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ON (p, q)-EIGENVALUES OF SUBELLIPTIC OPERATORS ON HOMOGENEOUS LIE GROUPS

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Abstract. In this article, we study the nonlinear Dirichlet (p, q)-eigenvalue problem for subelliptic operators defined by the left-invariant vector which satisfy the Hörmander condition. We prove both the solvability of the eigenvalue problem and the existence of the minimizer of the corresponding variational problem.

1. Introduction

In this article, we consider the Dirichlet (p, q)-eigenvalue problem, $1 < p < \nu$, $1 < q < p^* = \nu p(\nu - p)$ for subelliptic operators

$$-\mathrm{div}_{\mathrm{H}} \left(|\nabla_{\mathrm{H}}\, u|^{p-2}\nabla_{\mathrm{H}}\, u\right) = \lambda \|u\|^{p-q}_{L^q(\Omega)} |u|^{q-2}u \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega, \tag{1.1}$$

where $\nabla_{\mathrm{H}}\ u = (X_{11}u, \dots, X_{1n_1}u)$ is the horizontal (weak) subgradient of u defined by left-invariant vector fields $X_{11}, \dots, X_{1n_1}$ which satisfy the Hörmander condition [19]. Since the vector fields $X_{11}u$, $\dots, X_{1n_1}$ satisfy the Hörmander condition, they generate a Lie algebra V, and we consider Ω as a bounded domain on a corresponding stratified homogeneous Lie group $\mathbb{G}$. The number ν is called the homogeneous dimension of $\mathbb{G}$. Note that the Kohn–Laplace operator $\Delta_{\mathrm{H}} = \mathrm{div}_{\mathrm{H}}\nabla_{\mathrm{H}}$ induced by left-invariant vector fields on Heisenberg group $\mathbb{H}^n$ is a subelliptic operator which plays an important role in physics.

The eigenvalue problems for subelliptic operators defined by the left-invariant vector which satisfy the Hörmander condition were considered first in [11]. Remark that in the recent decades the eigenvalue problems for p-sub-Laplace operators

$$-\mathrm{div}_{\mathrm{H}} \left(|\nabla_{\mathrm{H}}\, u|^{p-2}\nabla_{\mathrm{H}}\, u\right) = \lambda |u|^{p-2}u \ \text{ in } \ \Omega, \ \ u = 0 \ \text{ on } \ \partial\Omega,$$

were intensively studied, for example, in [4, 23, 31].

The eigenvalue problem (in the commutative case, $\mathbb{G} = \mathbb{R}^n$) traces back to the works of Lord Rayleigh [28], where the author established the variational formulation of this problem in the linear case ($p = q = 2$) which is based on the Dirichlet integral

$$\|u\|^2_{W^{1,2}_0(\Omega)} = \int\limits_{\Omega} |\nabla u|^2 \ dx.$$

We note also classical works [26, 27] devoted to eigenvalues of linear elliptic operators and their connections with the problems of continuum mechanics.

The non-linear commutative case $p = q \neq 2$ was investigated by many authors as a typical non-linear eigenvalue boundary value problem in Euclidean domains of $\mathbb{R}^n$ (see, for example, [1,2,15,16,18], for extensive references we refer to [21]). In the case $p \neq q$, the non-linear eigenvalue boundary value problems in domains $\Omega \subset \mathbb{R}^n$ were considered in [9, 14, 15, 24]. Unfortunately, standard methods of the non-linear spectral theory of elliptic operators (see, for example [14]), do not work in the case of subelliptic operators. Therefore in the present work we adapted the inverse iteration method, which was suggested in [10]. On the base of this adapted method we study the non-linear eigenvalue boundary value problem for subelliptic operators.

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In the present work, we consider the Dirichlet (p,q)-eigenvalue problem (1.1) in the weak formulation: a function u solves the eigenvalue problem, iff $u \in W_0^{1,p}(\Omega)$ and

$$\int\limits_{\Omega} |\nabla_{\mathrm{H}} u|^{p-2} \nabla_{\mathrm{H}} u \nabla_{\mathrm{H}} v \, dx = \lambda \|u\|_{L^q(\Omega)}^{p-q} \int\limits_{\Omega} |u|^{q-2} uv \, dx \tag{1.2}$$

for all $v \in W_0^{1,p}(\Omega)$. In this case, we refer to λ as an eigenvalue and to u as the corresponding eigenfunction.

We prove the solvability of the Dirichlet (p,q)-eigenvalue problem (1.1) (see Theorem 3.1 and Theorem 3.2). Indeed, in Theorem 3.2, we have considered the following minimizing problem given by

$$\lambda = \inf_{u \in W_0^{1,p}(\Omega) \colon \|u\|_{L^q(\Omega)}=1} \int\limits_{\Omega} |\nabla_{\mathrm{H}} u|^p \, dx$$

and proved the existence of a function $v \in W_0^{1,p}(\Omega)$, $\|v\|_{L^q(\Omega)} = 1$, such that

$$\lambda = \int\limits_{\Omega} |\nabla_{\mathrm{H}} v|^p \, dx.$$

Moreover, we observe that v is an eigenfunction corresponding to λ and its associated eigenfunctions are precisely the scalar multiple of those vectors at which λ is reached. Finally, in Theorem 3.3, we establish some qualitative properties of the eigenfunctions of (1.1).

2. Homogeneous Lie Groups and Sobolev Spaces

Recall that a stratified homogeneous group [13], or, in another terminology, a Carnot group [25] is a connected simply connected nilpotent Lie group $\mathbb{G}$ whose Lie algebra V is decomposed into the direct sum $V_1 \oplus \cdots \oplus V_m$ of vector spaces such that $\dim V_1 \geqslant 2$, $[V_1, \ V_i] = V_{i+1}$ for $1 \leqslant i \leqslant m-1$ and $[V_1, \ V_m] = \{0\}$. Let $X_{11}, \ldots, X_{1n_1}$ be left-invariant basis vector fields of V_1. Since they generate V, for each i, $1 < i \leqslant m$, one can choose a basis X_{ik} in V_i, $1 \leqslant k \leqslant n_i = \dim V_i$, consisting of commutators of order $i-1$ of fields $X_{1k} \in V_1$. We identify the elements g of $\mathbb{G}$ with the vectors $x \in \mathbb{R}^N$, $N = \sum\limits_{i=1}^{m} n_i$, $x = (x_{ik})$, $1 \leqslant i \leqslant m$, $1 \leqslant k \leqslant n_i$ by means of exponential map $\exp(\sum x_{ik} X_{ik}) = g$. Dilations δ_t defined by the formula

$$\begin{aligned} \delta_t x &= (t^i x_{ik})_{1 \leqslant i \leqslant m, \, 1 \leqslant k \leqslant n_j} \\ &= (t x_{11}, \ldots, t x_{1n_1}, t^2 x_{21}, \ldots, t^2 x_{2n_2}, \ldots, t^m x_{m1}, \ldots, t^m x_{mn_m}), \end{aligned}$$

are automorphisms of $\mathbb{G}$ for each $t > 0$. The Lebesgue measure dx on $\mathbb{R}^N$ is the bi-invariant Haar measure on $\mathbb{G}$ (which is generated by the Lebesgue measure by means of the exponential map), and $d(\delta_t x) = t^\nu \, dx$, where the number $\nu = \sum\limits_{i=1}^{m} i n_i$ is called the homogeneous dimension of the group $\mathbb{G}$. The measure $|E|$ of a measurable subset E of $\mathbb{G}$ is defined by $|E| = \int\limits_E \, dx$.

The system of basis vectors $X_1, X_2, \ldots, X_{n_1}$ of the space V_1 satisfies the Hörmander hypoellipticity condition [19].

Euclidean space $\mathbb{R}^n$ with the standard structure is an example of an abelian group: the vector fields $\partial/\partial x_i$, $i = 1, \ldots, n$, have no non-trivial commutation relations and form the basis of the corresponding Lie algebra. One example of a non-abelian stratified group is the Heisenberg group $\mathbb{H}^n$. The non-commutative multiplication is defined as

$$hh' = (x,y,z)(x',y',z') = (x+x', y+y', z+z'-2xy'+2yx'),$$

where $x, x', y, y' \in \mathbb{R}^n$, $z, z' \in \mathbb{R}$. The left-translation $L_h(\cdot)$ is defined as $L_h(h') = hh'$. The left-invariant vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial z}, \ \ Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial z}, \ \ i = 1, \ldots, n, \ \ Z = \frac{\partial}{\partial z},$$

constitute the basis of the Lie algebra V of the Heisenberg group $\mathbb{H}^n$. All non-trivial relations are only of the form $[X_i, Y_i] = -4Z$, $i = 1, \dots, n$, and all other commutators vanish.

The Lie algebra of the Heisenberg group $\mathbb{H}^n$ has dimension $2n+1$ and splits into the direct sum $V = V_1 \oplus V_2$. The vector space V_1 is generated by the vector fields X_i, Y_i, $i = 1, \dots, n$, and the space V_2 is the one-dimensional center which is spanned by the vector field Z.

Recall that a homogeneous norm on the group $\mathbb{G}$ is a continuous function $|\cdot| : \mathbb{G} \to [0, \infty)$ that is C^∞-smooth on $\mathbb{G} \setminus \{0\}$ and has the following properties:

(a) $|x| = |x^{-1}|$ and $|\delta_t(x)| = t|x|$;

(b) $|x| = 0$ if and only if $x = 0$;

(c) there exists a constant $\tau_0 > 0$ such that $|x_1 x_2| \leqslant \tau_0(|x_1| + |x_2|)$ for all $x_1, x_2 \in \mathbb{G}$.

The homogeneous norm on the group $\mathbb{G}$ defines a homogeneous (quasi)metric

$$\rho(x, y) = |y^{-1}x|.$$

Recall that a continuous map $\gamma : [a, b] \to \mathbb{G}$ is called a continuous curve on $\mathbb{G}$. This continuous curve is rectifiable if

$$\sup \left\{ \sum_{k=1}^{m} \left| (\gamma(t_k))^{-1} \gamma(t_{k+1}) \right| \right\} < \infty,$$

where the supremum is taken over all partitions $a = t_1 < t_2 < \dots < t_m = b$ of the segment $[a, b]$. The rectifiable curve is called a horizontal rectifiable curve if its tangent vector $\dot{\gamma}(t)$ lies in the horizontal tangent space V_1, i.e., there exist functions $a_i(t)$, $t \in [a, b]$, such that $\sum\limits_1^{n_1} a_i^2 \leq 1$ and

$$\dot{\gamma}(t) = \sum_{i=1}^{n_1} a_i(t) X_{1i}(\gamma(t)).$$

The length $l(\gamma)$ of a horizontal rectifiable curve $\gamma : [a, b] \to \mathbb{G}$ can be calculated by the formula

$$l(\gamma) = \int_a^b \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_0^{\frac{1}{2}} \, dt = \int_a^b \left(\sum_{i=1}^{n} |a_i(t)|^2 \right)^{\frac{1}{2}} dt,$$

where $\langle \cdot, \cdot \rangle_0$ is the inner product on V_1. The result of [5] implies that one can connect two arbitrary points $x, y \in \mathbb{G}$ by a horizonal rectifiable curve. The Carnot-Carathéodory distance $d(x, y)$ is the infimum of the lengths over all horizontal rectifiable curves with endpoints x and y in $\mathbb{G}$. The Hausdorff dimension of the metric space $(\mathbb{G}, d)$ coincides with the homogeneous dimension ν of the group $\mathbb{G}$.

2.1. **Sobolev spaces on Carnot groups.** Let $\mathbb{G}$ be a Carnot group with one-parameter dilatation group δ_t, $t > 0$, and a homogeneous norm ρ, and let E be a measurable subset of $\mathbb{G}$. The Lebesgue space $L^p(E)$, $p \in [1, \infty]$, is the space of pth-power integrable functions $f : E \to \mathbb{R}$ with the standard norm

$$\|f\|_{L^p(E)} = \left(\int_E |f(x)|^p \, dx \right)^{\frac{1}{p}}, \ 1 \leq p < \infty, \tag{2.1}$$

and $\|f\|_{L^\infty(E)} = \operatorname{esssup}_E |f(x)|$ for $p = \infty$. We denote by $L^p_{\mathrm{loc}}(E)$ the space of functions $f : E \to \mathbb{R}$ such that $f \in L^p(F)$ for each compact subset F of E.

Let Ω be an open set in $\mathbb{G}$. The (horizontal) Sobolev space $W^{1,p}(\Omega)$, $1 \leqslant p \leqslant \infty$, consists of the functions $f : \Omega \to \mathbb{R}$ which are locally integrable in Ω, having the weak derivatives $X_{1i}f$ along the horizontal vector fields X_{1i}, $i = 1, \dots, n_1$, and the finite norm

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla_{\mathrm{H}} f\|_{L^p(\Omega)},$$

where $\nabla_{\mathrm{H}} f = (X_{11}f, \dots, X_{1n_1}f)$ is the horizontal subgradient of f. If $f \in W^{1,p}(U)$ for each bounded open set U such that $\overline{U} \subset \Omega$, then we say that f belongs to the class $W^{1,p}_{\mathrm{loc}}(\Omega)$.

The Sobolev space $W^{1,p}_0(\Omega)$ is defined to be the closure of $C^\infty_c(\Omega)$ under the norm

$$\|f\|_{W^{1,p}_0(\Omega)} = \|f\|_{L^p(\Omega)} + \|\nabla_{\mathrm{H}} f\|_{L^p(\Omega)}.$$

For the following result, refer to [12, 29, 30, 32].

Lemma 2.1. *The space $W_0^{1,p}(\Omega)$ is a real separable and uniformly convex Banach space.*

The following embedding result follows from [8, (2.8)] and [12], [17, Theorem 8.1], see also [3, Theorem 2.3]. We denote by $p^* = \nu p/(\nu - p)$.

Lemma 2.2. *Let $\Omega \subset \mathbb{G}$ be a bounded domain and $1 \leq p < \nu$. Then $W_0^{1,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for every $1 \leq q \leq p^*$. Moreover, the embedding is compact for every $1 \leq q < p^*$.*

Hence, in the case $1 \leq p < \nu$, we can consider the Sobolev space $W_0^{1,p}(\Omega)$ as Banach spaces with the norm

$$\|f\|_{W_0^{1,p}(\Omega)} = \|\nabla_{\mathrm{H}} f\|_{L^p(\Omega)}. \tag{2.2}$$

Next, we state the following result, which follows from [6, Theorem 9.14] on bounded, continuous, coercive and monotone operators on Banach spaces.

Theorem 2.3. *Let V be a real separable reflexive Banach space and V^* be the dual of V. Assume that $A : V \to V^*$ is a bounded, continuous, coercive and monotone operator. Then A is surjective, i.e., given any $f \in V^*$, there exists $u \in V$ such that $A(u) = f$. If A is strictly monotone, then A is also injective.*

3. Dirichlet (p, q)-eigenvalue Problems

We assume that $1 < p < \nu$, $1 < q < p^*$ and the spaces $L^q(\Omega)$ and $W_0^{1,p}(\Omega)$ are endowed with the norms (2.1) and (2.2), respectively, unless otherwise mentioned. In this article, we study the non-linear eigenvalue problem defined the vector fields satisfying the Hörmander hypoellipticity condition [19].

Let $1 < p < \nu$, $\lambda \in \mathbb{R}$, and consider the following subelliptic equation:

$$-\operatorname{div}_{\mathrm{H}} \left(|\nabla_{\mathrm{H}} u|^{p-2} \nabla_{\mathrm{H}} u\right) = \lambda \|u\|_{L^q(\Omega)}^{p-q} |u|^{q-2} u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{3.1}$$

where $1 < q < p^* = \frac{\nu p}{\nu - p}$. We say that $(\lambda, u) \in \mathbb{R} \times W_0^{1,p}(\Omega) \setminus \{0\}$ is an eigenpair of (3.1) if for every $v \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega |\nabla_{\mathrm{H}} u|^{p-2} \nabla_{\mathrm{H}} u \nabla_{\mathrm{H}} v \, dx = \lambda \|u\|_{L^q(\Omega)}^{p-q} \int_\Omega |u|^{q-2} u v \, dx. \tag{3.2}$$

Moreover, we refer to λ as an eigenvalue and to u as the corresponding eigenfunction.

3.1. Main results.

Let us formulate in this section the main results of the present work which are stated as follows.

Theorem 3.1. *Let $1 < p < \nu$ and $1 < q < p^*$. Then the following properties hold:*

(a) *There exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \cap L^q(\Omega)$ such that $\|w_n\|_{L^q(\Omega)} = 1$ and for every $v \in W_0^{1,p}(\Omega)$, we have*

$$\int_\Omega |\nabla_H w_{n+1}|^{p-2} \nabla_H w_{n+1} \nabla_H v \, dx = \mu_n \int_\Omega |w_n|^{q-2} w_n v \, dx, \tag{3.3}$$

where

$$\mu_n \geq \lambda := \inf \left\{ \int_\Omega |\nabla_H u|^p \, dx : u \in W_0^{1,p}(\Omega) \cap L^q(\Omega), \, \|u\|_{L^q(\Omega)} = 1 \right\}.$$

(b) *Moreover, the sequences $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\|w_{n+1}\|_{W_0^{1,p}(\Omega)}^p\}_{n\in\mathbb{N}}$ given by (3.3) are nonincreasing and converge to the same limit μ, which is bounded below by λ. Further, there exists a subsequence $\{n_j\}_{j\in\mathbb{N}}$ such that both $\{w_{n_j}\}_{j\in\mathbb{N}}$ and $\{w_{n_j+1}\}_{j\in\mathbb{N}}$ converge in $W_0^{1,p}(\Omega)$ to the same limit $w \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ with $\|w\|_{L^q(\Omega)} = 1$, and (μ, w) is an eigenpair of (3.1).*

Theorem 3.2. *Let $1 < p < \nu$ and $1 < q < p^*$. Suppose $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \cap L^q(\Omega)$ such that $\|u_n\|_{L^q(\Omega)} = 1$ and $\lim\limits_{n\to\infty} \|u_n\|^p_{W_0^{1,p}(\Omega)} = \lambda$.*

Then there exists a subsequence $\{u_{n_j}\}_{j\in\mathbb{N}}$ which converges weakly in $W_0^{1,p}(\Omega)$ to $u \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ with $\|u\|_{L^q(\Omega)} = 1$ such that $\lambda = \int\limits_\Omega |\nabla_H u|^p\,dx$. Moreover, (λ, u) is an eigenpair of (3.1) and any associated eigenfunction of λ are precisely the scalar multiple of those vectors at which λ is reached.

Our final main result concerns the following qualitative properties of the eigenfunctions of (3.1).

Theorem 3.3. *Let $1 < p < \nu$ and $1 < q < p^*$. Assume that $\lambda > 0$ is an eigenvalue of problem (3.1) and $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ is a corresponding eigenfunction. Then* (a) *$u \in L^\infty(\Omega)$.* (b) *Moreover, if u is nonnegative in Ω, then $u > 0$ in Ω. Further, for every $\omega \Subset \Omega$, there exists a positive constant c depending on ω such that $u \geq c > 0$ in ω.*

4. Auxiliary Results

In this section, we establish some auxiliary results that are crucial to prove our main results. To this end, we define the operators $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ by

$$\langle Av, w\rangle = \langle \operatorname{div}\left(|\nabla_{\mathrm{H}} v|^{p-2}\nabla_{\mathrm{H}} v\right), w\rangle = \int\limits_\Omega |\nabla_{\mathrm{H}} v|^{p-2}\nabla_{\mathrm{H}} v \nabla_{\mathrm{H}} w\,dx \tag{4.1}$$

and $B : L^q(\Omega) \to (L^q(\Omega))^*$ by

$$\langle B(v), w\rangle = \int\limits_\Omega |v|^{q-2} v w\,dx. \tag{4.2}$$

The symbols $(W_0^{1,p}(\Omega))^*$ and $(L^q(\Omega))^*$ denote the dual of $W_0^{1,p}(\Omega)$ and $L^q(\Omega)$, respectively. First, we have the following result.

Lemma 4.1. (i) *The operators A defined by* (4.1) *and B defined by* (4.2) *are continuous.* (ii) *Moreover, A is bounded, coercive and monotone.*

Proof. (i) **Continuity:** We only prove the continuity of A, since the continuity of B would follow similarly. To this end, suppose $\{v_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ such that $v_n \to v$ in the norm of $W_0^{1,p}(\Omega)$. Thus, up to a subsequence $\{v_{n_j}\}_{j\in\mathbb{N}}$, it follows that $\nabla_{\mathrm{H}} v_{n_j} \to \nabla_{\mathrm{H}} v$ pointwise almost everywhere in Ω. We observe that

$$\left\| |\nabla_{\mathrm{H}} v_{n_j}|^{p-2}\nabla_{\mathrm{H}} v_{n_j}\right\|_{L^{\frac{p}{p-1}}(\Omega)} \leq \|\nabla_{\mathrm{H}} v_{n_j}\|^{p-1}_{W_0^{1,p}(\Omega)} \leq C,$$

for some constant $C > 0$, which is independent of n. Therefore

$$|\nabla_{\mathrm{H}} v_{n_j}|^{p-2}\nabla_{\mathrm{H}} v_{n_j} \rightharpoonup |\nabla_{\mathrm{H}} v|^{p-2}\nabla_{\mathrm{H}} v$$

weakly in $L^{\frac{p}{p-1}}(\Omega)$. Since the weak limit is independent of the choice of the subsequence, it follows that

$$|\nabla_{\mathrm{H}} v_n|^{p-2}\nabla_{\mathrm{H}} v_n \rightharpoonup |\nabla_{\mathrm{H}} v|^{p-2}\nabla_{\mathrm{H}} v$$

weakly in $L^{\frac{p}{p-1}}(\Omega)$. As a consequence, we have

$$\langle Av_n, w\rangle \to \langle Av, w\rangle,$$

for every $w \in W_0^{1,p}(\Omega)$. Thus A is a continuous operator.

(ii) **Boundedness:** Using Hölder's inequality, we have

$$\|Av\|_{(W_0^{1,p}(\Omega))^*} = \sup_{\|w\|_{W_0^{1,p}(\Omega)}\leq 1} |\langle Av, w\rangle| \leq \|v\|^{p-1}_{W_0^{1,p}(\Omega)}\|w\|_{W_0^{1,p}(\Omega)} \leq \|v\|^{p-1}_{W_0^{1,p}(\Omega)}.$$

Thus A is bounded.
Coercivity: We observe that

$$\langle Av, v\rangle = \int\limits_\Omega |\nabla_{\mathrm{H}} v|^p\,dx = \|v\|^p_{W_0^{1,p}(\Omega)}.$$

Since $p > 1$, the operator A is a coercive operator.
Monotonicity: Recall the following algebraic inequality from [7, Lemma 2.1]: there exists a constant $C = C(p) > 0$ such that

$$\langle |a|^{p-2}a - |b|^{p-2}b, a-b\rangle \geq C(p)(|a|+|b|)^{p-2}|a-b|^2,\ 1<p<\infty, \tag{4.3}$$

for any $a, b \in \mathbb{R}^N$.

Hence for every $v, w \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\langle Av - Aw, v-w\rangle &= \int\limits_\Omega \langle |\nabla_{\mathrm{H}}\, v|^{p-2}\nabla_{\mathrm{H}}\, v - |\nabla_{\mathrm{H}}\, w|^{p-2}\nabla_{\mathrm{H}}\, w, \nabla_{\mathrm{H}}\,(v-w)\rangle\, dx \\ &= \int\limits_\Omega \langle |\nabla_{\mathrm{H}}\, v|^{p-2}\nabla_{\mathrm{H}}\, v - |\nabla_{\mathrm{H}}\, w|^{p-2}\nabla_{\mathrm{H}}\, w, \nabla_{\mathrm{H}}\, v - \nabla_{\mathrm{H}}\, w\rangle\, dx \\ &\geq C(p)\int\limits_\Omega (|\nabla_{\mathrm{H}}\, v| + |\nabla_{\mathrm{H}}\, w|)^{p-2}|\nabla_{\mathrm{H}}\, v - \nabla_{\mathrm{H}}\, w|^2\, dx \geq 0.\end{aligned}$$

Thus A is a monotone operator. □

Lemma 4.2. *The operators A defined by* (4.1) *and B defined by* (4.2) *satisfy the following properties:*

(H$_1$) $A(tv) = |t|^{p-2}tA(v) \quad \forall t \in \mathbb{R} \quad$ *and* $\quad \forall v \in W_0^{1,p}(\Omega)$.

(H$_2$) $B(tv) = |t|^{q-2}tB(v) \quad \forall t \in \mathbb{R} \quad$ *and* $\quad \forall v \in L^q(\Omega)$.

(H$_3$) $\langle A(v), w\rangle \leq \|v\|^{p-1}_{W_0^{1,p}(\Omega)}\|w\|_{W_0^{1,p}(\Omega)}$ *for all $v, w \in W_0^{1,p}(\Omega)$, where the equality holds if and only if $v = 0$ or $w = 0$ or $v = tw$ for some $t > 0$.*

(H$_4$) $\langle B(v), w\rangle \leq \|v\|^{q-1}_{L^q(\Omega)}\|w\|_{L^q(\Omega)}$ *for all $v, w \in L^q(\Omega)$, where the equality holds if and only if $v = 0$ or $w = 0$ or $v = tw$ for some $t \geq 0$.*

(H$_5$) *For every $w \in L^q(\Omega) \setminus \{0\}$, there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that*

$$\langle A(u), v\rangle = \langle B(w), v\rangle \quad \forall \quad v \in W_0^{1,p}(\Omega).$$

Proof. (H$_1$) Follows by the definition of A.

(H$_2$) Follows by the definition of B.

(H$_3$) First, using Cauchy–Schwartz inequality and then by Hölder's inequality with exponents $\frac{p}{p-1}$ and p, we obtain

$$\begin{aligned}\langle Av, w\rangle = \int\limits_\Omega |\nabla_{\mathrm{H}}\, v|^{p-2}\nabla_{\mathrm{H}}\, v\nabla_{\mathrm{H}}\, w\, dx &\leq \int\limits_\Omega |\nabla_{\mathrm{H}}\, v|^{p-1}|\nabla_{\mathrm{H}}\, w|\, dx \\ &\leq \|v\|^{p-1}_{W_0^{1,p}(\Omega)}\|w\|_{W_0^{1,p}(\Omega)}.\end{aligned}$$

If $v = 0$ or $w = 0$, then the equality $\langle Av, w\rangle = \|v\|^{p-1}_{W_0^{1,p}(\Omega)}\|w\|_{W_0^{1,p}(\Omega)}$ holds. So, we assume this equality such that both $v \neq 0$ and $w \neq 0$. Then the equality of Cauchy–Schwartz and Hölder's inequality hold simultaneously. That is, at one end (due to the equality of the Cauchy–Schwartz inequality), we get

$$\int\limits_\Omega |\nabla_{\mathrm{H}}\, v|^{p-2}\nabla_{\mathrm{H}}\, v\nabla_{\mathrm{H}}\, w\, dx = \int\limits_\Omega |\nabla_{\mathrm{H}}\, v|^{p-1}|\nabla_{\mathrm{H}}\, w|\, dx,$$

which gives $|\nabla_{\mathrm{H}}\, v|^{p-2}\nabla_{\mathrm{H}}\, v\nabla_{\mathrm{H}}\, w = |\nabla_{\mathrm{H}}\, v|^{p-1}|\nabla_{\mathrm{H}}\, w|$ and hence $\nabla_{\mathrm{H}}\, v(x) = c(x)\nabla_{\mathrm{H}}\, w(x)$ for almost every $x \in \Omega$ for some $c(x) \geq 0$. Also, due to the equality in Hölder's inequality, we have

$$\int\limits_\Omega |\nabla_{\mathrm{H}}\, v|^{p-2}\nabla_{\mathrm{H}}\, v\nabla_{\mathrm{H}}\, w\, dx = \|v\|^{p-1}_{W_0^{1,p}(\Omega)}\|w\|_{W_0^{1,p}(\Omega)},$$

which gives $|\nabla_{\mathrm{H}} v| = t|\nabla_{\mathrm{H}} w|$ in Ω for some constant $t > 0$. Therefore $c(x) = t$ in Ω. Hence $\nabla_{\mathrm{H}} v = t\nabla_{\mathrm{H}} w$ in Ω and therefore $v = tw$ in Ω for some $t > 0$. Thus (H_3) holds.

(H_4) This property can be verified similarly as in (H_3).

(H_5) Note that by Lemma 2.1, it follows that $W_0^{1,p}(\Omega)$ is a separable and reflexive Banach space. By Lemma 4.1, the operator $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ is bounded, continuous, coercive and monotone.

By Lemma 2.2, we have $W_0^{1,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$. Therefore $B(w) \in (W_0^{1,p}(\Omega))^*$ for every $w \in L^q(\Omega) \setminus \{0\}$.

Hence by Theorem 2.3, for every $w \in L^q(\Omega) \setminus \{0\}$, there exists $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\langle A(u), v\rangle = \langle B(w), v\rangle \quad \forall v \in W_0^{1,p}(\Omega).$$

Hence the property (H_5) holds. This completes the proof. □

The next result is useful to prove the boundedness of the eigenfunctions of (3.1).

Lemma 4.3. *Let $\Omega \subset \mathbb{G}$ be such that $|\Omega| < \infty$ and $1 < p < \nu$, $1 < r < p^* = \frac{\nu p}{\nu - p}$. Then for every $u \in W_0^{1,p}(\Omega)$, there exists a positive constant $C = C(r,p,\nu)$ such that*

$$\left(\int_\Omega |u|^r\,dx\right)^{\frac{1}{r}} \le C|\Omega|^{\frac{1}{r}-\frac{1}{p}+\frac{1}{\nu}}\left(\int_\Omega |\nabla_H u|^p\,dx\right)^{\frac{1}{p}}. \tag{4.4}$$

Proof. Proceeding as in [22, Corollary 1.57], we set

$$s = \begin{cases} 1, & \text{if } \ r\nu \le \nu + r \\ \frac{\nu r}{\nu + r}, & \text{if } \ \nu r > \nu + r. \end{cases}$$

Then $1 \le s \le p$, $s < \nu$ and $s^* = \frac{\nu s}{\nu - s} \ge r$. Using Hölder's inequality along with Lemma 2.2, we obtain

$$\begin{aligned} \|u\|_{L^r(\Omega)} \le \|u\|_{L^{s^*}(\Omega)}|\Omega|^{\frac{1}{r}-\frac{1}{s}+\frac{1}{\nu}} &\le C\|\nabla_{\mathrm{H}} u\|_{L^s(\Omega)}|\Omega|^{\frac{1}{r}-\frac{1}{s}+\frac{1}{\nu}} \\ &\le C\|\nabla_{\mathrm{H}} u\|_{L^p(\Omega)}|\Omega|^{\frac{1}{r}-\frac{1}{p}+\frac{1}{\nu}}. \end{aligned} \tag{4.5}$$

Hence the proof is complete. □

5. Proof of the Main Results

Proof of Theorem 3.1.

(a) First, we recall the definition of the operators $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ from (4.1) and $B : L^q(\Omega) \to (L^q(\Omega))^*$ from (4.2), respectively. Then, noting the property (H_5) from Lemma 4.2 and proceeding along the lines of the proof in [10, page 579 and pages 584 − 585], the result follows.

(b) Note that by Lemma 2.1, $W_0^{1,p}(\Omega)$ is a uniformly convex Banach space and by Lemma 2.2, $W_0^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$. Next, using Lemma 4.1-(i), the operators $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ and $B : L^q(\Omega) \to (L^q(\Omega))^*$ are continuous and by Lemma 4.2, the properties $(\mathrm{H}_1) - (\mathrm{H}_5)$ hold. Noting these facts, the result follows from [10, page 579, Theorem 1]. □

Proof of Theorem 3.2. The proof follows due to the same reasoning as in the proof of Theorem 3.1-(b) except that here we apply [10, page 583, Proposition 2] in place of [10, page 579, Theorem 1]. □

Proof of Theorem 3.3.

(a) Due to the homogeneity of equation (3.1), without loss of generality, we assume that $\|u\|_{L^q(\Omega)} = 1$. Let $k \ge 1$ and set $L(k) := \{x \in \Omega : u(x) > k\}$. Choosing $v = (u-k)^+$ as a test function in (3.2), we obtain

$$\int_{L(k)} |\nabla_{\mathrm{H}} u|^p\,dx = \lambda \int_{L(k)} |u|^{q-2}u(u-k)\,dx \le \lambda \int_{L(k)} |u|^{q-1}(u-k)\,dx. \tag{5.1}$$

We prove the result in the following two cases:

Case I. Let $q \leq p$, then since $k \geq 1$, over the set $L(k)$, we have $|u|^{q-1} \leq |u|^{p-1}$. Therefore from (5.1), we have

$$\int_{L(k)} |\nabla_{\mathrm{H}} u|^p \, dx \leq \lambda \int_{L(k)} |u|^{p-1}(u-k)\, dx$$
$$\leq \lambda \int_{L(k)} (2^{p-1}(u-k)^p + 2^{p-1}k^{p-1}(u-k))\, dx, \tag{5.2}$$

where to obtain the last inequality above, we have used the inequality $(a+b)^{p-1} \leq 2^{p-1}(a^{p-1}+b^{p-1})$ for $a, b \geq 0$. Using Sobolev's inequality (4.4) with $r = p$ in (5.2), we obtain

$$(1 - S\lambda 2^{p-1}|L(k)|^{\frac{p}{\nu}}) \int_{L(k)} (u-k)^p \, dx \leq \lambda S 2^{p-1} k^{p-1} |L(k)|^{\frac{p}{\nu}} \int_{L(k)} (u-k)\, dx, \tag{5.3}$$

where $S > 0$ is the Sobolev constant. Note that $\|u\|_{L^1(\Omega)} \geq k|L(k)|$ and therefore for every $k \geq k_0 = (2^p S\lambda)^{\frac{\nu}{p}} \|u\|_{L^1(\Omega)}$, we have $S\lambda 2^{p-1}|L(k)|^{\frac{p}{\nu}} \leq \frac{1}{2}$. Using this fact in (5.3), for every $k \geq \max\{k_0, 1\}$, we get

$$\int_{L(k)} (u-k)^p \, dx \leq \lambda S 2^p k^{p-1} |L(k)|^{\frac{p}{\nu}} \int_{L(k)} (u-k)\, dx. \tag{5.4}$$

Using Hölder's inequality and estimate (5.4), we obtain

$$\int_{L(k)} (u-k)\, dx \leq (\lambda S 2^p)^{\frac{1}{p-1}} k |L(k)|^{1+\frac{p}{\nu(p-1)}}. \tag{5.5}$$

Noting (5.5), by [20, Lemma 5.1], we get $u \in L^\infty(\Omega)$.

Case II. Let $q > p$, then using the inequality $(a+b)^{q-1} \leq 2^{q-1}(a^{q-1}+b^{q-1})$ for $a, b \geq 0$ in (5.1), we get

$$\int_{L(k)} |\nabla_{\mathrm{H}} u|^p \, dx \leq \lambda \int_{L(k)} (2^{q-1}(u-k)^q + 2^{q-1}k^{q-1}(u-k))\, dx. \tag{5.6}$$

Now, using Sobolev's inequality (4.4) with $r = q$ in estimate (5.6), we obtain

$$\left(\int_{L(k)} (u-k)^q \, dx\right)^{\frac{p}{q}} \leq S\lambda |L(k)|^{p(\frac{1}{q}-\frac{1}{p}+\frac{1}{\nu})} \int_{L(k)} (2^{q-1}(u-k)^q + 2^{q-1}k^{q-1}(u-k))\, dx, \tag{5.7}$$

where $S > 0$ is the Sobolev constant. Since $\int_{L(k)} (u-k)^q \, dx \leq \|u\|^q_{L^q(\Omega)} = 1$ and $q > p$, the quantity in the left-hand side of (5.7) can be estimated from below as

$$\left(\int_{L(k)} (u-k)^q \, dx\right)^{\frac{p}{q}} = \left(\int_{L(k)} (u-k)^q \, dx\right)^{\frac{p-q}{q}+1} \geq \int_{L(k)} (u-k)^q \, dx. \tag{5.8}$$

Using (5.8) in (5.7), we get

$$\left(1 - S\lambda 2^{q-1}|L(k)|^{p(\frac{1}{q}-\frac{1}{p}+\frac{1}{\nu})}\right) \int_{L(k)} (u-k)^q \, dx$$
$$\leq S\lambda 2^{q-1} k^{q-1} |L(k)|^{p(\frac{1}{q}-\frac{1}{p}+\frac{1}{\nu})} \int_{L(k)} (u-k)\, dx. \tag{5.9}$$

Let $\alpha = p(\frac{1}{q} - \frac{1}{p} + \frac{1}{\nu})$, which is positive, since $1 < q < p^*$. Choosing $k_1 = (S\lambda 2^q)^{\frac{1}{\alpha}} \|u\|_{L^1(\Omega)}$, due to the fact that $k|L(k)| \leq \|u\|_{L^1(\Omega)}$, we obtain for every $k \geq k_1$ that $S\lambda 2^{q-1}|L(k)|^\alpha \leq \frac{1}{2}$. Using this property in (5.9), we have

$$\int_{L(k)} (u-k)^q \, dx \leq S\lambda 2^q k^{q-1} |L(k)|^\alpha \int_{L(k)} (u-k) \, dx. \tag{5.10}$$

By Hölder's inequality and estimate (5.10), we arrive at

$$\int_{L(k)} (u-k) \, dx \leq (\lambda S 2^q)^{\frac{1}{q-1}} k |L(k)|^{1+\frac{\alpha}{q-1}}. \tag{5.11}$$

Noting (5.11), by [20, Lemma 5.1], we get $u \in L^\infty(\Omega)$.

(b) By [29, Theorem 5], the result follows. □

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