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SIMPLE CONTINUED FRACTIONS FOR CENTERED POLYGONAL NUMBERS

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Abstract. In this paper, basic information about centered polygonal numbers and continuous fractions are given. By using the continuous fraction algorithm, the relations between centered polygonal numbers and continued fractions have been studied. A general form for continued fractions of ratios of centered polygonal numbers of consecutive order is explained.

1. Introduction and Motivation

Continued fractions have played an important role in many algebraic operations. Since ancient times many mathematicians used continued fractions in their works. We can see many examples and traces of continued fractions in the studies of ancient Greek, Indian and Arab mathematicians [4]. It was proved that a real number is a rational number if and only if it has finite continued fractions expansion and every rational number can be represented as a finite simple continued fraction.

A set of rational numbers by using polygonal numbers represent continued fractions as is seen in [1, 2, 5, 6]. Using similar methods as in aforementioned studies, we give the generalized formula for continued fractions expansion of some centered polygonal numbers.

2. Preliminaries

Definition 1. For $m = 3, 4, \dots$ and $n \in \mathbb{N}$, the n-th centered m-gonal numbers formulas are given as follows [3]:

$$CS_m(n) := \frac{mn^2 - mn + 2}{2}.$$

Definition 2. Algebraically, the n-th centered m-gonal number $CS_m(n)$ is obtained as the sum of the first n elements of the sequence $1, m, 2m, 3m, \dots$. So, by Definition 1,

$$\begin{aligned} CS_m(n) &= 1 + m + 2m + 3m + \cdots + (n-1)m \\ &= 1 + m\frac{(n-1)n}{2} \\ &= \frac{mn^2 - mn + 2}{2} \end{aligned}$$

holds [3].

Example 1. For $m = 3, 4, \dots, 8$ and $n \in \mathbb{N}$, some centered polygonal numbers formulas are written as follows:

$$\begin{aligned} CS_3(n) &= \frac{3n^2 - 3n + 2}{2}, \\ CS_4(n) &= 2n^2 - 2n + 1, \\ CS_5(n) &= \frac{5n^2 - 5n + 2}{2}, \\ CS_6(n) &= 3n^2 - 3n + 1, \end{aligned}$$

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$$CS_7(n) = \frac{7n^2 - 7n + 2}{2},$$
$$CS_8(n) = 4n^2 - 4n + 1.$$

Example 2. For $m = 3, 4, 5, 6$ and $n = 1, 2, 3, 4, 5$ some centered polygonal numbers are given as follows [7, A005448, A001844, A005891, A003215]:

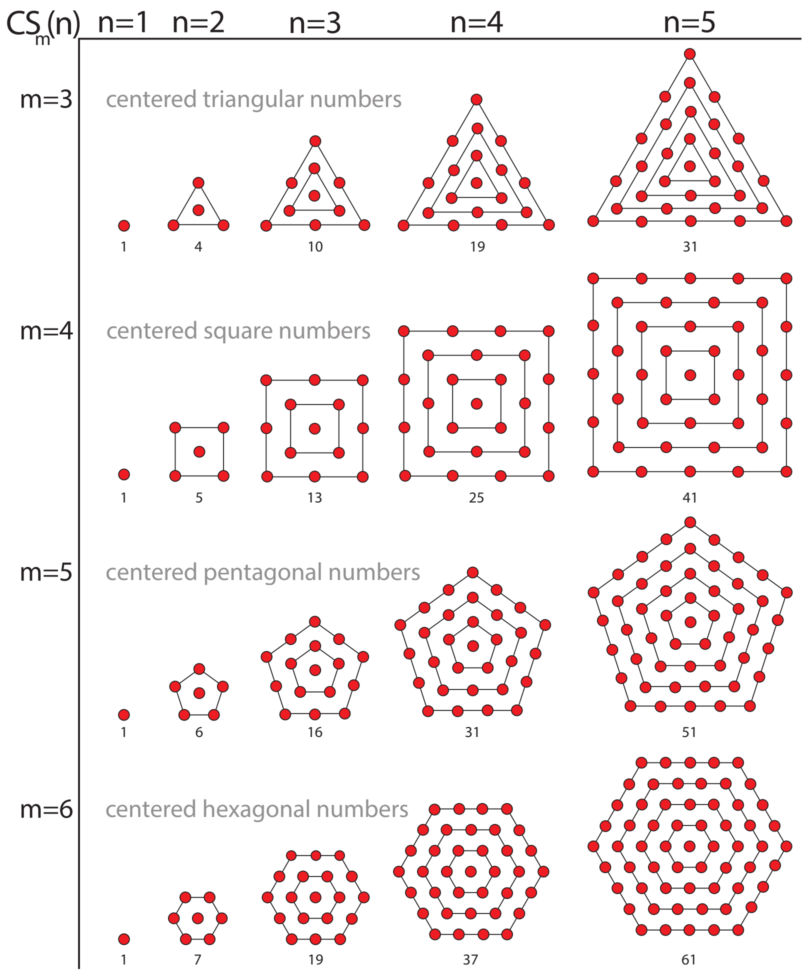

FIGURE 1. Some centered polygonal numbers.

Definition 3. A simple continued fraction is an expression of the form

$$\frac{p}{q} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where a_i are non-negative integers, for $i > 0$ and a_0, there may be any integer. The above expression is cumbrous to write and is usually written in one of these two forms: $\frac{p}{q} = a_0 + \frac{1}{a_1} \frac{1}{+a_2} \frac{1}{+a_3+} \cdots$, or using the list notation, $\langle a_0; a_1, a_2, a_3, \ldots \rangle$ [5].

Definition 4 (The Continued Fraction Algorithm)**.** Suppose we wish to find a continued fraction expansion of $x \in \mathbb{R}$.

Let $x_0 = x$ and set $a_0 = [x_0]$.

Define $x_1 = \frac{1}{x_0-[x_0]}$ and a set $a_1 = [x_1]$.

Also, $x_2 = \frac{1}{x_1-[x_1]} \Rightarrow a_2 = [x_2], \ldots, x_k = \frac{1}{x_{k-1}-[x_{k-1}]} \Rightarrow a_k = [x_k], \ldots,$

where $[x_i]$ is an integer part of x_i. This process is continued infinitely or to some finite stage till an $x_i \in \mathbb{N}$ exists such that $a_i = [x_i]$ [6].

Corollary 1. *For $m = 3, 4, \ldots$ and $n \in \mathbb{N}$,*

$$CS_m(n) < CS_m(n+1).$$

Proof. It is trivial by Definition 1. □

3. Main Results

In this section, we give some lemmas and theorems that the continued fraction expansions of the consecutive terms ratio of the central polygonal numbers $CS_m(n)$ are examined according to the odd or even case of n and m.

Theorem 1. *Let $m \geq 3$ and $n \geq 3$ be positive integers, then*

$$\frac{CS_m(n)}{CS_m(n+1)} = \begin{cases} \left\langle 0; 1, \frac{n-1}{2}, mn \right\rangle, & \text{if } n \text{ is odd,} \\ \left\langle 0; 1, \frac{n}{2}-1, 1, 1, \frac{mn}{4}-1, 2 \right\rangle, & \text{if } n \text{ is even and } n = 4k, \quad k = 1, 2, 3, \ldots, \\ \left\langle 0; 1, \frac{n}{2}-1, 1, 1, \frac{mn}{4}-1, 2 \right\rangle, & \text{if } n \text{ and } m \text{ are even and } n = 4k-2, \quad k = 2, 3, 4, \ldots, \\ \left\langle 0; 1, \frac{n}{2}-1, 1, 1, \frac{mn}{4}-\frac{1}{2} \right\rangle, & \text{if } m \text{ is odd, } n \text{ is even and } n = 4k-2, \quad k = 2, 3, 4, \ldots . \end{cases}$$

Lemma 1. *Let $m, n \geq 3$. If n is an odd integer, then*

$$\frac{CS_m(n)}{CS_m(n+1)} = \left\langle 0; 1, \frac{n-1}{2}, mn \right\rangle.$$

Proof. By using the continued fraction algorithm and the induction on n and m, we consider

Case i) For $m = n = 3$, we have

$$\frac{CS_3(3)}{CS_3(4)} = \frac{10}{19}.$$

Let $x_0 = \frac{10}{19}$, thus $a_0 = 0$. The algorithm follows

$$\begin{aligned} x_1 &= \frac{1}{x_0-[x_0]} = \frac{19}{10} = 1 + \frac{9}{10} \Rightarrow a_1 = 1, \\ x_2 &= \frac{1}{x_1-[x_1]} = \frac{10}{9} = 1 + \frac{1}{9} \Rightarrow a_2 = 1, \\ x_3 &= \frac{1}{x_2-[x_2]} = 9 \Rightarrow a_3 = 9. \end{aligned}$$

So, we get $\frac{CS_3(3)}{CS_3(4)} = \frac{10}{19} = \langle 0; 1, 1, 9 \rangle = \left\langle 0; 1, \frac{3-1}{2}, 3.3 \right\rangle$. Hence for $m = n = 3$, the result is true.

Case ii) Let us suppose that the result is true for $n = 2k-1$ and $2 < k \leq n$. Then we get

$$\frac{CS_m(n)}{CS_m(n+1)} = \frac{CS_m(2k-1)}{CS_m(2k)} = \langle 0; 1, k-1, m(2k-1) \rangle .$$

Case iii) We need to show that the result is true for $m+1$ and $n = 2k+1$.

$$\begin{aligned} \frac{CS_{m+1}(2k+1)}{CS_{m+1}(2k+2)} &= \frac{2(m+1)k^2 + (m+1)k + 1}{2(m+1)k^2 + 3(m+1)k + 2(m+1) + 1} \\ &= \frac{(2m+2)k^2 + (m+1)k + 1}{(2m+2)k^2 + (3m+3)k + 2m + 3}. \end{aligned}$$

Let $x_0 = \frac{(2m+2)k^2+(m+1)k+1}{(2m+2)k^2+(3m+3)k+2m+3}$, thus $a_0 = 0$. The algorithm follows

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{(2m+2)\,k^2 + (3m+3)\,k + m + 2}{(2m+2)\,k^2 + (m+1)\,k + 1} = 1 + \frac{(2m+2)\,k + m + 1}{(2m+2)\,k^2 + (m+1)\,k + 1} \Rightarrow a_1 = 1,$$
$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{(2m+2)\,k^2 + (m+1)\,k + 1}{(2m+2)\,k + m + 1} = k + \frac{1}{(2m+2)\,k + m + 1} \Rightarrow a_2 = k,$$
$$x_3 = \frac{1}{x_2 - [x_2]} = (2m+2)\,k + m + 1 \in \mathbb{Z} \Rightarrow a_3 = (m+1)\,(2k+1)\,.$$

Thus we get

$$x_3 = (2m+2)\,k + m + 1 \in \mathbb{Z} \text{ so } \frac{CS_{m+1}(2k+1)}{CS_{m+1}\,(2k+2)} = \langle 0; 1, k, (m+1)\,(2k+1)\rangle\,.$$

Hence the result is true. □

Lemma 2. *Let $m \geq 3$ and $n = 4k$ $(k = 1, 2, 3, \dots)$, then*

$$\frac{CS_m(n)}{CS_m\,(n+1)} = \left\langle 0; 1, \frac{n}{2} - 1, 1, 1, \frac{mn}{4} - 1, 2\right\rangle.$$

Proof. Using the continued fraction algorithm and the induction on n and m, we consider

Case i) For $m = 3$ and $n = 4$, we have

$$\frac{CS_3(4)}{CS_3\,(5)} = \frac{19}{31}.$$

Let $x_0 = \frac{19}{31}$, thus $a_0 = 0$. The algorithm follows

$$x_1 = \frac{1}{x_0 - [x_0]} = \frac{31}{19} = 1 + \frac{12}{19} \Rightarrow a_1 = 1,$$
$$x_2 = \frac{1}{x_1 - [x_1]} = \frac{19}{12} = 1 + \frac{7}{12} \Rightarrow a_2 = 1,$$
$$x_3 = \frac{1}{x_2 - [x_2]} = \frac{12}{7} = 1 + \frac{5}{7} \Rightarrow a_3 = 1,$$
$$x_4 = \frac{1}{x_3 - [x_3]} = \frac{7}{5} = 1 + \frac{2}{5} \Rightarrow a_4 = 1,$$
$$x_5 = \frac{1}{x_4 - [x_4]} = \frac{5}{2} = 2 + \frac{1}{2} \Rightarrow a_5 = 2,$$
$$x_6 = \frac{1}{x_5 - [x_5]} = \frac{2}{1} = 2 \Rightarrow a_6 = 2.$$

So, we get $\frac{CS_3(4)}{CS_3(5)} = \frac{19}{31} = \langle 0; 1, 1, 1, 1, 2, 2\rangle = \left\langle 0; 1, \frac{4}{2} - 1, 1, 1, \frac{3.4}{4} - 1, 2\right\rangle$. Hence for $m = 3$ and $n = 4$, the results is true.

Case ii) Let us suppose that the result is true for $n = 4k$ and $1 < k \leq n$. Then we get

$$\frac{CS_m(n)}{CS_m\,(n+1)} = \frac{CS_m(4k)}{CS_m\,(4k+1)} = \langle 0; 1, 2k-1, 1, 1, mk-1, 2\rangle\,.$$

Case iii) We need to show that the result is true for $m+1$ and $n = 4k+4$.

$$\frac{CS_{m+1}(4k+4)}{CS_{m+1}\,(4k+5)} = \frac{(16m+16)\,k^2 + (28m+28)\,k + 12m + 14}{(16m+16)\,k^2 + (36m+36)\,k + 20m + 22}.$$

Let $x_0 = \frac{(16m+16)k^2 + (28m+28)k + 12m + 14}{(16m+16)k^2 + (36m+36)k + 20m + 22}$, thus $a_0 = 0$. The algorithm follows

$$\begin{aligned} x_1 &= \frac{1}{x_0 - [x_0]} \\ &= \frac{(16m+16)\,k^2 + (36m+36)\,k + 20m + 22}{(16m+16)\,k^2 + (28m+28)\,k + 12m + 14} \end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{(8m+8)\,k + 8m + 8}{(16m+16)\,k^2 + (28m+28)\,k + 12m + 14} \\
&\Rightarrow a_1 = 1, \\
x_2 &= \frac{1}{x_1 - [x_1]} \\
&= \frac{(16m+16)\,k^2 + (28m+28)\,k + 12m + 14}{(8m+8)\,k + 8m + 8} \\
&= 2k + 1 + \frac{(4m+4)\,k + 4m + 6}{(8m+8)\,k + 8m + 8} \\
&\Rightarrow a_2 = 2k + 1, \\
x_3 &= \frac{1}{x_2 - [x_2]} \\
&= \frac{(8m+8)\,k + 8m + 8}{(4m+4)\,k + 4m + 6} \\
&= 1 + \frac{(4m+4)\,k + 4m + 2}{(4m+4)\,k + 4m + 6} \\
&\Rightarrow a_3 = 1, \\
x_4 &= \frac{1}{x_3 - [x_3]} \\
&= \frac{(4m+4)\,k + 4m + 6}{(4m+4)\,k + 4m + 2} \\
&= 1 + \frac{4}{(4m+4)\,k + 4m + 2} \\
&\Rightarrow a_4 = 1, \\
x_5 &= \frac{1}{x_4 - [x_4]} \\
&= \frac{(4m+4)\,k + 4m + 2}{4} \\
&= (m+1)\,k + m + \frac{2}{4} \\
&\Rightarrow a_5 = (m+1)\,k + m = (m+1)\,(k+1) - 1, \\
x_6 &= \frac{1}{x_5 - [x_5]} = \frac{4}{2} = 2 \\
&\Rightarrow a_5 = 2.
\end{aligned}$$

Thus we get $x_6 = 2 \in \mathbb{Z}$ so,

$$\frac{CS_{m+1}\,(4\,(k+1))}{CS_{m+1}\,(4\,(k+1)+1)} = \frac{CS_{m+1}(4k+4)}{CS_{m+1}\,(4k+5)} = \langle 0; 1, 2k+1, 1, 1, (m+1)\,(k+1) - 1, 2\rangle\,.$$

Hence the result is true. □

Lemma 3. *Let $m > 3$ be an even integer and $n = 4k - 2$ $(k = 1, 2, 3, \ldots)$, then*

$$\frac{CS_m(n)}{CS_m\,(n+1)} = \left\langle 0; 1, \frac{n}{2} - 1, 1, 1, \frac{mn}{4} - 1, 2 \right\rangle.$$

Proof. Using the continued fraction algorithm and the induction on n and m, we consider

Case i) For $m = 4$ and $n = 6$,

$$\frac{CS_4(6)}{CS_4\,(7)} = \frac{61}{85}.$$

Let $x_0 = \frac{61}{85}$, thus $a_0 = 0$. The algorithm follows

$$\begin{aligned}
x_1 &= \frac{1}{x_0 - [x_0]} = \frac{85}{61} = 1 + \frac{24}{61} \Rightarrow a_1 = 1, \\
x_2 &= \frac{1}{x_1 - [x_1]} = \frac{61}{24} = 2 + \frac{13}{24} \Rightarrow a_2 = 2, \\
x_3 &= \frac{1}{x_2 - [x_2]} = \frac{24}{13} = 1 + \frac{11}{13} \Rightarrow a_3 = 1, \\
x_4 &= \frac{1}{x_3 - [x_3]} = \frac{13}{11} = 1 + \frac{2}{11} \Rightarrow a_4 = 1, \\
x_5 &= \frac{1}{x_4 - [x_4]} = \frac{11}{2} = 5 + \frac{1}{2} \Rightarrow a_5 = 5, \\
x_6 &= \frac{1}{x_5 - [x_5]} = \frac{2}{1} = 2 \Rightarrow a_6 = 2.
\end{aligned}$$

So, we get $\frac{CS_4(6)}{CS_4(7)} = \frac{61}{85} = \langle 0; 1, 2, 1, 1, 5, 2 \rangle = \left\langle 0; 1, \frac{6}{2} - 1, 1, 1, \frac{4.6}{4} - 1, 2 \right\rangle$. Hence for $m = 4$ and $n = 6$, the results is true.

Case ii) Let us suppose that the result is true for $m = 2l$ $(m > 3)$, $n = 4k - 2$ and $2 < k \leq n$, $2 < l \leq n$. Then we get

$$\frac{CS_m(n)}{CS_m(n+1)} = \frac{CS_{2l}(4k-2)}{CS_{2l}(4k-1)} = \langle 0; 1, 2(k-1), 1, 1, l(2k-1) - 1, 2 \rangle .$$

Case iii) We need to show that the result is true for $m = 2l + 2$ and $n = 4k + 2$.

$$\frac{CS_{2l+2}(4k+2)}{CS_{2l+2}(4k+3)} = \frac{(32l+32)k^2 + (24l+24)k + 4l + 6}{(32l+32)k^2 + (40l+40)k + 12l + 14}.$$

Let $x_0 = \frac{(32l+32)k^2+(24l+24)k+4l+6}{(32l+32)k^2+(40l+40)k+12l+14}$, thus $a_0 = 0$. The algorithm follows

$$\begin{aligned}
x_1 &= \frac{1}{x_0 - [x_0]} \\
&= \frac{(32l+32)k^2 + (40l+40)k + 12l + 14}{(32l+32)k^2 + (24l+24)k + 4l + 6} \\
&= 1 + \frac{(16l+16)k + 8l + 8}{(32l+32)k^2 + (24l+24)k + 4l + 6} \\
&\Rightarrow a_1 = 1, \\
x_2 &= \frac{1}{x_1 - [x_1]} \\
&= \frac{(32l+32)k^2 + (24l+24)k + 4l + 6}{(16l+16)k + 8l + 8} \\
&= 2k + \frac{(8l+8)k + 4l + 6}{(16l+16)k + 8l + 8} \\
&\Rightarrow a_2 = 2k, \\
x_3 &= \frac{1}{x_2 - [x_2]} \\
&= \frac{(16l+16)k + 8l + 8}{(8l+8)k + 4l + 6} \\
&= 1 + \frac{(8l+8)k + 4l + 2}{(8l+8)k + 4l + 6} \\
&\Rightarrow a_3 = 1,
\end{aligned}$$

$$\begin{aligned}
x_4 &= \frac{1}{x_3 - [x_3]} \\
&= \frac{(8l+8)\,k + 4l + 6}{(8l+8)\,k + 4l + 2} \\
&= 1 + \frac{4}{(8l+8)\,k + 4l + 2} \\
&\Rightarrow a_4 = 1, \\
x_5 &= \frac{1}{x_4 - [x_4]} \\
&= \frac{(8l+8)\,k + 4l + 2}{4} \\
&= (2l+2)\,k + l + \frac{2}{4} \\
&\Rightarrow a_5 = (2l+2)\,k + l, \\
x_6 &= \frac{1}{x_5 - [x_5]} = \frac{4}{2} = 2 \\
&\Rightarrow a_5 = 2.
\end{aligned}$$

Thus we get $x_6 = 2 \in \mathbb{Z}$ so,

$$\frac{CS_{2(l+1)}\left(4\,(k+1)-2\right)}{CS_{2(l+1)}\left(4\,(k+1)-1\right)} = \frac{CS_{2l+2}(4k+2)}{CS_{2l+2}\,(4k+3)} = \left\langle 0; 1, 2k, 1, 1, (l+1)\,(2k+1) - 1, 2 \right\rangle .$$

Hence the result is true. □

Lemma 4. *Let $m \geq 3$ be an odd integer and $n = 4k-2$ $(k = 1, 2, 3, \ldots)$, then*

$$\frac{CS_m(n)}{CS_m\,(n+1)} = \left\langle 0; 1, \frac{n}{2} - 1, 1, 1, \frac{mn}{4} - \frac{1}{2} \right\rangle.$$

Proof. Using the continued fraction algorithm and the induction on n and m, we condider

Case i) For $m = 3$ and $n = 6$,

$$\frac{CS_3(6)}{CS_3\,(7)} = \frac{46}{64}.$$

Let $x_0 = \frac{46}{64}$, thus $a_0 = 0$. The algorithm follows

$$\begin{aligned}
x_1 &= \frac{1}{x_0 - [x_0]} = \frac{64}{46} = 1 + \frac{18}{46} \Rightarrow a_1 = 1, \\
x_2 &= \frac{1}{x_1 - [x_1]} = \frac{46}{18} = 2 + \frac{10}{18} \Rightarrow a_2 = 2, \\
x_3 &= \frac{1}{x_2 - [x_2]} = \frac{18}{10} = 1 + \frac{8}{10} \Rightarrow a_3 = 1, \\
x_4 &= \frac{1}{x_3 - [x_3]} = \frac{10}{8} = 1 + \frac{2}{8} \Rightarrow a_4 = 1, \\
x_5 &= \frac{1}{x_4 - [x_4]} = \frac{8}{2} = 4 \Rightarrow a_5 = 4.
\end{aligned}$$

So, we get $\frac{CS_3(6)}{CS_3(7)} = \frac{46}{64} = \langle 0; 1, 2, 1, 1, 4 \rangle = \left\langle 0; 1, \frac{6}{2} - 1, 1, 1, \frac{3.6}{4} - \frac{1}{2} \right\rangle$. Hence for $m = 3$ and $n = 6$, the results is true.

Case ii) Let us suppose that the result is true for $m = 2l-1$ $(m \geq 3)$, $n = 4k-2$ and $2 < k \leq n$, $2 < l \leq n$. Then we get

$$\frac{CS_m(n)}{CS_m\,(n+1)} = \frac{CS_{2l-1}(4k-2)}{CS_{2l-1}\,(4k-1)} = \left\langle 0; 1, 2\,(k-1)\,, 1, 1, (2l-1)\left(k - \frac{1}{2}\right) - \frac{1}{2} \right\rangle.$$

Case iii) We need to show that the result is true for $n = 4k + 2$ and $m = 2l + 1$.

$$\begin{aligned}
\frac{CS_m(n)}{CS_m\,(n+1)} &= \frac{16mk^2 + 12mk + 2m + 2}{16mk^2 + 20mk + 6m + 2}, \\
\frac{CS_{2l+1}(4k+2)}{CS_{2l+1}\,(4k+3)} &= \frac{16\,(2l+1)\,k^2 + 12\,(2l+1)\,k + 2\,(2l+1) + 2}{16\,(2l+1)\,k^2 + 20\,(2l+1)\,k + 6\,(2l+1) + 2} \\
&= \frac{(32l+16)\,k^2 + (24l+12)\,k + 4l + 4}{(32l+16)\,k^2 + (40l+20)\,k + 12l + 8}.
\end{aligned}$$

Let $x_0 = \frac{(32l+16)k^2 + (24l+12)k + 4l + 4}{(32l+16)k^2 + (40l+20)k + 12l + 8}$, thus $a_0 = 0$. The algorithm follows

$$\begin{aligned}
x_1 &= \frac{1}{x_0 - [x_0]} \\
&= \frac{(32l+16)\,k^2 + (40l+20)\,k + 12l + 8}{(32l+16)\,k^2 + (24l+12)\,k + 4l + 4} \\
&= 1 + \frac{(16l+8)\,k + 8l + 4}{(32l+16)\,k^2 + (24l+12)\,k + 4l + 4} \\
&\Rightarrow a_1 = 1, \\
x_2 &= \frac{1}{x_1 - [x_1]} \\
&= \frac{(32l+16)\,k^2 + (24l+12)\,k + 4l + 4}{(16l+8)\,k + 8l + 4} \\
&= 2k + \frac{(8l+4)\,k + 4l + 4}{(16l+8)\,k + 8l + 4} \\
&\Rightarrow a_2 = 2k, \\
x_3 &= \frac{1}{x_2 - [x_2]} \\
&= \frac{(16l+8)\,k + 8l + 4}{(8l+4)\,k + 4l + 4} \\
&= 1 + \frac{(8l+4)\,k + 4l}{(8l+4)\,k + 4l + 4} \\
&\Rightarrow a_3 = 1, \\
x_4 &= \frac{1}{x_3 - [x_3]} \\
&= \frac{(8l+4)\,k + 4l + 4}{(8l+4)\,k + 4l} \\
&= 1 + \frac{4}{(8l+4)\,k + 4l} \\
&\Rightarrow a_4 = 1, \\
x_5 &= \frac{1}{x_4 - [x_4]} \\
&= \frac{(8l+4)\,k + 4l}{4} \\
&= (2l+1)\,k + l \in \mathbb{Z} \\
&\Rightarrow a_5 = (2l+1)\left((k+1) - \frac{1}{2}\right) - \frac{1}{2}.
\end{aligned}$$

Thus we get $x_5 = (2l+1)\left(k+\frac{1}{2}\right) - \frac{1}{2} \in \mathbb{Z}$ so,

$$\frac{CS_{2(l+1)-1}\left(4\left(k+1\right)-2\right)}{CS_{2(l+1)-1}\left(4\left(k+1\right)-1\right)} = \frac{CS_{2l+1}(4k+2)}{CS_{2l+1}\left(4k+3\right)} = \left\langle 0; 1, 2k, 1, 1, (2l+1)\left(k+\frac{1}{2}\right) - \frac{1}{2}\right\rangle.$$

Hence the result is true. □

Proof of Theorem 1. Owing to Lemma 1, Lemma 2, Lemma 3 and Lemma 4, we complete the proof of Theorem 1. □

4. Conclusions

In this study, the continued fractions of the ratio of two consecutive order of central polygonal numbers are given. Similar studies can be conducted in the light of this study by making use of higher order figurate numbers such as cubic numbers.

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