U}'$. So, $F(x_0) \nsubseteq St_{cl}(F(x_0), \mathcal{U})$, which is a contradiction and the assertion follows. □

3. Slight B^*-continuity of the Limit Multifunction

In this section we provide sufficient conditions under which the upper (lower) slight B^*-continuity for the limit multifunction is preserved. First we give the following definition:

Definition 3.1. *A net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is said to be frequently upper (lower) slightly B^*-continuous at $x \in X$ if for each $j \in \mathcal{J}$ there exists $j_0 \in \mathcal{J}$, $j_0 > j$ such that F_{j_0} is upper (lower) slightly B^*-continuous at x. The net $\{F_j : j \in \mathcal{J}\}$ is said to be frequently upper (lower) slightly B^*-continuous on X if it is so at every point of X.*

The net $\{F_j : j \in \mathcal{J}\}$ is said to be frequently slightly B^*-continuous if it is both frequently upper and lower slightly B^*-continuous.

We prove the following:

Theorem 3.2. *Let a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ be τ_{cl}^{+}-pointwise as well as lower cl-convergent to a multifunction $F : X \to Y$ and $\{F_j : j \in \mathcal{J}\}$ be frequently upper slightly B^*-continuous. Then F is also upper slightly B^*-continuous.*

Proof. Let, if possible, F be not upper slightly B^*-continuous at $x_0 \in X$. Then there exists a clopen set V in Y with $F(x_0) \subset V$ and an open set U containing x_0 such that each B^*-set B with $x_0 \in B \subset U$ contains a point x_B such that

$$F(x_B) \cap (Y \setminus V) \neq \emptyset.$$

Put $V_1 = Y \setminus V$. Since $F(x_0) \in \tau_{cl}^{+}\text{-}\lim F_j(x_0)$, there exists $j_1 \in \mathcal{J}$ such that $F_j(x_0) \subset V$ for all $j > j_1$. Now $\{F_j : j \in \mathcal{J}\}$ being lower cl-convergent and $F^{-}(V_1) \neq \emptyset$, there exists $j_2 \in \mathcal{J}$ such that for all $j > j_2$

$$F^{-}(V_1) \subset F_j^{-}(V_1),$$

so that,

$$F_j(x_B) \cap V_1 \neq \emptyset. \tag{3.1}$$

Since $\{F_j : j \in \mathcal{J}\}$ is frequently upper slightly B^*-continuous at x_0, there exists $j \in \mathcal{J}$ with $j > \max\{j_1, j_2\}$ such that F_j is upper slightly B^*-continuous at x_0. Hence $F_j(x_0) \subset V$ and there exists a B^*-set B such that for all $b \in B$,

$$x_0 \in B \subset U \qquad \text{with} \qquad F_j(b) \subset V,$$

which contradicts (3.1) and the assertion follows. □

Similarly, the following can be proved:

Theorem 3.3. *Let a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ be τ_{cl}^{-}-pointwise as well as upper cl-convergent to a multifunction $F : X \to Y$ and $\{F_j : j \in \mathcal{J}\}$ be frequently lower slightly B^*-continuous. Then F is also lower slightly B^*-continuous.*

In view of Theorems 2.4, 2.5, 3.2 and 3.3 we immediately get the following:

Theorem 3.4. *If a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ cl-converges to $F : X \to Y$ and if it is frequently slightly B^*-continuous then F is also slightly B^*-continuous.*

The following theorems show that nearly-strong convergence preserves the upper (lower) slight B^*-continuity of the limit multifunction.

Theorem 3.5. *Let $\{F_j : j \in \mathcal{J}\}$, be a net of multifunctions $F_j : X \to Y$ that converges nearly-strongly to a multifunction $F : X \to Y$. If $\{F_j\}$ is frequently upper slightly B^*-continuous then F is also upper slightly B^*-continuous.*

Proof. Let $x_0 \in X$, V be clopen in Y with $F(x_0) \subset V$ and U be an open set containing x_0. Then $\mathcal{A} := \{V, Y \setminus V\}$ is a clopen cover of Y. Since $\{F_j : j \in \mathcal{J}\}$ is nearly-strongly convergent to F, there exists $j_1 \in \mathcal{J}$ such that for all $j > j_1$ and for all $x \in X$,

$$F(x) \subseteq St_{cl}(F_j(x), \mathcal{A}) \qquad \text{and} \qquad F_j(x) \subseteq St_{cl}(F(x), \mathcal{A}).$$

Again, since $F_j : X \to Y$ is frequently upper slightly B^*-continuous at x_0, there exists $j_2 \in \mathcal{J}$ with $j_2 > j_1$ such that F_{j_2} is upper slightly B^*-continuous at x_0. Thus there exists a B^*-set $B \subset U$ containing x_0 such that for all $b \in B$

$$F_{j_2}(b) \subset V$$

which implies that

$$St_{cl}(F_{j_2}(b), \mathcal{A}) = V.$$

Now

$$F_{j_2}(x_0) \subseteq St_{cl}(F(x_0), \mathcal{A}) = V,$$

which gives that

$$St_{cl}(F_{j_2(x_0)}, \mathcal{A}) = V.$$

Consequently, we have that for all $b \in B$

$$F(b_2) \subseteq St_{cl}(F_{j_2(b)}, \mathcal{A}) = V$$

and we are done. □

The following can also be proved on the similar lines:

Theorem 3.6. *Let $\{F_j : j \in \mathcal{J}\}$ be a net of multifunctions $F_j : X \to Y$ converging nearly-strongly to a multifunction $F : X \to Y$. If $\{F_j : j \in \mathcal{J}\}$ is frequently lower slightly B^*-continuous then F is also lower slightly B^*-continuous.*

4. Set of Points of Slight B^*-continuity

For a multifunction $F : X \to Y$, we write,

$$B^+(F) = \{x \in X : F \text{ is upper slightly } B^*\text{-continuous at } x\}$$

and

$$B^-(F) = \{x \in X : F \text{ is lower slightly } B^*\text{-continuous at } x\}.$$

Theorem 4.1. *If a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is τ_{cl}^+-pointwise as well as lower nearly-strongly convergent to a multifunction $F : X \to Y$, then*

$$\bigcap_{i\in\mathcal{J}} \bigcup_{j\geq i} B^+(F_j) \subseteq B^+(F).$$

Proof. Let $x_0 \in \bigcap_{i\in\mathcal{J}} \bigcup_{j\geq i} B^+(F_j)$, V be clopen in Y with $F(x_0) \subseteq V$ and U be an open neighborhood of x_0. Since $F_j : X \to Y$ is τ_{cl}^+-pointwise convergent, we have

$$F(x_0) \in \tau_{cl}^+\text{-}\lim F_j(x_0)$$

which implies that V^+ is an open neighborhood of $F(x_0)$ in $\big(S(X), \tau_{cl}^+\big)$ and therefore there exists $j_1 \in \mathcal{J}$ such that for all $j \geq j_1$

$$F_j(x_0) \in V^+.$$

Again, since $\{F_j : j \in \mathcal{J}\}$ is lower nearly-strongly convergent converging to F, corresponding to the clopen cover $\mathcal{A} = \{V, Y \setminus V\}$ of Y, there exists $j_2 \in \mathcal{J}$ such that for every $j \in \mathcal{J}$, $j \geq j_2$ and $x \in X$, we have

$$F(x) \subset St_{cl}\Big(F_j(x), \mathcal{A}\Big).$$

Choose $j \geq \max\{j_1, j_2\}$ such that $x_0 \in B^+(F_j)$. Then there exists a B^*-set $B \subset U$ containing x_0 such that for all $x \in B$

$$F_j(x) \subset V$$

so that

$$St(F_j(x), \mathcal{A}) = V.$$

This implies that $F(x) \subseteq V$ for every $x \in B$. Hence $x_0 \in B^+(F)$ and we are done. □

On the similar lines, we prove the following:

Theorem 4.2. *If a net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is τ_{cl}^--pointwise as well as upper nearly-strongly convergent to $F : X \to Y$ then*

$$\bigcap_{i\in\mathcal{J}} \bigcup_{j\geq i} B^-(F_j) \subseteq B^-(F).$$

Proof. Let $x_0 \in \bigcap_{i\in\mathcal{J}} \bigcup_{j\geq i} B^-(F_j)$, V be clopen in Y with $F(x_0) \cap V \neq \emptyset$ and let U be an open neighborhood of x_0. Since $F_j : X \to Y$ is τ_{cl}^--pointwise convergent, we have

$$F(x_0) \in \tau_{cl}^-\text{-}\lim F_j(x_0)$$

which implies that V^- is an open neighborhood of $F(x_0)$ in $\big(S(X), \tau_{cl}^-\big)$ so that there exists $j_1 \in \mathcal{J}$ such that for all $j \geq j_1$,

$$F_j(x_0) \in V^-.$$

Again, since $\{F_j : j \in \mathcal{J}\}$ is upper nearly-strongly convergent converging to F, corresponding to the clopen cover $\mathcal{A} = \{V, Y \setminus V\}$ of Y, there exists $j_2 \in \mathcal{J}$ such that for every $j \in \mathcal{J}$, $j \geq j_2$ and $x \in X$, we have

$$F(x) \subset St_{cl}\Big(F_j(x), \mathcal{A}\Big).$$

Choose $j \geq \max\{j_1, j_2\}$ such that $x_0 \in B^-(F_j.)$ Then there exists a B^*-set $B \subset U$ containing x_0 such that for all $x \in B$

$$F_j(x) \cap V \neq \emptyset.$$

Suppose that $F(x^{'}) \cap V = \emptyset$ for some $x \in B$. Then

$$F(x^{'}) \subset Y \setminus V$$

which implies that

$$St_{cl}(F(x^{'}), \mathcal{A}) = Y \setminus V.$$

But $F_j(x^{'}) \cap V \neq \emptyset$ which is a contradiction and we are done. □

The notion of upper (lower) slight continuity [16] is stronger than that of upper (lower) slight B^*-continuity. In what follows, we provide the conditions under which the limit multifunction becomes upper (lower) slightly continuous. First we recall the following notion for functions [1]:

A net $\{f_j : j \in \mathcal{J}\}$ of functions $f_j : X \to Y$ is called continuously convergent at $x_0 \in X$ to a function $f : X \to Y$ if for each net $\{x_\alpha : \alpha \in \mathcal{A}\}$ in X

$$x_0 \in \lim_{\mathcal{A}} x_\alpha \implies f(x_0) \in \lim_{\mathcal{J}\times\mathcal{A}} f_j(x_\alpha).$$

If the convergence is at each point $x_0 \in X$, then the net is said to be continuously convergent on X.

The above notion of convergence of functions has been generalized to multifunctions in [2] which is as follows:

Let τ denotes a topology on a family $S(Y)$ of all subsets of Y. A net $\{F_j : j \in \mathcal{J}\}$ of multifunctions $F_j : X \to Y$ is said to be τ-continuously convergent to $F : X \to Y$ if the net of functions $F_j : X \to (S(Y), \tau)$ is continuously convergent to a function $F : X \to (S(Y), \tau)$.

Now, we prove the following:

Theorem 4.3. *Let $\{F_j : j \in \mathcal{J}\}$ be a net of multifunctions $F_j : X \to Y$. If $\{F_j : j \in \mathcal{J}\}$ is lower nearly-strongly as well as τ^+_{cl}-continuously convergent to a multifunction $F : X \to Y$ at the point x_0, then $x_0 \in C^+(F)$, where, $C^+(F) = \{x \in X : F$ is upper slightly continuous at $x\}$.*

Proof. On the contrary, let if possible, $x_0 \notin C^+(F)$. Then, there exists a clopen set $V \subset Y$ such that $F(x_0) \subset V$ and for each neighborhood U of X_0, there exists a point $x_U \in U$ with the property

$$F(x_U) \cap (Y \setminus V) \neq \emptyset. \tag{4.1}$$

Let $\mathcal{A}$ denote the family of all neighborhoods of x_0. We define an ordering " $\leq$ " in $\mathcal{A}$ and say that for $U_1, U_2 \in \mathcal{A}$, $U_1 \leq U_2$ if $U_2 \subset U_1$. Then $(\mathcal{A}, \leq)$ is a directed set. Let us denote

$$\Sigma = \{\sigma := (U, j) : U \in \mathcal{A} \text{ and } j \in \mathcal{J}\}.$$

For $\sigma_1 = (U_1, j_1)$ and $\sigma_2 = (U_2, j_2)$, we write $\sigma_1 \leq \sigma_2$ if $U_2 \subset U_1$ and $j_1 \leq j_2$. We consider

$$\{F_j(x_U) : \sigma = (U, j) \in \Sigma\}.$$

Since the net $\{F_j : j \in \mathcal{J}\}$ is τ^+_{cl}-continuously convergent to F at the point x_0, we have

$$F(x_0) \in \tau^+_{cl}\text{-}\lim_{\Sigma} F_j(x_U),$$

i.e, we get $\sigma_0 = (U_0, j_0)$ such that $F_j(x_U) \subset V$ for every $\sigma \geq \sigma_0$, $\sigma = (U, j)$. Now, for the clopen cover $\mathcal{U} = \{V, Y \setminus V\}$ of Y, we get for all $\sigma \geq \sigma_0$,

$$St_{cl}(F_j(x_U), \mathcal{U}) = V. \tag{4.2}$$

From (4.1) and (4.2), we have

$$F(x_U) \nsubseteq V = St_{cl}\big(F_j(x_U), \mathcal{U}\big)$$

so that the net $\{F_j : j \in \mathcal{J}\}$ is not lower strongly convergent to F which is a contradiction and we are done. □

The following can also be proved on similar lines:

Theorem 4.4. *Let $\{F_j : j \in \mathcal{J}\}$ be a net of multifunctions $F_j : X \to Y$. If $\{F_j : j \in \mathcal{J}\}$ is upper nearly-strongly as well as τ_{cl}^{-}-continuously convergent to a multifunction $F : X \to Y$ at the point x_0, then $x_0 \in C^{-}(F)$, where, $C^{-}(F) = \{x \in X : F$ is lower slightly continuous at $x\}$.*

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