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SOME FUNDAMENTAL PROPERTIES OF BANACH SPACES $l_p(\mathbb{BC}(N))$ WITH THE $*-$NORM $\ddot{\|}\ .\ \ddot{\|}_{2,l_p(\mathbb{BC}(N))}$

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Abstract. In this paper, we define non-Newtonian bicomplex versions of some known topological properties of sequence spaces and also, we introduce these new topological and geometric properties of $*-$bicomplex sequence spaces $l_p(\mathbb{BC}(N))$ for $\ddot{0}\ \ddot{<}\ p\ \ddot{\leq}\ \infty$. Our obtained results extend some existing facts in the literature.

1. Introduction and Preliminaries

Non-Newtonian calculus [7] is a popular and effective tool for its application in technology and mathematics. Up to now, researchers used non-Newtonian calculus in many areas including engineering, economy, biology, approximation theory, probability theory, weighted calculus etc. and extended some concepts in classical calculus to those in non-Newtonian calculus.

On the other hand, in search for development of special algebras, in 1892, Corrado Segre [20] defined the concept of bicomplex numbers which are a generalization of complex numbers. In 1991, Price [15] published a book on bicomplex numbers, multicomplex numbers, and their function theory. Since this subject has been developed very fast in recent times due to huge applications in different fields of mathematical sciences, it has attracted considerable interest from many authors. Alpay *et al.* [1], one of them, laid the foundations for a rigorous theory of functional analysis with bicomplex scalars. For other related studies on bicomplex analysis, we recommend [10, 11, 16].

Firstly, after introducing non-Newtonian bicomplex numbers as a generalization of both bicomplex numbers and non-Newtonian complex numbers, Sager and Sağır established $*-$bicomplex sequence spaces $l_p(\mathbb{BC}(N))$ for $\ddot{0}\ \ddot{<}\ p\ \ddot{\leq}\ \infty$ and also, studied non-Newtonian completeness property of the spaces [17,18]. Also, in the literature, there are many results dealing with the geometric and topological properties of various sequence spaces. Some of these works are noted in [2, 4, 5, 8, 12–14, 19, 21].

Motivated by the above studies, our purpose in this study is to investigate some fundamental geometric and topological properties of $*-$bicomplex sequence spaces $l_p(\mathbb{BC}(N))$ for $\ddot{0}\ \ddot{<}\ p\ \ddot{\leq}\ \infty$. We also explain the properties not satisfied with some illustrative examples. We hope that some properties of the spaces considered in this work will be hammering away in studying other aspects of such spaces.

Now, we summarize a number of known results which will be needed in other sections.

Let i and j be independent imaginary units such that $i^2 = j^2 = -1$, $ij = ji$ and let $\mathbb{C}(i)$ be the set of complex numbers with the imaginary unit i. The set of bicomplex numbers $\mathbb{BC}$ is defined by

$$\mathbb{BC} = \{z = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}(i)\}.$$

The set $\mathbb{BC}$ forms a Banach space with respect to the algebraic operations for all $z, w \in \mathbb{BC}$, $\lambda \in \mathbb{R}$ and Euclidean norm defined as [15]

$$\begin{aligned} z + w &= (z_1 + jz_2) + (w_1 + jw_2) = (z_1 + w_1) + j(z_2 + w_2), \\ \lambda.z &= \lambda.(z_1 + jz_2) = \lambda z_1 + j\lambda z_2, \\ \|.\| &: \mathbb{BC} \to \mathbb{R}, z \to \|z\| = \sqrt{|z_1|^2 + |z_2|^2}. \end{aligned}$$

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A complete ordered field is called arithmetic if its realm is a subset of $\mathbb{R}$. A generator is a one-to-one function whose domain $\mathbb{R}$ and whose range is a subset of $\mathbb{R}$. Let α be a generator with range A. We denote by $\mathbb{R}_\alpha$ the range of generator α. Also, the elements of $\mathbb{R}_\alpha$ (or $\mathbb{R}(N)_\alpha$) are called non-Newtonian real numbers.

Let α and β be arbitrarily chosen generators which image is the set $\mathbb{R}$ to A and B, respectively, and let $*-$ ("star-") calculus be also the ordered pair of arithmetics ($\alpha-$ arithmetic, $\beta-$ arithmetic). The following notations will be used. All definitions given for $\alpha-$arithmetic are also valid for $\beta-$arithmetic.

	$\alpha-$arithmetic	$\beta-$ arithmetic
Realm	$A\,(=\mathbb{R}_\alpha=\mathbb{R}(N)_\alpha)$	$B\,(=\mathbb{R}_\beta=\mathbb{R}(N)_\beta)$
Summation	$y\dot{+}z=\alpha\left\{\alpha^{-1}(y)+\alpha^{-1}(z)\right\}$	$\ddot{+}$
Subtraction	$y\dot{-}z=\alpha\left\{\alpha^{-1}(y)-\alpha^{-1}(z)\right\}$	$\ddot{-}$
Multiplication	$y\dot{\times}z=\alpha\left\{\alpha^{-1}(y)\times\alpha^{-1}(z)\right\}$	$\ddot{\times}$
Division	$y\dot{/}z=\frac{y}{z}\alpha=\alpha\left\{\frac{\alpha^{-1}(y)}{\alpha^{-1}(z)}\right\}\ (z\neq\dot{0})$	$\ddot{/}$
Ordering	$y\dot{\leq}z\Longleftrightarrow\alpha^{-1}(y)\leq\alpha^{-1}(z)$	$\ddot{\leq}$

There are the following three properties for the isomorphism from $a-$arithmetic to $\beta-$arithmetic that is the unique function $\imath$(iota).

1. $\imath$ is one-to-one.
2. $\imath$ is on A and onto B.
3. For all u,$v\in A$,

$$\begin{aligned}
\iota\left(u\dot{+}v\right)&=\iota(u)\,\ddot{+}\iota(v)\,,\ \iota\left(u\dot{-}v\right)=\iota(u)\,\ddot{-}\iota(v)\,,\\
\iota\left(u\dot{\times}v\right)&=\iota(u)\,\ddot{\times}\iota(v)\,,\ \iota\left(u\dot{/}v\right)=\iota(u)\,\ddot{/}\iota(v)\,,\ v\neq\dot{0},\\
u\,\dot{<}v&\Longleftrightarrow\iota(u)\,\ddot{<}\iota(v)\,,\\
\iota(u)&=\beta\left\{\alpha^{-1}(u)\right\}.
\end{aligned}$$

Also, for every integer n, we set $\iota(\dot{n})=\ddot{n}$.

An $\alpha-$ positive number is a number x with $\dot{0}\,\dot{<}\,x$ and an $\alpha-$negative number is a number with $x\,\dot{<}\,\dot{0}$. $\alpha-$zero and $\alpha-$one numbers are denoted by $\dot{0}=\alpha(0)$ and $\dot{1}=\alpha(1)$, and the set of $\alpha-$ positive numbers is denoted by $\mathbb{R}_\alpha^+$ (or $\mathbb{R}(N)_\alpha^+$). Also, $\alpha(-p)=\alpha\left\{\alpha^{-1}\left(\ddot{p}\right)\right\}=\dot{-}\dot{p}$ for all $p\in\mathbb{Z}^+$. An open interval on $\mathbb{R}(N)_\alpha$ for $a,b\in\mathbb{R}(N)_\alpha$ with $a\,\dot{<}\,b$ is represented by $(a,b)_N=\left\{x\in\mathbb{R}(N)_\alpha : a\,\dot{<}\,x\,\dot{<}\,b\right\}$. The $\alpha-$absolute value of $x\in A$ is defined by

$$\dot{|}\,x\,\dot{|}=|x|_\alpha=\begin{cases}x, & \text{if } \dot{0}\,\dot{<}\,x,\\ \dot{0}, & \text{if } \dot{0}=x,\\ \dot{0}\,\dot{-}\,x, & \text{if } x\,\dot{<}\,\dot{0}.\end{cases}$$

Let $\ddot{b}\in B\subseteq\mathbb{R}$. Then the number $\ddot{b}\,\ddot{\times}\,\ddot{b}$ is called the $\beta-$square of $\ddot{b}$ and denoted by $b^{\ddot{2}}$. Let $\ddot{b}$ be a nonnegative number in B. Then $\beta\left[\sqrt{\beta^{-1}(\ddot{b})}\right]$ is called the $\beta-$square root of $\ddot{b}$ and denoted by $\sqrt[\cdot\cdot]{\ddot{b}}$ [2,7].

The definitions of $\alpha-$convergence of a sequence of elements in A, $\alpha-$convergent series, non-Newtonian metric space, non-Newtonian normed space, non-Newtonian completeness, non-Newtonian upper bound, non-Newtonian supremum are found in [2,3,6,7,9].

Note that we use the notations $\dot{\sup}$ and $\dot{\sum}$ for non-Newtonian supremum and $\alpha-$series, respectively in this article.

Let $\dot{a}\in\left(A,\dot{+},\dot{-},\dot{\times},\dot{/},\dot{\leq}\right)$ and $\ddot{b}\in\left(B,\ddot{+},\ddot{-},\ddot{\times},\ddot{/},\ddot{\leq}\right)$ be arbitrarily chosen elements from the corresponding arithmetics. Then the ordered pair $\left(\dot{a},\ddot{b}\right)$ is called as a $*-$point. The set of all $*-$points is called the set of $*-$complex numbers (non-Newtonian complex numbers) and denoted by $\mathbb{C}^*$ or $\mathbb{C}(N)$, that is,

$$\mathbb{C}(N)=\left\{\left(\dot{a},\ddot{b}\right):\dot{a}\in A\subseteq\mathbb{R},\ddot{b}\in B\subseteq\mathbb{R}\right\}.$$

The set $\mathbb{C}(N)$ forms a Banach space and a field with the algebraic operations $\oplus_1$, $\odot_1$ and $\otimes_1$ defined on $\mathbb{C}(N)$ and the $*-$norm $\ddot{\|}\ .\ \ddot{\|}_1$ defined by

$$\ddot{\|}\ z^*\ \ddot{\|}_1 = \ddot{\sqrt{\left[\imath(\dot{a}\dot{-}\dot{0})\right]^{\ddot{2}} \ddot{+} (\ddot{b}\ddot{-}\ddot{0})^{\ddot{2}}}} = \beta\Big(\sqrt{a^2+b^2}\Big),$$

where $z^* = \big(\dot{a}, \ddot{b}\big) \in \mathbb{C}(N)$ [21].

The following definitions and properties related to $\mathbb{BC}(N)$ and appearing in [17] will be needed in the sequel.

Let $\dot{a}, \dot{c} \in \big(A, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{\leq}\ \big)$ and $\ddot{b}, \ddot{d} \in \big(B, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{\leq}\ \big)$. Then $\big(\dot{a}, \ddot{b}, \dot{c}, \ddot{d}\big)$ is called as a $*-$bicomplex point. The set of all $*-$bicomplex points is called the set of $*-$bicomplex numbers (non-Newtonian bicomplex numbers) and denoted by $\mathbb{BC}^*$ or $\mathbb{BC}(N)$; that is,

$$\begin{aligned}\mathbb{BC}(N) &= \Big\{\big(\dot{a}, \ddot{b}, \dot{c}, \ddot{d}\big) : \dot{a}, \dot{c} \in A \subseteq \mathbb{R}, \ddot{b}, \ddot{d} \in B \subseteq \mathbb{R}\Big\}\\ &= \Big\{\big(z^*, w^*\big) : z^* = \big(\dot{a}, \ddot{b}\big), w^* = \Big(\dot{c}, \ddot{d}\Big), \dot{a}, \dot{c} \in A \subseteq \mathbb{R}, \ddot{b}, \ddot{d} \in B \subseteq \mathbb{R}\Big\}.\end{aligned}$$

The algebraic operations addition $\oplus_2$, multiplication $\otimes_2$ and scalar multiplication $\odot_2$ defined on $\mathbb{BC}(N)$ as follows:

$$\begin{aligned}\oplus_2 &: \mathbb{BC}(N) \times \mathbb{BC}(N) \to \mathbb{BC}(N),\\ (\zeta_1^*, \zeta_2^*) &\to \zeta_1^* \oplus_2 \zeta_2^* = (z_1^*, w_1^*) \oplus_2 (z_2^*, w_2^*) = (z_1^* \oplus_1 z_2^*, w_1^* \oplus_1 w_2^*),\\ \otimes_2 &: \mathbb{BC}(N) \times \mathbb{BC}(N) \to \mathbb{BC}(N),\\ (\zeta_1^*, \zeta_2^*) &\to \zeta_1^* \otimes_2 \zeta_2^* = (z_1^*, w_1^*) \otimes_2 (z_2^*, w_2^*)\\ &\quad= ((z_1^* \otimes_1 z_2^*) \ominus_1 (w_1^* \otimes_1 w_2^*), (z_1^* \otimes_1 w_2^*) \oplus_1 (z_2^* \otimes_1 w_1^*)),\\ \odot_2 &: \mathbb{C}(N) \times \mathbb{BC}(N) \to \mathbb{BC}(N),\\ (z^*, \zeta_1^*) &\to z^* \odot_2 \zeta_1^* = z^* \odot_2 (z_1^*, w_1^*) = (z^* \otimes_1 z_1^*, z^* \otimes_1 w_1^*),\end{aligned}$$

where $\zeta_1^* = (z_1^*, w_1^*)$, $\zeta_2^* = (z_2^*, w_2^*) \in \mathbb{BC}(N)$ and $z^* \in \mathbb{C}(N)$. According to these operations, it can simply be shown that the set $\mathbb{BC}(N)$ forms a vector space over the field $\mathbb{C}(N)$ and a ring. We can denote the non-Newtonian complex number $z^* = \big(\dot{a}, \ddot{b}\big)$ by $\big(\dot{a}, \ddot{0}\big) \oplus_1 i^* \otimes_1 \big(\dot{0}, \ddot{b}\big) = \dot{a} \oplus_1 i^* \otimes_1 \ddot{b}$, where $i^* = \big(\dot{0}, \ddot{1}\big)\big(= \big(\dot{0}, \ddot{1}, \dot{0}, \ddot{0}\big)\big), \left(i^*\right)^2 = \ominus_1 1^* \big(= \big(\dot{-}\dot{1}, \ddot{0}\big)\big)$. Also, we can denote the non-Newtonian bicomplex number $\zeta^* = (z^*, w^*)$ by $(z^*, 0^*) \oplus_2 j^* \otimes_2 (w^*, 0^*) = z^* \oplus_2 j^* \otimes_2 w^* = \big(\dot{a}, \ddot{b}\big) \oplus_2 j^* \otimes_2 \big(\dot{c}, \ddot{d}\big)$, where $z^* = \big(\dot{a}, \ddot{b}\big), w^* = \big(\dot{c}, \ddot{d}\big), j^* = \big(\dot{0}, \ddot{0}, \dot{1}, \ddot{0}\big) = (0^*, 1^*), \left(j^*\right)^2 = \ominus_2 1^* \big(= \big(\ominus_1 1^*, 0^*\big) = \ominus_2 (1^*, 0^*) = \ominus_2 1^*\big)$, and also define z^* and w^* by $\operatorname{Re}\zeta^*$ and $\operatorname{Im}\zeta^*$, respectively.

The $*-$distance $d_{\mathbb{BC}(N)}$ between two arbitrarily elements $\zeta_1^* = z_1^* \oplus_2 j^* \otimes_2 w_1^*, \zeta_2^* = z_2^* \oplus_2 j^* \otimes_2 w_2^*$ of the set $\mathbb{BC}(N)$ is defined by

$$\begin{aligned}d_{\mathbb{BC}(N)} &: \mathbb{BC}(N) \times \mathbb{BC}(N) \to \ddot{[}\ddot{0}, \infty\ddot{)} \subset B,\\ (\zeta_1^*, \zeta_2^*) &\to d_{\mathbb{BC}(N)}(\zeta_1^*, \zeta_2^*) = \ddot{\sqrt{\ddot{\|}\ z_1^* \ominus_1 z_2^*\ \ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|}\ w_1^* \ominus_1 w_2^*\ \ddot{\|}_1^{\ddot{2}}}}.\end{aligned}$$

The $*-$distance $d_{\mathbb{BC}(N)}$ is a non-Newtonian metric on $\mathbb{BC}(N)$.

For definitions of $*-$bicomplex sequence, $*-$limit of $*-$bicomplex sequence, $*-$bicomplex Cauchy sequence, $*-$bicomplex series and convergence of $*-$bicomplex series, we refer to [17].

Then $\mathbb{BC}(N)$ is a Banach space with respect to the $*-$norm $\ddot{\|}\ .\ \ddot{\|}_2$ defined by

$$\ddot{\|}\ \zeta^*\ \ddot{\|}_2 = \ddot{\sqrt{\ddot{\|}\ z^*\ \ddot{\|}_1^{\ddot{2}} \ddot{+} \ddot{\|}\ w^*\ \ddot{\|}_1^{\ddot{2}}}},$$

for $\zeta^* = z^* \oplus_2 j^* \otimes_2 w^* \in \mathbb{BC}(N)$.

Lemma 1. *Let $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$ and $z^* \in \mathbb{C}(N)$. Then the following statements hold:*

i) $\ddot{\|}\ \zeta_1^* \oplus_2 \zeta_2^*\ \ddot{\|}_2 \ddot{\leq} \ddot{\|}\ \zeta_1^*\ \ddot{\|}_2 \ddot{+} \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2$.

ii) $\ddot{\|} \zeta_1^* \ominus_2 \zeta_2^* \ddot{\|}_2 \ddot{\leq} \ddot{\|} \zeta_1^* \ddot{\|}_2 \ddot{+} \ddot{\|} \zeta_2^* \ddot{\|}_2$.
iii) $\ddot{\|} z^* \odot_2 \zeta_1^* \ddot{\|}_2 = \ddot{\|} z^* \ddot{\|}_1 \ddot{\times} \ddot{\|} \zeta_1^* \ddot{\|}_2$ [17].

In [10], Sager and Sağır constructed the $*-$bicomplex sequence spaces $l_\infty(\mathbb{BC}(N))$ and $l_p(\mathbb{BC}(N))$ of all bounded and absolutely $p-$ summable $*-$bicomplex sequences over the field $\mathbb{C}(N)$ by using the $*-$ norm $\ddot{\|} \, . \, \ddot{\|}_2$ as follows:

$$w(\mathbb{BC}(N)) := \left\{\zeta = (\zeta_n^*) : \zeta_n^* \in \mathbb{BC}(N) \text{ for all } n \in \mathbb{N}\right\},$$

$$l_\infty(\mathbb{BC}(N)) := \left\{\zeta = (\zeta_n^*) \in w(\mathbb{BC}(N)) : \ddot{\sup_{n\in\mathbb{N}}} \ddot{\|} \zeta_n^* \ddot{\|}_2 \ddot{<} \infty\right\},$$

$$l_p(\mathbb{BC}(N)) := \left\{\zeta = (\zeta_n^*) \in w(\mathbb{BC}(N)) : \ddot{\sum_{n=1}^{\infty}} \ddot{\|} \zeta_n^* \ddot{\|}_2^{p} \ddot{<} \infty\right\}$$

for $\ddot{0} \ddot{<} p \ddot{<} \infty, \; p \in \mathbb{R}(N)_\beta$.

The algebraic operations addition $\oplus$, scalar multiplication $\odot$ and multiplication $\otimes$ defined on $w(\mathbb{BC}(N))$ as follows, respectively:

$$\oplus : w(\mathbb{BC}(N)) \times w(\mathbb{BC}(N)) \to w(\mathbb{BC}(N)), (s,t) \to s \oplus t = (s_n^* \oplus_2 t_n^*),$$
$$\odot : \mathbb{C}(N) \times w(\mathbb{BC}(N)) \to w(\mathbb{BC}(N)), (z^*, s) \to z^* \odot s = (z^* \odot_2 s_n^*),$$

where $s = (s_n^*), t = (t_n^*) \in w(\mathbb{BC}(N))$ *and* $z^* \in \mathbb{C}(N)$. The set $w(\mathbb{BC}(N))$ forms a vector space over the non-Newtonian complex field $\mathbb{C}(N)$ with respect to the addition $\oplus$ and scalar multiplication $\odot$.

Corollary 1. *$l_\infty(\mathbb{BC}(N))$ is a Banach space with the $*-$norm $\ddot{\|} \, . \, \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))}$ defined by*

$$\ddot{\|} s \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} = \ddot{\sup_{n\in\mathbb{N}}} \ddot{\|} s_n^* \ddot{\|}_2 ; s = (s_n^*) \in l_\infty(\mathbb{BC}(N))$$

[18].

Corollary 2. *For $\ddot{1} \ddot{\leq} p \ddot{<} \infty$, the space $l_p(\mathbb{BC}(N))$ is a Banach space with the $*-$norm $\ddot{\|} \, . \, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}$ defined by*

$$\ddot{\|} s \ddot{\|}_{2,l_p(\mathbb{BC}(N))} = \left(\ddot{\sum_{n=1}^{\infty}} \ddot{\|} s_n^* \ddot{\|}_2^{p}\right)^{\frac{1}{p}} ; \; s = (s_n^*) \in l_p(\mathbb{BC}(N))$$

[18].

Definition 1. Let X be a vector space over the field $\mathbb{C}(N)$. A map $\ddot{\|} |.| \ddot{\|} : X \to \ddot{[}\ddot{0}, \infty\ddot{)} = B' \subset B$ is said to be a $p, *-$norm for $\ddot{0} \ddot{<} p \ddot{\leq} \ddot{1}$ if it satisfies the following properties:
(i) $\ddot{\|} |x| \ddot{\|} = \ddot{0}$ if and only if $x = 0$.
(ii) $\ddot{\|} |\mu^* \odot x| \ddot{\|} = \ddot{\|} |\mu^*| \ddot{\|}_1^{p} \ddot{\times} \ddot{\|} |x| \ddot{\|}$ for all $x \in X, \mu^* \in \mathbb{C}(N)$.
(iii) $\ddot{\|} |x \oplus y| \ddot{\|} \ddot{\leq} \ddot{\|} |x| \ddot{\|} \ddot{+} \ddot{\|} |y| \ddot{\|}$ for all $x, y \in X$.
Then X is said to be a $p, *-$ normed space [18].

Corollary 3. *For $\ddot{0} \ddot{<} p \ddot{<} \ddot{1}$, the space $l_p(\mathbb{BC}(N))$ is a $p,*-$Banach space with the $p,*-$norm $\ddot{\|} \, . \, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}$ defined by*

$$\ddot{\|} s \ddot{\|}_{2,l_p(\mathbb{BC}(N))} = \ddot{\sum_{n=1}^{\infty}} \ddot{\|} s_n^* \ddot{\|}_2^{p} ; \; s = (s_n^*) \in l_p(\mathbb{BC}(N))$$

[18].

2. Main Results

We begin this section by introducing non-Newtonian versions of the concepts of convex, strictly convex and uniformly convex sets which will be needed in the sequel.

Definition 2. Let X be a vector space over the non-Newtonian real field $\mathbb{R}(N)$ and C be a subset of X. If $\lambda \otimes x \oplus (\dot{1} \dot{-} \lambda) \otimes y \in C$ for all $x, y \in C$ and $\lambda \in (\dot{0}, \dot{1})_N$, then C is said to be non-Newtonian convex [8].

In the following Definition 3, Definition 4, Theorem 1, Theorem 2 and Theorem 3, $\|.\|_N$ denotes the non-Newtonian norm on the non-Newtonian Banach space X.

Definition 3. A non-Newtonian-Banach space X is said to be non-Newtonian strictly convex, if $\|\lambda \otimes x \oplus (\dot{1} \dot{-} \lambda) \otimes y\|_N \dot{<} \dot{1}$ for all $x, y \in S_X = \{a \in X : \|a\|_N = \dot{1}\}$ with $x \neq y$ and $\lambda \in (\dot{0}, \dot{1})_N$ [8].

Definition 4. A non-Newtonian Banach space X is said to be non-Newtonian uniformly convex, if for any ε with $\dot{0} \dot{<} \varepsilon \dot{\leq} \dot{2}$, the inequalities $\|x\|_N \dot{\leq} \dot{1}$, $\|y\|_N \dot{\leq} \dot{1}$ and $\|x \ominus y\|_N \dot{\geq} \varepsilon$ imply that there exists a $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $\left\|\frac{x \oplus y}{\dot{2}}\right\|_N \dot{\leq} \dot{1} \dot{-} \delta$ [8].

Theorem 1. *Let X be a non-Newtonian Banach space. Then the following statements are equivalent:*
(i) *X is non-Newtonian strictly convex.*
(ii) *For all $x, y \in X$, $x \neq y$ and $\lambda \in (\dot{0}, \dot{1})_N$, $\|\lambda \otimes x \oplus (\dot{1} \dot{-} \lambda) \otimes y\|_N^p \dot{<} \lambda \dot{\times} \|x\|_N^p \dot{+} (\dot{1} \dot{-} \lambda) \dot{\times} \|y\|_N^p$ with $\dot{1} \dot{<} p \dot{<} \infty$* [8].

The following two theorems determine the relationship between the strictly convexity and uniformly convexity.

Theorem 2. *Let X be a non-Newtonian uniformly convex non-Newtonian Banach space. Then the following statements hold:*
(i) *For any r, ε with $r \dot{\geq} \varepsilon \dot{>} \dot{0}$ and $x, y \in X$, the inequalities $\|x\|_N \dot{\leq} r$, $\|y\|_N \dot{\leq} r$ and $\|x \ominus y\|_N \dot{\geq} \varepsilon$ imply that there exists a $\delta = \delta\left(\frac{\varepsilon}{r}\right) \dot{>} \dot{0}$ such that $\left\|\frac{x \oplus y}{\dot{2}}\right\|_N \dot{\leq} r \dot{\times} \left(\dot{1} \dot{-} \delta\left(\frac{\varepsilon}{r}\alpha\right)\right)$.*
(ii) *For any r, ε with $r \dot{\geq} \varepsilon \dot{>} \dot{0}$ and $x, y \in X$, the inequalities $\|x\|_N \dot{\leq} r$, $\|y\|_N \dot{\leq} r$ and $\|x \ominus y\|_N \dot{\geq} \varepsilon$ imply that there exists a $\delta = \delta\left(\frac{\varepsilon}{r}\right) \dot{>} \dot{0}$ such that $\left\|\lambda \otimes x \oplus (\dot{1} \dot{-} \lambda) \otimes y\right\|_N \dot{\leq} r \dot{\times} \left(\dot{1} \dot{-} \dot{2} \dot{\times} \min\{\lambda, \dot{1} \dot{-} \lambda\} \dot{\times} \delta\left(\frac{\varepsilon}{r}\alpha\right)\right)$ for all $\lambda \in (\dot{0}, \dot{1})_N$.*

Proof. (i) Let $\|x\|_N \dot{\leq} r$, $\|y\|_N \dot{\leq} r$ *and* $\|x \ominus y\|_N \dot{\geq} \varepsilon$. Then we have $\left\|\frac{x}{r}\right\|_N \dot{\leq} \dot{1}$, $\left\|\frac{y}{r}\right\|_N \dot{\leq} \dot{1}$ *and* $\left\|\frac{x}{r} \ominus \frac{y}{r}\right\|_N \dot{\geq} \frac{\varepsilon}{r}\alpha \dot{>} \dot{0}$. So, by the definition of a non-Newtonian uniformly convex set, there exists a $\delta = \delta\left(\frac{\varepsilon}{r}\alpha\right) \dot{>} \dot{0}$ such that

$$\left\|\frac{\frac{x}{r} \oplus \frac{y}{r}}{\dot{2}}\right\|_N = \left\|\frac{x \oplus y}{\dot{2} \dot{\times} r}\right\|_N \dot{\leq} \dot{1} \dot{-} \delta\Big(\frac{\varepsilon}{r}\alpha\Big)$$

and thus

$$\left\|\frac{x \oplus y}{\dot{2}}\right\|_N \dot{\leq} r \dot{\times} \left(\dot{1} \dot{-} \delta\Big(\frac{\varepsilon}{r}\alpha\Big)\right).$$

This completes the proof of (i).

(ii) If $\lambda = \frac{\dot{1}}{\dot{2}}\alpha$, then we obtain (i). If $\lambda \in \left(\dot{0}, \frac{\dot{1}}{\dot{2}}\alpha\right)_N$, we conclude that by (i) there exists a $\delta = \delta\left(\frac{\varepsilon}{r}\right) \dot{>} \dot{0}$ such that

$$\begin{aligned}
\left\|\lambda \otimes x \oplus (\dot{1} \ddot{-} \lambda) \otimes y\right\|_N &= \left\|\lambda \otimes (x \oplus y) \oplus \left(\dot{1} \dot{-} \dot{2} \dot{\times} \lambda\right) \otimes y\right\|_N \\
&\dot{\leq} \dot{2} \dot{\times} \lambda \dot{\times} \left\|\frac{x \oplus y}{\dot{2}}\right\|_N \dot{+} \left(\dot{1} \dot{-} \dot{2} \dot{\times} \lambda\right) \dot{\times} \|y\|_N \\
&\dot{\leq} \dot{2} \dot{\times} \lambda \dot{\times} r \dot{\times} \left(\dot{1} \dot{-} \delta\left(\frac{\varepsilon}{r}\alpha\right)\right) \dot{+} \left(\dot{1} \dot{-} \dot{2} \dot{\times} \lambda\right) \dot{\times} r
\end{aligned}$$

$$=r \dot{\times} \left(\dot{1} \dot{-} \dot{2} \dot{\times} \lambda \dot{\times} \delta \left(\frac{\varepsilon}{r}\alpha\right)\right).$$

On the other hand, if $\lambda \in \left(\frac{\dot{1}}{\dot{2}}\alpha, \dot{0}\right)_N$, we conclude that by (i) there exists a $\delta = \delta\left(\frac{\varepsilon}{r}\right) \dot{>} \dot{0}$ such that

$$\begin{aligned}\left\|\lambda \otimes x \oplus \left(\dot{1} \dot{-} \lambda\right) \otimes y\right\|_N &= \left\|\left(\dot{2} \dot{\times} \lambda \dot{-} \dot{1}\right) \otimes x \oplus \left(\dot{1} \dot{-} \lambda\right) \otimes (x \ominus y)\right\|_N \\ &\dot{\leq} \left(\dot{2} \dot{\times} \lambda \dot{-} \dot{1}\right) \dot{\times} \|x\|_N \dot{+} \dot{2} \dot{\times} \left(\dot{1} \dot{-} \lambda\right) \dot{\times} \left\|\frac{x \oplus y}{\dot{2}}\right\|_N \\ &\dot{\leq} \left(\dot{2} \dot{\times} \lambda \dot{-} \dot{1}\right) \dot{\times} r \dot{+} \dot{2} \dot{\times} \dot{\times} r \dot{\times} \left(\dot{1} \dot{-} \delta\left(\frac{\varepsilon}{r}\alpha\right)\right) \\ &= r \dot{\times} \left(\dot{1} \dot{-} \dot{2} \dot{\times} \left(\dot{1} \dot{-} \lambda\right) \dot{\times} \delta\left(\frac{\varepsilon}{r}\alpha\right)\right).\end{aligned}$$

This completes the proof of (ii). □

Theorem 3. *Let X be a non-Newtonian Banach space. If X is non-Newtonian uniformly convex, then X is non-Newtonian strictly convex.*

Proof. Let X be a non-Newtonian uniformly convex non-Newtonian Banach space. Then for any ε with $\dot{0} \dot{<} \varepsilon \dot{\leq} \dot{2}$, the inequalities $\|x\|_N \dot{\leq} \dot{1}, \|y\|_N \dot{\leq} \dot{1}$ and $\|x \ominus y\|_N \dot{\geq} \varepsilon$ imply that there exists a $\delta = \delta(\varepsilon) \dot{>} \dot{0}$ such that $\left\|\frac{x \oplus y}{\dot{2}}\right\|_N \dot{\leq} \dot{1} \dot{-} \delta$. Thus, by Theorem 2 (ii), in case of $r = \dot{1}$, we can write $\left\|\lambda \otimes x \oplus \left(\dot{1} \dot{-} \lambda\right) \otimes y\right\|_N \dot{\leq} \dot{1} \dot{-} \dot{2} \dot{\times} \dot{\min}\{\lambda, \dot{1} \dot{-} \lambda\} \dot{\times} \delta(\varepsilon)$ for all $\lambda \in \left(\dot{0}, \dot{1}\right)_N$. So, since $\dot{2} \dot{\times} \dot{\min}\{\lambda, \dot{1} \dot{-} \lambda\} \dot{\times} \delta(\varepsilon) \dot{>} \dot{0}$, we get $\left\|\lambda \otimes x \oplus \left(\dot{1} \dot{-} \lambda\right) \otimes y\right\|_N \dot{<} \dot{1}$. This shows that X is non-Newtonian strictly convex. □

Now, let us examine some geometric properties of $*-$bicomplex sequence spaces $l_\infty(\mathbb{BC}(N))$ and $l_p(\mathbb{BC}(N))$ for $\ddot{0} \ddot{<} p \ddot{<} \infty,\ p \in \mathbb{R}(N)_\beta$. Firstly, we give two properties related to a non-Newtonian supremum that will be used frequently in the rest of this section.

Proposition 1. *Let $A, B \subset \mathbb{R}(N)_\beta^+$. Define $A \ddot{+} B = \{a \ddot{+} b : a \in A, b \in B\}$, $AB = A \ddot{\times} B = \{a \ddot{\times} b : a \in A, b \in B\}$. If A and B are non-Newtonian bounded above, the followings are true:*

(i) $\ddot{\sup}\left(A \ddot{+} B\right) = \ddot{\sup}A \ddot{+} \ddot{\sup}B$.
(ii) $\ddot{\sup}(AB) = \ddot{\sup}A \ddot{\times} \ddot{\sup}B$.

Proof. The proof depends on the definitions and some properties of a non-Newtonian supremum given in [6, 9]. □

Lemma 2. *Let $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$. Then we have*

$$\ddot{\|}\, \zeta_1^* \oplus_2 \zeta_2^* \,\ddot{\|}_2^{\ddot{2}} \ddot{+} \ddot{\|}\, \zeta_1^* \ominus_2 \zeta_2^* \,\ddot{\|}_2^{\ddot{2}} = \ddot{2} \ddot{\times} \left(\ddot{\|}\, \zeta_1^* \,\ddot{\|}_2^{\ddot{2}} \ddot{+} \ddot{\|}\, \zeta_2^* \,\ddot{\|}_2^{\ddot{2}}\right).$$

Proof. The proof is a direct application of the definition of a non-Newtonian real valued norm $\ddot{\|} \cdot \ddot{\|}_2$. □

Theorem 4. *The sets $\mathbb{BC}(N)$ and $w(\mathbb{BC}(N))$ are non-Newtonian convex.*

Proof. The proof is clear from the definition of a non-Newtonian convex set. □

Theorem 5. *The set $\mathbb{BC}(N)$ is non-Newtonian uniformly convex and strictly convex.*

Proof. Let $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$, $\varepsilon \in \left(\ddot{0}, \ddot{2}\right]_N$, $\ddot{\|}\, \zeta_1^* \,\ddot{\|}_2 \ddot{\leq} \ddot{1}$, $\ddot{\|}\, \zeta_2^* \,\ddot{\|}_2 \ddot{\leq} \ddot{1}$ and $\varepsilon \ddot{\leq} \ddot{\|}\, \zeta_1^* \ominus_2 \zeta_2^* \,\ddot{\|}_2$. Then by using Lemma 2, we have

$$\ddot{\|}\, \zeta_1^* \oplus_2 \zeta_2^* \,\ddot{\|}_2^{\ddot{2}} = \ddot{2} \ddot{\times} \left(\ddot{\|}\, \zeta_1^* \,\ddot{\|}_2^{\ddot{2}} \ddot{+} \ddot{\|}\, \zeta_2^* \,\ddot{\|}_2^{\ddot{2}}\right) \ddot{-} \ddot{\|}\, \zeta_1^* \ominus_2 \zeta_2^* \,\ddot{\|}_2^{\ddot{2}} \ddot{\leq} \ddot{4} \ddot{-} \varepsilon^{\ddot{2}}$$

and so,

$$\begin{aligned}\ddot{\|}\ \frac{\zeta_1^* \oplus_2 \zeta_2^*}{\ddot{2}}\ \ddot{\|}_2 &= \left[\frac{\ddot{1}}{\ddot{2}^{\ddot{2}}}\beta \ddot{\times}\ \ddot{\|}\ \zeta_1^* \oplus_2 \zeta_2^*\ \ddot{\|}_2^{\ddot{2}}\right]^{\frac{\ddot{1}}{\ddot{2}}\beta}\\
&\ddot{\leq}\left[\frac{\ddot{1}}{\ddot{2}^{\ddot{2}}}\beta\ \ddot{\times}\left(\ddot{4}\ \ddot{-}\ \varepsilon^{\ddot{2}}\right)\right]^{\frac{\ddot{1}}{\ddot{2}}\beta}\\
&\ddot{\leq}\left[\ddot{1}\ \ddot{-}\left(\frac{\varepsilon}{\ddot{2}}\beta\right)^{\ddot{2}}\right]^{\frac{\ddot{1}}{\ddot{2}}\beta}.\end{aligned}$$

If we take $\delta(\varepsilon) = \ddot{1}\ \ddot{-}\left[\ddot{1}\ \ddot{-}\left(\frac{\varepsilon}{\ddot{2}}\beta\right)^{\ddot{2}}\right]^{\frac{\ddot{1}}{\ddot{2}}\beta}$, we say that $\mathbb{BC}(N)$ is non-Newtonian uniformly convex and strictly convex by Theorem 3. □

Lemma 3. *Let p be a non-Newtonian real number with $\ddot{1}\ \ddot{<}\ p\ \ddot{<}\ \infty$, $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$, $\zeta_1^* \neq \zeta_2^*$ and $\lambda \in \left(\ddot{0}, \ddot{1}\right)_N$. Then we have*

$$\ddot{\|}\ \lambda \odot_2 \zeta_1^* \oplus_2 \left(\ddot{1}\ \ddot{-}\ \lambda\right) \odot_2 \zeta_2^*\ \ddot{\|}_2^{p} \ddot{<}\ \lambda \ddot{\times}\ \ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{p}\ \ddot{+}\left(\ddot{1}\ \ddot{-}\ \lambda\right)\ddot{\times}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{p}.$$

Proof. The proof is a consequence of Theorem 1 and Theorem 5. □

Lemma 4. *Let p be a non-Newtonian real number with $\ddot{2}\ \ddot{\leq}\ p\ \ddot{<}\ \infty$ and $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$. Then we have*

$$\ddot{\|}\ \zeta_1^* \oplus_2 \zeta_2^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_1^* \ominus_2 \zeta_2^*\ \ddot{\|}_2^{p} \ddot{\leq}\ \ddot{2}^{p \ddot{-} \ddot{1}}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{p}\right).$$

Proof. From Lemma 2.1 in [8], we know

$$\left(\ddot{\|}\ \zeta_1^* \oplus_2 \zeta_2^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_1^* \ominus_2 \zeta_2^*\ \ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}\beta}\ \ddot{\leq}\left(\ddot{\|}\ \zeta_1^* \oplus_2 \zeta_2^*\ \ddot{\|}_2^{\ddot{2}}\ \ddot{+}\ \ddot{\|}\ \zeta_1^* \ominus_2 \zeta_2^*\ \ddot{\|}_2^{\ddot{2}}\right)^{\frac{\ddot{1}}{\ddot{2}}\beta}$$

for all $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$ and $\ddot{2}\ \ddot{\leq}\ p\ \ddot{<}\ \infty$. By Lemma 2,

$$\begin{aligned}&\left(\ddot{\|}\ \zeta_1^* \oplus_2 \zeta_2^*\ \ddot{\|}_2^{\ddot{2}}\ \ddot{+}\ \ddot{\|}\ \zeta_1^* \ominus_2 \zeta_2^*\ \ddot{\|}_2^{\ddot{2}}\right)^{\frac{\ddot{1}}{\ddot{2}}\beta}\ \ddot{\leq}\left(\ddot{2}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{\ddot{2}}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{\ddot{2}}\right)\right)^{\frac{\ddot{1}}{\ddot{2}}\beta}\\
&= \ddot{2}^{\frac{\ddot{1}}{\ddot{2}}\beta}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{\ddot{2}}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{\ddot{2}}\right)^{\frac{\ddot{1}}{\ddot{2}}\beta}.\end{aligned}$$

Then by the non-Newtonian Hölder inequality in [8] for $\frac{\ddot{2}}{p}\beta\ \ddot{+}\ \frac{p \ddot{-} \ddot{2}}{p}\beta = \ddot{1}$, we have

$$\begin{aligned}\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{\ddot{2}}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{\ddot{2}} &\ddot{\leq}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{p}\right)^{\frac{\ddot{2}}{p}\beta}\ \ddot{\times}\left(\ddot{1}\ \ddot{+}\ \ddot{1}\right)^{\frac{p \ddot{-} \ddot{2}}{p}\beta}\\
&= \ddot{2}^{\frac{p \ddot{-} \ddot{2}}{p}\beta}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{p}\right)^{\frac{\ddot{2}}{p}\beta}\end{aligned}$$

and so,

$$\begin{aligned}\ddot{2}^{\frac{\ddot{1}}{\ddot{2}}\beta}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{\ddot{2}}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{\ddot{2}}\right)^{\frac{\ddot{1}}{\ddot{2}}\beta} &\ddot{\leq}\ \ddot{2}^{\frac{\ddot{1}}{\ddot{2}}\beta\ \ddot{+}\ \frac{p \ddot{-} \ddot{2}}{\ddot{2} \ddot{\times} p}\beta}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}\beta}\\
= \quad &\ddot{2}^{\frac{p \ddot{-} \ddot{1}}{p}\beta}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}\beta}.\end{aligned}$$

This implies that $\left(\ddot{\|}\ \zeta_1^* \oplus_2 \zeta_2^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_1^* \ominus_2 \zeta_2^*\ \ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}\beta}\ \ddot{\leq}\ \ddot{2}^{\frac{p \ddot{-} \ddot{1}}{p}\beta}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{p}\right)^{\frac{\ddot{1}}{p}\beta}$. Therefore $\ddot{\|}\ \zeta_1^* \oplus_2 \zeta_2^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_1^* \ominus_2 \zeta_2^*\ \ddot{\|}_2^{p} \ddot{\leq}\ \ddot{2}^{p \ddot{-} \ddot{1}}\ \ddot{\times}\left(\ddot{\|}\ \zeta_1^*\ \ddot{\|}_2^{p}\ \ddot{+}\ \ddot{\|}\ \zeta_2^*\ \ddot{\|}_2^{p}\right)$ for all $\zeta_1^*, \zeta_2^* \in \mathbb{BC}(N)$ and $\ddot{2}\ \ddot{\leq}\ p\ \ddot{<}\ \infty$. The proof is completed. □

Theorem 6. *$l_p(\mathbb{BC}(N))$ for $\ddot{0} \ddot{<} p \ddot{<} \infty$ and $l_\infty(\mathbb{BC}(N))$ are non-Newtonian convex.*

Proof. Let $s = (s_n^*)$, $t = (t_n^*) \in l_p(\mathbb{BC}(N))$ and $\lambda \in \left(\ddot{0}, \ddot{1}\right)_N$. Then $\ddot{\sum}_{n=1}^{\infty} \ddot{\|}\, s_n^* \,\ddot{\|}_2^{\ddot{p}} \ddot{<} \infty$ and $\ddot{\sum}_{n=1}^{\infty} \ddot{\|}\, t_n^* \,\ddot{\|}_2^{\ddot{p}} \ddot{<} \infty$. Therefore we have

$$\begin{aligned}
\ddot{\|}\, \lambda \odot_2 s_n^* \oplus_2 \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2^{\ddot{p}} &\ddot{\leq} \left(\ddot{\|}\, \lambda \odot_2 s_n^* \,\ddot{\|}_2 \ddot{+} \ddot{\|}\, \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2\right)^{p} \\
&\ddot{\leq} \left(\ddot{2} \ddot{\times} \ddot{\max}\left\{\ddot{\|}\, \lambda \odot_2 s_n^* \,\ddot{\|}_2, \ddot{\|}\, \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2\right\}\right)^{p} \\
&= \ddot{2}^{\ddot{p}} \ddot{\times} \ddot{\max}\left\{\ddot{\|}\, \lambda \odot_2 s_n^* \,\ddot{\|}_2^{\ddot{p}}, \ddot{\|}\, \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2^{\ddot{p}}\right\} \\
&= \ddot{2}^{\ddot{p}} \ddot{\times} \left(\ddot{\|}\, \lambda \odot_2 s_n^* \,\ddot{\|}_2^{\ddot{p}} \ddot{+} \ddot{\|}\, \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2^{\ddot{p}}\right) \\
&= \ddot{2}^{\ddot{p}} \ddot{\times} \left(\lambda^{p} \ddot{\times} \ddot{\|}\, s_n^* \,\ddot{\|}_2^{\ddot{p}} \ddot{+} \left(\ddot{1} \ddot{-} \lambda\right)^{p} \ddot{\times} \ddot{\|}\, t_n^* \,\ddot{\|}_2^{\ddot{p}}\right)
\end{aligned}$$

which implies that $\lambda \odot s \oplus \left(\ddot{1} \ddot{-} \lambda\right) \odot t \in l_p(\mathbb{BC}(N))$.

Let $s = (s_n^*)$, $t = (t_n^*) \in l_\infty(\mathbb{BC}(N))$ and $\lambda \in \left(\ddot{0}, \ddot{1}\right)_N$. Then $\ddot{\sup}_{n\in\mathbb{N}} \ddot{\|}\, s_n^* \,\ddot{\|}_2 \ddot{<} \infty$ and $\ddot{\sup}_{n\in\mathbb{N}} \ddot{\|}\, t_n^* \,\ddot{\|}_2 \ddot{<} \infty$. Then by Proposition 1 (i) and (ii), we have

$$\begin{aligned}
\ddot{\sup}_{n\in\mathbb{N}} \ddot{\|}\, \lambda \odot_2 s_n^* \oplus_2 \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2 &\ddot{\leq} \ddot{\sup}_{n\in\mathbb{N}} \left(\ddot{\|}\, \lambda \odot_2 s_n^* \,\ddot{\|}_2 \ddot{+} \ddot{\|}\, \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2\right) \\
&\ddot{\leq} \ddot{\sup}_{n\in\mathbb{N}} \left(\lambda \ddot{\times} \ddot{\|}\, s_n^* \,\ddot{\|}_2 \ddot{+} \left(\ddot{1} \ddot{-} \lambda\right) \ddot{\times} \ddot{\|}\, t_n^* \,\ddot{\|}_2\right) \\
&= \lambda \ddot{\times} \ddot{\sup}_{n\in\mathbb{N}} \ddot{\|}\, s_n^* \,\ddot{\|}_2 \ddot{+} \left(\ddot{1} \ddot{-} \lambda\right) \ddot{\times} \ddot{\sup}_{n\in\mathbb{N}} \ddot{\|}\, t_n^* \,\ddot{\|}_2,
\end{aligned}$$

which implies that $\lambda \odot s \oplus \left(\ddot{1} \ddot{-} \lambda\right) \odot t \in l_\infty(\mathbb{BC}(N))$. Consequently, $l_p(\mathbb{BC}(N))$ for $\ddot{0} \ddot{<} p \ddot{<} \infty$ and $l_\infty(\mathbb{BC}(N))$ are non-Newtonian convex. □

Theorem 7. *$l_p(\mathbb{BC}(N))$ for $\ddot{1} \ddot{<} p \ddot{<} \infty$ is non-Newtonian strictly convex.*

Proof. Let $s = (s_n^*)$, $t = (t_n^*) \in S_{l_p(\mathbb{BC}(N))}$, $s \neq t$ and $\lambda \in \left(\ddot{0}, \ddot{1}\right)_N$. Then by Lemma 3, we get

$$\begin{aligned}
\ddot{\|}\, \lambda \odot s \oplus \left(\ddot{1} \ddot{-} \lambda\right) \odot t \,\ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}} &= \ddot{\sum_{n=1}^{\infty}} \ddot{\|}\, \lambda \odot_2 s_n^* \oplus_2 \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2^{\ddot{p}} \\
&\ddot{<} \ddot{\sum_{n=1}^{\infty}} \left(\lambda \ddot{\times} \ddot{\|}\, s_n^* \,\ddot{\|}_2^{\ddot{p}} \ddot{+} \left(\ddot{1} \ddot{-} \lambda\right) \ddot{\times} \ddot{\|}\, t_n^* \,\ddot{\|}_2^{\ddot{p}}\right) \\
&= \lambda \ddot{\times} \ddot{\sum_{n=1}^{\infty}} \ddot{\|}\, s_n^* \,\ddot{\|}_2^{\ddot{p}} \ddot{+} \left(\ddot{1} \ddot{-} \lambda\right) \ddot{\times} \ddot{\sum_{n=1}^{\infty}} \ddot{\|}\, t_n^* \,\ddot{\|}_2^{\ddot{p}} \\
&= \lambda \ddot{\times} \ddot{\|}\, s \,\ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}} \ddot{+} \left(\ddot{1} \ddot{-} \lambda\right) \ddot{\times} \ddot{\|}\, t \,\ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}} \\
&= 1,
\end{aligned}$$

which implies that $l_p(\mathbb{BC}(N))$ for $\ddot{1} \ddot{<} p \ddot{<} \infty$ is non-Newtonian strictly convex. □

Example 1. $l_\infty(\mathbb{BC}(N))$ is not non-Newtonian strictly convex.

Solution 1. Let

$$(s_n^*) = (1^*, j^*, 0^*, 0^*, \dots), \quad (t_n^*) = (\ominus_2 1^*, j^*, 0^*, 0^*, \dots).$$

Then $\ddot{\|}\, s \,\ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} = \ddot{\|}\, t \,\ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} = \ddot{1}$ and

$$\ddot{\|}\, \lambda \odot s \oplus \left(\ddot{1} \ddot{-} \lambda\right) \odot t \,\ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} = \ddot{\sup}_{n\in\mathbb{N}} \ddot{\|}\, \lambda \odot_2 s_n^* \oplus_2 \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^* \,\ddot{\|}_2$$

$$
\begin{aligned}
&= \ddot{\sup_{n\in\mathbb{N}}} \ddot{\|} \left(\left(\ddot{2} \ddot{\times} \lambda \ddot{-} \ddot{1}\right) \odot_2 1^*, j^*, 0^*, 0^*, \dots\right) \ddot{\|}_2 \\
&= \ddot{\sup_{n\in\mathbb{N}}} \left\{ \left|\ddot{2} \ddot{\times} \lambda \ddot{-} \ddot{1}\right|_\beta, \ddot{1} \right\} = \ddot{1}
\end{aligned}
$$

for all $\lambda \in \left(\ddot{0}, \ddot{1}\right)_N$. That is to say that $l_\infty(\mathbb{BC}(N))$ is not non-Newtonian strictly convex.

Example 2. $l_1(\mathbb{BC}(N))$ is not non-Newtonian strictly convex.

Solution 2. Let

$$
(s_n^*) = (i^*, 0^*, 0^*, \dots), \quad (t_n^*) = (0^*, \ominus_2 i^*, 0^*, 0^*, \dots).
$$

Then $\ddot{\|}\, s\, \ddot{\|}_{2,l_1(\mathbb{BC}(N))} = \ddot{\|}\, t\, \ddot{\|}_{2,l_1(\mathbb{BC}(N))} = \ddot{1}$ and

$$
\begin{aligned}
\ddot{\|}\, \lambda \odot s \oplus \left(\ddot{1} \ddot{-} \lambda\right) \odot t\, \ddot{\|}_{2,l_1(\mathbb{BC}(N))} &= \ddot{\sum_{n=1}^{\infty}} \ddot{\|}\, \lambda \odot_2 s_n^* \oplus_2 \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 t_n^*\, \ddot{\|}_2 \\
&= \ddot{\|}\, \lambda \odot_2 i^*\, \ddot{\|}_2 \ddot{+} \ddot{\|} \left(\ddot{1} \ddot{-} \lambda\right) \odot_2 (\ominus_2 i^*)\, \ddot{\|}_2 \\
&= \lambda \ddot{+} \left(\ddot{1} \ddot{-} \lambda\right) = \ddot{1}
\end{aligned}
$$

for all $\lambda \in \left(\ddot{0}, \ddot{1}\right)_N$. That is to say that $l_1(\mathbb{BC}(N))$ is not non-Newtonian strictly convex.

Theorem 8. *$l_p(\mathbb{BC}(N))$ for $\ddot{2} \ddot{\leq} p \ddot{<} \infty$ is non-Newtonian uniformly convex.*

Proof. Let $s = (s_n^*), t = (t_n^*) \in l_p(\mathbb{BC}(N)), \varepsilon \in \left(\ddot{0}, \ddot{2}\right]_N, \ddot{\|}\, s\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))} \ddot{\leq} \ddot{1}, \ddot{\|}\, t\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))} \ddot{\leq} \ddot{1}$ and $\varepsilon \ddot{\leq} \ddot{\|}\, s \ominus t\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}$. Then by Lemma 4, we have

$$
\begin{aligned}
\ddot{\|}\, s \oplus t\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}} \ddot{+} \ddot{\|}\, s \ominus t\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}} &= \ddot{\sum_{n=1}^{\infty}} \ddot{\|}\, s_n^* \oplus_2 t_n^*\, \ddot{\|}_2^{\ddot{p}} \ddot{+} \ddot{\sum_{n=1}^{\infty}} \ddot{\|}\, s_n^* \ominus_2 t_n^*\, \ddot{\|}_2^{\ddot{p}} \\
&= \ddot{\sum_{n=1}^{\infty}} \left(\ddot{\|}\, s_n^* \oplus_2 t_n^*\, \ddot{\|}_2^{\ddot{p}} \ddot{+} \ddot{\|}\, s_n^* \ominus_2 t_n^*\, \ddot{\|}_2^{\ddot{p}}\right) \\
&\ddot{\leq} \ddot{\sum_{n=1}^{\infty}} \ddot{2}^{p \ddot{-} \ddot{1}} \ddot{\times} \left(\ddot{\|}\, s_n^*\, \ddot{\|}_2^{\ddot{p}} \ddot{+} \ddot{\|}\, t_n^*\, \ddot{\|}_2^{\ddot{p}}\right) \\
&= \ddot{2}^{p \ddot{-} \ddot{1}} \ddot{\times} \left(\ddot{\sum_{n=1}^{\infty}} \ddot{\|}\, s_n^*\, \ddot{\|}_2^{\ddot{p}} \ddot{+} \ddot{\sum_{n=1}^{\infty}} \ddot{\|}\, s_n^*\, \ddot{\|}_2^{\ddot{p}}\right) \\
&= \ddot{2}^{p \ddot{-} \ddot{1}} \ddot{\times} \left[\ddot{\|}\, s\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}} \ddot{+} \ddot{\|}\, t\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}}\right] \\
&\ddot{\leq} \ddot{2}^{\ddot{p}}.
\end{aligned}
$$

Thus we can write

$$
\ddot{\|}\, s \oplus t\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}} \ddot{\leq} \ddot{2}^{\ddot{p}} \ddot{-} \ddot{\|}\, s \ominus t\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}} \ddot{\leq} \ddot{2}^{\ddot{p}} \ddot{-} \varepsilon^p,
$$

and so,

$$
\begin{aligned}
\ddot{\|}\, \frac{s \oplus t}{\ddot{2}}\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))} &= \left[\frac{\ddot{1}}{\ddot{2}^{\ddot{p}}} \beta \ddot{\times} \ddot{\|}\, s \oplus t\, \ddot{\|}_{2,l_p(\mathbb{BC}(N))}^{\ddot{p}}\right]^{\frac{\ddot{1}}{p}\beta} \\
&\ddot{\leq} \left[\ddot{1} \ddot{-} \left(\frac{\varepsilon}{\ddot{2}}\beta\right)^p\right]^{\frac{\ddot{1}}{p}\beta}.
\end{aligned}
$$

If we take $\delta(\varepsilon) = \ddot{1} \ddot{-} \left[\ddot{1} \ddot{-} \left(\frac{\varepsilon}{\ddot{2}}\beta\right)^p\right]^{\frac{\ddot{1}}{p}\beta}$, we say that $l_p(\mathbb{BC}(N))$ for $\ddot{2} \ddot{\leq} p \ddot{<} \infty$ is non-Newtonian uniformly convex. $\square$

Example 3. $l_\infty(\mathbb{BC}(N))$ is not non-Newtonian uniformly convex.

Solution 3. Let

$$(s_n^*) = (i^*, j^*, i^*, 0^*, 0^*, \dots), \quad (t_n^*) = (i^*, j^*, \ominus_2 i^*, 0^*, 0^*, \dots).$$

Then, $\ddot{\|}\ s\ \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} = \ddot{\|}\ t\ \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} = \ddot{1}$ and

$$\begin{aligned}\ddot{\|}\ s \ominus t\ \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} &= \ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ s_n^* \ominus_2 t_n^*\ \ddot{\|}_2 \\ &= \ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ \left(0^*, 0^*, \ddot{2} \odot_2 i^*, 0^*, 0^*, \dots\right)\ \ddot{\|}_2 \\ &= \ddot{\sup_{n\in\mathbb{N}}}\left\{\ddot{0}, \ddot{2}\right\} = \ddot{2}\end{aligned}$$

and $\varepsilon\ \ddot{\leq}\ \ddot{\|}\ s \ominus t\ \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} = \ddot{2}$. On the other hand,

$$\begin{aligned}\ddot{\|}\ \frac{s \oplus t}{\ddot{2}}\ \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} &= \ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ \frac{s_n^* \oplus_2 t_n^*}{\ddot{2}}\ \ddot{\|}_2 \\ &= \ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ (i^*, j^*, 0^*, 0^*, \dots)\ \ddot{\|}_2 = \ddot{1}.\end{aligned}$$

Thus there doesn't exist $\delta(\varepsilon)\ \ddot{>}\ \ddot{0}$ such that $\ddot{\|}\ \frac{s\oplus t}{\ddot{2}}\ \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))}\ \ddot{\leq}\ \ddot{1}\ \ddot{-}\ \delta$. That is to say that $l_\infty(\mathbb{BC}(N))$ is not non-Newtonian uniformly convex.

Example 4. $l_1(\mathbb{BC}(N))$ is not non-Newtonian uniformly convex.

Solution 4. Let

$$(s_n^*) = (i^*, 0^*, 0^*, \dots), \quad (t_n^*) = (0^*, \ominus_2 j^*, 0^*, 0^*, \dots).$$

Then $\ddot{\|}\ s\ \ddot{\|}_{2,l_1(\mathbb{BC}(N))} = \ddot{\|}\ t\ \ddot{\|}_{2,l_1(\mathbb{BC}(N))} = \ddot{1}$ and

$$\ddot{\|}\ s \ominus t\ \ddot{\|}_{2,l_1(\mathbb{BC}(N))} = \ddot{\sum_{n=1}^{\infty}}\ \ddot{\|}\ s_n^* \ominus_2 t_n^*\ \ddot{\|}_2 = \ddot{\|}\ i^*\ \ddot{\|}_2\ \ddot{+}\ \ddot{\|}\ \ominus_2 j^*\ \ddot{\|}_2 = \ddot{2}$$

and $\varepsilon\ \ddot{\leq}\ \ddot{\|}\ s \ominus t\ \ddot{\|}_{2,l_1(\mathbb{BC}(N))} = \ddot{2}$. On the other hand,

$$\ddot{\|}\ \frac{s \oplus t}{\ddot{2}}\ \ddot{\|}_{2,l_1(\mathbb{BC}(N))} = \ddot{\sum_{n=1}^{\infty}}\ \ddot{\|}\ \frac{s_n^* \oplus_2 t_n^*}{\ddot{2}}\ \ddot{\|}_2 = \ddot{\|}\ \frac{i^*}{\ddot{2}}\ \ddot{\|}_2\ \ddot{+}\ \ddot{\|}\ \frac{\ominus_2 j^*}{\ddot{2}}\ \ddot{\|}_2 = \ddot{1}.$$

Thus there doesn't exist $\delta(\varepsilon)\ \ddot{>}\ \ddot{0}$ such that $\ddot{\|}\ \frac{s\oplus t}{\ddot{2}}\ \ddot{\|}_{2,l_1(\mathbb{BC}(N))}\ \ddot{\leq}\ \ddot{1}\ \ddot{-}\ \delta$. That is to say that $l_1(\mathbb{BC}(N))$ is not non-Newtonian uniformly convex.

Now, we define the new topological concepts as follows:

Definition 5. Let X be a $*-$bicomplex sequence space with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_2$ and

$$\widetilde{X} := \left\{(s_n^*) \in w(\mathbb{BC}(N)) : \text{there exists } (t_n^*) \in X \text{ such that } \ddot{\|}\ s_n^*\ \ddot{\|}_2\ \ddot{\leq}\ \ddot{\|}\ t_n^*\ \ddot{\|}_2 \text{ for all } n \in \mathbb{N}\right\}.$$

If $\widetilde{X} \subset X$, then X is called a non-Newtonian bicomplex solid space.

Definition 6. Let X be a Banach $*-$bicomplex sequence space with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_2$. If $\zeta_l^{*(n)} \to \zeta_l^*$ as $n \to \infty$ for all $l \in \mathbb{N}$ with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_2$, whenever $\zeta^{(n)} \to \zeta$ as $n \to \infty$ with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_{2,X}$, then X is called a non-Newtonian bicomplex $BK-$space.

Definition 7. Let X be a $*-$bicomplex sequence space with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_2$ and $\pi := \{f | f : \mathbb{N} \to \mathbb{N}\text{is one-to-one and onto}\}$. If $s_\sigma = \left(s^*_{\sigma(n)}\right) \in X$, whenever $\left(s^*_n\right) \in X$ and $\sigma \in \pi$, then X is called a non-Newtonian bicomplex symmetric space.

The following theorems present the properties of being non-Newtonian bicomplex solid space, non-Newtonian bicomplex BKspace, non-Newtonian bicomplex symmetric space of $*-$bicomplex sequence spaces $l_\infty(\mathbb{BC}(N))$ and $l_p(\mathbb{BC}(N))$ for $\ddot{0} \ddot{<} p \ddot{<} \infty$, $p \in \mathbb{R}(N)_\beta$ by using the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_2$.

Theorem 9. *$l_\infty(\mathbb{BC}(N))$ is a non-Newtonian bicomplex solid space.*

Proof. Let $(s^*_n) \in l_\infty(\widetilde{\mathbb{BC}}(N))$ be arbitrary. Then there exists $(t^*_n) \in l^{\Bbbk}_\infty(\mathbb{BC})$ such that $\ddot{\|}\ s^*_n\ \ddot{\|}_2 \ddot{\leq} \ddot{\|}\ t^*_n\ \ddot{\|}_2$ for all $n \in \mathbb{N}$. Therefore $\ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ t^*_n\ \ddot{\|}_2 \ddot{<} \infty$, and so, $\ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ s^*_n\ \ddot{\|}_2 \ddot{<} \infty$. This shows that $(s^*_n) \in l_\infty(\mathbb{BC}(N))$. Then we have the inclusion $l_\infty(\widetilde{\mathbb{BC}}(N)) \subset l_\infty(\mathbb{BC}(N))$ which means that $l_\infty(\mathbb{BC}(N))$ is a non-Newtonian bicomplex solid space. □

Theorem 10. *$l_\infty(\mathbb{BC}(N))$ is a non-Newtonian bicomplex BK$-$space.*

Proof. Let $\left(\zeta^{(n)}\right) \in l_\infty(\mathbb{BC}(N))$ such that $\zeta^{(n)} \to \zeta$ as $n \to \infty$ with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))}$. Then for every $\varepsilon \ddot{>} \ddot{0}$, there exists $n_0 \in \mathbb{N}$ such that $\ddot{\|}\ \zeta^{(n)} - \zeta\ \ddot{\|}_{2,l_\infty(\mathbb{BC}(N))} \ddot{<} \varepsilon$ for all $n \geq n_0$. Thus we have that $\ddot{\sup_{l\in\mathbb{N}}}\ \ddot{\|}\ \zeta^{*(n)}_l \ominus_2 \zeta^*_l\ \ddot{\|}_2 \ddot{<} \varepsilon$ for every $\varepsilon \ddot{>} \ddot{0}$ and for all $n \geq n_0$. So, for any fixed $l \in \mathbb{N}$, we write $\ddot{\|}\ \zeta^{*(n)}_l \ominus_2 \zeta^*_l\ \ddot{\|}_2 \ddot{<} \varepsilon$ for every $\varepsilon \ddot{>} \ddot{0}$ and for all $n \geq n_0$ which means that $\left(\zeta^{*(n)}_l\right)$ converges to the non-Newtonian bicomplex number ζ^*_l with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_2$. Thus the coordinates are continuous on $l_\infty(\mathbb{BC}(N))$, as required. □

Theorem 11. *$l_\infty(\mathbb{BC}(N))$ is a non-Newtonian bicomplex symmetric space.*

Proof. Let $(s^*_n) \in l_\infty(\mathbb{BC}(N))$ and $\sigma \in \pi$. Then since $\sigma : \mathbb{N} \to \mathbb{N}$ is an injective and surjective function, we have $\left\{\ \ddot{\|}\ s^*_{\sigma(n)}\ \ddot{\|}_2 : n \in \mathbb{N}\right\} = \left\{\ \ddot{\|}\ s^*_n\ \ddot{\|}_2 : n \in \mathbb{N}\right\}$. Thus the equality $\ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ s^*_{\sigma(n)}\ \ddot{\|}_2 = \ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ s^*_n\ \ddot{\|}_2$ holds. Since $\ddot{\sup_{n\in\mathbb{N}}}\left\{\ \ddot{\|}\ s^*_n\ \ddot{\|}_2 : n \in \mathbb{N}\right\} \ddot{<} \infty$, we have $\ddot{\sup_{n\in\mathbb{N}}}\ \ddot{\|}\ s^*_{\sigma(n)}\ \ddot{\|}_2 \ddot{<} \infty$. This means that $\left(s^*_{\sigma(n)}\right) \in l_\infty(\mathbb{BC}(N))$ and we get the required result. □

Theorem 12. *$l_p(\mathbb{BC}(N))$ for $\ddot{0} \ddot{<} p \ddot{<} \infty$ is a non-Newtonian bicomplex solid space.*

Proof. Let $(s^*_n) \in l_p(\widetilde{\mathbb{BC}}(N))$ be arbitrary. Then there exists $(t^*_n) \in l_p(\mathbb{BC}(N))$ such that $\ddot{\|}\ s^*_n\ \ddot{\|}_2 \ddot{\leq} \ddot{\|}\ t^*_n\ \ddot{\|}_2$ for all $n \in \mathbb{N}$ and so, $\ddot{\|}\ s^*_n\ \ddot{\|}_2^{\ddot{p}} \ddot{\leq} \ddot{\|}\ t^*_n\ \ddot{\|}_2^{\ddot{p}}$ for all $n \in \mathbb{N}$. Therefore the $\beta-$series $\ddot{\sum_{n=1}^{\infty}}\ \ddot{\|}\ t^*_n\ \ddot{\|}_2^{\ddot{p}}$ is convergent, the comparison test implies that $\ddot{\sum_{n=1}^{\infty}}\ \ddot{\|}\ s^*_n\ \ddot{\|}_2^{\ddot{p}}$ converges. This shows that $(s^*_n) \in l_p(\mathbb{BC}(N))$. Then we have the inclusion $l_p(\widetilde{\mathbb{BC}}(N)) \subset l_p(\mathbb{BC}(N))$ which means that $l_p(\mathbb{BC}(N))$ for $\ddot{0} \ddot{<} p \ddot{<} \infty$ is a non-Newtonian bicomplex solid space. □

Theorem 13. *$l_p(\mathbb{BC}(N))$ for $\ddot{1} \ddot{\leq} p \ddot{<} \infty$ is a non-Newtonian bicomplex BK$-$space.*

Proof. Let $\left(\zeta^{(n)}\right) \in l_p(\mathbb{BC}(N))$ such that $\zeta^{(n)} \to \zeta$ as $n \to \infty$ with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_{2,l_p(\mathbb{BC}(N))}$. Then for every $\ddot{0} \ddot{<} \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that $\ddot{\|}\ \zeta^{(n)} - \zeta\ \ddot{\|}_{2,l_p(\mathbb{BC}(N))} \ddot{<} \varepsilon$ for all $n \geq n_0$. Thus we have $\left(\ddot{\sum_{n=1}^{\infty}}\ \ddot{\|}\ \zeta^{*(n)}_l \ominus_2 \zeta^*_l\ \ddot{\|}_2^{\ddot{p}}\right)^{\frac{\ddot{1}}{p}\beta} \ddot{<} \varepsilon$ for every $\ddot{0} \ddot{<} \varepsilon$ and for all $n \geq n_0$.

So, for any fixed $l \in \mathbb{N}$, we write $\ddot{\|}\ \zeta_l^{*(n)} \ominus_2 \zeta_l^* \ \ddot{\|}_2^{\ddot{p}} \ddot{<} \varepsilon^{\ p}$ and hence $\ddot{\|}\ \zeta_l^{*(n)} \ominus_2 \zeta_l^* \ \ddot{\|}_2 \ddot{<} \varepsilon$ for every $\ddot{0} \ddot{<} \varepsilon$ and for all $n \geq n_0$ which means that $\left(\zeta_l^{*(n)}\right)$ converges to the non-Newtonian bicomplex number ζ_l^* with respect to the non-Newtonian real valued norm $\ddot{\|}\ .\ \ddot{\|}_2$. Thus the coordinates are continuous on $l_p\left(\mathbb{BC}\left(N\right)\right)$ for $\ddot{1} \ddot{\leq} p \ddot{<} \infty$, and we get the required result. □

Theorem 14. *$l_p\left(\mathbb{BC}\left(N\right)\right)$ for $\ddot{0} \ddot{<} p \ddot{<} \infty$ is a non-Newtonian bicomplex symmetric space.*

Proof. Let $(s_n^*) \in l_p\left(\mathbb{BC}\left(N\right)\right)$ and $\sigma \in \pi$. Then since $\sigma : \mathbb{N} \to \mathbb{N}$ is an injective and surjective function, we have the equalities $\left\{\ \ddot{\|}\ s_{\sigma(n)}^* \ \ddot{\|}_2 : n \in \mathbb{N}\right\} = \left\{\ \ddot{\|}\ s_n^* \ \ddot{\|}_2 : n \in \mathbb{N}\right\}$ and $\left\{\ \ddot{\|}\ s_{\sigma(n)}^* \ \ddot{\|}_2^{\ddot{p}} : n \in \mathbb{N}\right\} = \left\{\ \ddot{\|}\ s_n^* \ \ddot{\|}_2^{\ddot{p}} : n \in \mathbb{N}\right\}$. So, we can write $\ddot{\sum\limits_{n=1}^{\infty}}\ \ddot{\|}\ s_{\sigma(n)}^* \ \ddot{\|}_2^{\ddot{p}} = \ddot{\sum\limits_{n=1}^{\infty}}\ \ddot{\|}\ s_n^* \ \ddot{\|}_2^{\ddot{p}}$. Thus, since $\ddot{\sum\limits_{n=1}^{\infty}}\ \ddot{\|}\ s_n^* \ \ddot{\|}_2^{\ddot{p}}$ converges, we obtain that $\ddot{\sum\limits_{n=1}^{\infty}}\ \ddot{\|}\ s_{\sigma(n)}^* \ \ddot{\|}_2^{\ddot{p}}$ converges. This means that $\left(s_{\sigma(n)}^*\right) \in l_p\left(\mathbb{BC}\left(N\right)\right)$, as required. □

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