Transactions of A. Razmadze
Mathematical Institute
Vol. **176** (2022), issue 2, 163–171

A BENDING PROBLEM OF AN INFINITE PLATE WEAKENED BY TWO IDENTICAL HOLES

ZURAB ABASHIDZE

Abstract. In this paper, we consider the bending problem for an infinite plate weakened by two square-shaped holes. The contours of the holes have the grooves at the vertices of the square along the smooth contours and are the sought part of the plate boundary. Applying the theory of functions of a complex variable and the conformal mapping theory, this problem is reduced to a boundary problem of the analytic function theory. A plate deflection and an unknown part of the boundary are found, assuming that the tangential normal moment on it is a constant value.

1. Introduction

Let us consider a homogeneous, isotropic, infinite plate weakened by two identical square-shaped holes. The contours of the holes have the grooves at the vertices of the square along the smooth contours and are the sought part of the plate boundary. The sides (straight lines) of the square are known and we denote them by l_1, while the unknown part of the boundary we denote by l_2. The entire boundary $l_1 \cup l_2$ of the plate is denoted by l.

It is assumed that the middle plane of the plate lies in the complex plane $z = x + iy$, where it occupies the domain S. Let S be a symmetric domain with respect to the coordinates of $Ox-$ and $Oy-$axes, and the lines of which the contour l_1 consists, are parallel to the coordinates of $Ox-$ and $Oy-$axes.

The points of connection of the contours l_1 and l_2, counted in the positive direction, are denote by $A_1, A_2, \dots, A_k$. Assume that A_1 is the starting point of some linear part of l_1. The arc abscissa of the point $t \in l$, calculated from point A_1, is denoted by s.

Assume that the normal bending moments acting at infinity are the known values and the torque is equal to zero:

$$M_x^\infty = M_1, \quad M_y^\infty = M_2, \quad M_{xy}^\infty = 0. \tag{1.1}$$

On the contour l, we have the conditions

$$\begin{aligned} &\frac{\partial w(t)}{\partial n} = d_k, \qquad d_k = tg\beta_k, \quad t \in l_1, \\ &N(t) = 0, \qquad\qquad\qquad\quad\; t \in l_1. \end{aligned} \tag{1.2}$$

$$\begin{aligned} &M_n(t) = 0, \qquad M_{ns}(t) = 0, \quad t \in l_2, \\ &N(t) = 0, \qquad\qquad\qquad\quad\;\; t \in l_2, \end{aligned} \tag{1.3}$$

where n is the outward normal; β_k are the constants (rotation angles); $N(t)$ is the cutting force; $M_n(t)$ is the normal bending moment; $M_{ns}(t)$ is the torque; t is a point of the contour; $w(x;y)$ is a plate deflection at the point (x, y).

2. Setting of the Problem and Reduction to Boundary Problem of the Theory of Analytical Functions

Let us consider the following problem: Under conditions (1.1)–(1.3) find a plate deflection and an unknown part of the boundary which is the contour l_2, assuming that the tangential normal

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2020 *Mathematics Subject Classification.* 74B05.
Key words and phrases. Cutting force; Normal bending moment; Plate deflection; Conformal mapping; Linear conjugation value problem.

moment on it is a constant value

$$M_s(t) = k, \quad t \in l_2. \tag{2.1}$$

According to the approximate theory of plate deflection, in the case under consideration, the function $w(x;y)$ satisfies the biharmonic equation

$$\Delta^2 w = 0, \quad z \in S. \tag{2.2}$$

As is known, the solution of a biharmonic equation has the form [1]:

$$w(x;y) = \operatorname{Re}\left(\overline{z}\phi(z) + \chi(z)\right), \quad z \in S, \tag{2.3}$$

where $\phi(z)$ and $\chi(x)$ are holomorphic functions in the domain S.

By (2.3) we obtain the equality

$$\frac{\partial w}{\partial n} = \operatorname{Re}\Big(i\frac{\partial \overline{t}}{\partial s}\big(\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)}\big)\Big), \tag{2.4}$$

where $\psi(z) = \chi'(z)$.

By virtue of condition (1.2), from equality (2.4), we obtain

$$\operatorname{Re}\Big(e^{-i\alpha(t)}\big(\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)}\big)\Big) = d(t), \quad t \in l_1, \tag{2.5}$$

where $\alpha(t)$ is the angle between the Ox-axis and the outward normal to the contour l_1 at the point t; $d(t) = d_k$ when $t \in A_k A_{k+1}$, $k = 1;3;5;7$.

Taking into account condition (1.2) and using the formula [2,3]

$$(1-\alpha)d\Big(\varkappa\phi(t) - t\overline{\phi'(t)} - \overline{\psi(t)}\Big) = \Big\{M_n + i\int\limits_{S_1}^{S} N ds\Big\}dz, \quad t \in l_1, \tag{$*$}$$

we have

$$\operatorname{Re}\Big(e^{-i\alpha(t)}\big(\varkappa\phi(t) - t\overline{\phi'(t)} - \overline{\psi(t)}\big)\Big) = C(t), \quad t \in l_1, \tag{2.6}$$

where $C(t)$ is the value of the piecewise-constant function at the point t; $C(t) = C_k$, when $t \in A_k A_{k+1}$, $k = 1;3;5;7$;

$$C_k = \sum_{j=1}^{k}{}' \sin(\alpha_k - \alpha_j) M_j,$$

where the prime at the sum sign means that the operation of summation involves only those values of j, for which the line $A_j A_{j+1}$ is included in the contour l_1;

$$M_j = \int\limits_{S_j}^{S_{j+1}} M_n ds$$

is the principal bending moment which acts on the contour $A_j A_{j+1}$ ($j = 1;3;5;7$); $\varkappa = \frac{\sigma+3}{\sigma-1}$; σ is Poisson's ratio. Summing equalities (2.5) and (2.6) and differentiating with respect to the arc abscissa s, we obtain the condition

$$\operatorname{Im}\phi'(t) = 0, \quad t \in l_1. \tag{2.7}$$

For the bending moment components, we have the equality [4]

$$M_n + M_s = M_x + M_y = -2D(1+\sigma)\big(\phi'(t) + \overline{\phi'(t)}\big), \tag{2.8}$$

where M_x and M_y are the bending moments; $D = \frac{Eh^3}{12(1-\sigma^2)}$ is the plate cylindrical rigidity; E is Young's modulus.

Using formula (2.8) and taking into account conditions (1.3) and (2.1), on the contour 1_2, we obtain the following condition:

$$\operatorname{Re}\phi'(t) = \frac{k}{-4D(1+\sigma)}, \quad t \in l_2.$$

By the same formula (2.8), with conditions (1.1) taken into account, at the point at infinity, we obtain the condition

$$\operatorname{Re}\phi'(t) = \frac{M_1 + M_2}{-4D(1+\sigma)} = \text{const}\,.$$

Thus for the function $\phi'(t)$, holomorphic in the domain S and bounded at the point at infinity, we obtain the following boundary conditions:

$$\operatorname{Im}\phi'(t) = 0, \qquad t \in l_1$$
$$\operatorname{Re}\phi'(t) = \frac{k}{-4D(1+\sigma)}, \qquad t \in l_2.$$

A solution of the obtained mixed problem have the form

$$\phi(z) = P \cdot z, \tag{2.9}$$

where $P = -\frac{k}{4D(1+\sigma)}$.

It remains to define the contour l_2 and the function $\psi(z)$.

For the cutting force, we have the formula [4, 5]

$$N(t) = -D(\Delta w)_x, \tag{2.10}$$

where the point t lies on the Ox-axis.

Due to the symmetry of the plate with respect to the Oy-axis, we can take the condition

$$w(x;y) = w(-x;y). \tag{2.11}$$

By virtue of (2.10) and (2.11), we obtain the equality

$$N(0;y) = 0. \tag{2.12}$$

Thus the cutting force on the Oy-axis is equal to zero.

From (2.11), we obtain

$$w_x(x;y) = -w_x(-x;y). \tag{2.13}$$

As a result, we obtain $w_x(0;y) = 0$.

Thus in the middle plane, on the straight line $\operatorname{Re} z = 0$, the normal to the middle plane does not rotate with respect to the Ox-axis.

By the symmetry of the plate with respect to the Ox-axis, we have the relation

$$w(x;y) = w(x;-y) \quad \text{and therefore} \quad w_y(x;0) = 0.$$

Thus in the middle plane, on the Ox-axis, the normal to the mean surface does not rotate with respect to the Oy-axis:

$$w(x;y) = w(-x;y) = w(-x;-y).$$

The obtained equality can be written also in the form $w(z) = w(-z)$, where z is a point on the mean plane.

Thus equality is fulfilled $w(ze^{i\beta}) = e^{-2i\beta}w(z)$, β is the cyclic symmetry angle and in our case, $\beta = \pi$.

The vectors of stresses acting at the symmetric points $(x;y)$ and $(-x;y)$ are symmetric, which implies that

$$\sigma_x(x;y) = -\sigma_x(-x;y).$$

When the points $(x;y)$ and $(-x;y)$, symmetric with respect to the Oy-axis, tend to one and the same point $(0;y)$ on the Oy-axis, we have

$$\sigma_x(0;y) = -\sigma_x(0;y) = 0.$$

Therefore for the bending moment on the Oy-axis, we have

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz = 0.$$

Thus, the bending moments and torques on the Oy-axis are equal to zero and on this Oy-axis the rotation of the normal with respect to the Ox-axis is $w_x(0;y)=0$. Hence it suffices to consider a half-plane $\operatorname{Re} z>0$ which we denote by S_1. Due to the cyclic symmetry of the problem, we have

$$\Psi(ze^{i\pi})=e^{-2i\pi}\Psi(z), \text{ and, which is the same } \Psi(z)=\Psi(-z). \tag{2.14}$$

By equality (2.9) and condition (1.3), from formula $(*)$, we obtain the relation

$$e^{2i\alpha(t)}\Psi(t)=P(1-\varkappa), \quad t\in l_2', \tag{2.15}$$

where $\Psi(z)=\psi'(z)$, while l_2' is that part of the contour l_2 which lies in the domain S_1.

Equality (2.5) with (2.9) taken into account is written in the form

$$\operatorname{Re}\Big\{e^{-i\alpha(t)}\big(2Pt+\overline{\psi(t)}\big)\Big\}=d(t), \quad t\in l_1', \tag{2.16}$$

where l_1' is the part of the contour l_1 which lies in the domain S_1.

By the differentiation of equation (2.16) with respect to the arc abscissa s, for the function $\Psi(z)$ on the contour l_1', we obtain the boundary condition

$$\operatorname{Im} e^{2i\alpha(t)}\Psi(t)=0, \quad t\in l_1'. \tag{2.17}$$

The angular points of the lines $l_1'\cup l_2'$ are denoted by A_k, $k=1;2;\dots;8$.

For a point t of the contour l_1', the equality

$$t-A_k=-i\rho\cdot e^{i\alpha_k}, \quad \rho=|t-A_k|$$

holds. From this equality, for the contour l_1', we easily obtain the equation

$$\operatorname{Re}\{t\cdot e^{-i\alpha(t)}\}=\operatorname{Re}\{A(t)\cdot e^{-i\alpha(t)}\}, \tag{2.18}$$

where $\alpha(t)$ is a piecewise-constant function on the contour l_1'; $\alpha(t)=\alpha_k$ when $t\in A_kA_{k+1}$ ($k=2n-1$ or k is an odd number), $A(t)=A_k$ when $t\in A_kA_{k+1}$ (here, k is an odd number).

Thus, by considering equality (2.18) together with equalities (2.15) and (2.17), for the function $\Psi(z)$ [6], we obtain the boundary conditions

$$e^{2i\alpha(t)}\Psi(t)=b, \qquad t\in l_2', \tag{2.19}$$

$$\operatorname{Im} e^{2i\alpha(t)}\Psi(t)=0, \qquad t\in l_1', \tag{2.20}$$

$$\operatorname{Re}\{t\cdot e^{-i\alpha(t)}\}=\operatorname{Re}\{A(t)\cdot e^{-i\alpha(t)}\}, \qquad t\in l_1'. \tag{2.21}$$

For the constant b contained in (2.19), we have $b=P\cdot(1-\varkappa)$.

3. Definition of the l_2 Contour and the Function $\psi(z)$

Denote by S_1' the external part of the unit circle (with center at the origin) of the plane ζ, cut along the real axis from the point $\zeta=m$ ($m>1$) up to infinity.

Assume that the domain S_1 of the plane z is conformally mapped onto the domain S_1' of the plane ζ by means of the function $z=-i\sqrt{\omega(\zeta)}$, where $\omega(\zeta)$ is an analytic function in a domain $|\zeta|>1$, which vanishes at the $\zeta=m$ and near the point at infinity, has the form

$$\omega(\zeta)=R\cdot\zeta+O(\zeta^{-1}), \quad R>0. \tag{3.1}$$

We mean that points A_k are displayed in points a_k, $k=1;2;\dots;8$, the images of contours l_1' and l_2' are indicated by L_1 and L_2, respectively.

By virtue of equality (2.14), the values of the function $\Psi_0(\zeta)=\Psi(-i\sqrt{\omega(\zeta)})$ on the cut of the domain S_1' from above and from below are equal to each other and thus the function $\Psi_0(\zeta)$ is analytic outside the unit circle in a domain $|\zeta|>1$. Equalities (2.19)–(2.21) will take the form

$$e^{2i\alpha_0(\sigma)}\Psi_0(\sigma)=b, \qquad \sigma\in L_2, \tag{3.2}$$

$$\operatorname{Im} e^{2i\alpha_0(\sigma)}\Psi_0(\sigma)=0, \qquad \sigma\in L_1, \tag{3.3}$$

$$\operatorname{Re}\{e^{-i\alpha_0(\sigma)}(-i\sqrt{\omega(\sigma)})\}=\operatorname{Re}\{A_0(\sigma)\cdot e^{-i\alpha_0(\sigma)}\}, \qquad \sigma\in L_1, \tag{3.4}$$

where $\alpha_0(\sigma) = \alpha(-i\sqrt{\omega(\sigma)})$ is the known piecewise-constant function on the contour L_1 and is the unknown function on L_2, since the contour itself is unknown, $A_0(\sigma) = A_k$, $\sigma \in a_k a_{k+1}$, $k = 1;3;5;7$.

To the expression $e^{2i\alpha_0(\sigma)}$, we have the equality

$$e^{2i\alpha_0(\sigma)} = -\frac{\sigma^2\omega'(\sigma)}{\sqrt{\omega(\sigma)}} \cdot \frac{\overline{\sqrt{\omega(\sigma)}}}{\overline{\omega'(\sigma)}}, \quad |\sigma| = 1. \tag{3.5}$$

If we use relation (3.5) in equality (3.2), then after differentiating equality (3.4) with respect to the variable ζ, we obtain the following boundary conditions:

$$\frac{-\sigma^2 i\omega'(\sigma)}{2\sqrt{\omega(\sigma)}} \cdot \Psi_0(\sigma) = b \cdot \frac{i\overline{\omega'(\sigma)}}{2\overline{\sqrt{\omega(\sigma)}}}, \qquad \sigma \in L_2, \tag{3.6}$$

$$\operatorname{Im}\left\{\sigma \cdot \frac{-i\omega'(\sigma)}{2\sqrt{\omega(\sigma)}} \cdot e^{-i\alpha_0(\sigma)}\right\} = 0, \qquad \sigma \in L_1, \tag{3.7}$$

$$\operatorname{Im}\left\{e^{2i\alpha_0(\sigma)}\Psi_0(\sigma)\right\} = 0, \qquad \sigma \in L_1. \tag{3.8}$$

Equality (3.6) can be written in the form

$$\frac{-\sigma^2 i\omega'(\sigma)}{2} \cdot \sqrt{\frac{\sigma - m}{\omega(\sigma)}} \cdot \Psi_0(\sigma) \cdot \overline{\sqrt{\sigma - m}} = \frac{b i\overline{\omega'(\sigma)}}{2} \cdot \overline{\sqrt{\frac{\sigma - m}{\omega(\sigma)}}} \cdot \sqrt{\sigma - m}. \tag{3.9}$$

Consider the function defined by the rule

$$F(\zeta) = \begin{cases} \frac{-\zeta^2 i\omega'(\zeta)}{2} \cdot \sqrt{\frac{\zeta - m}{\omega(\zeta)}} \cdot \Psi_0(\zeta) \cdot \overline{\sqrt{\frac{1}{\overline{\zeta}} - m}}, & |\zeta| > 1, \\ \frac{b i\overline{\omega'\left(\frac{1}{\overline{\zeta}}\right)}}{2} \cdot \overline{\sqrt{\frac{\frac{1}{\overline{\zeta}} - m}{\omega\left(\frac{1}{\overline{\zeta}}\right)}}} \cdot \sqrt{\zeta - m}, & |\zeta| < 1. \end{cases} \tag{3.10}$$

Here, $\zeta = m$ is a unique point in the external domain of a unit circle $|\zeta| > 1$, where the analytic function $\omega(\zeta)$ has a first order zero value and therefore $\sqrt{\frac{\zeta - m}{\omega(\zeta)}}$ will be an analytic function in this domain. Therefore, the function $F(\zeta)$ defined by equality (3.10) will be analytic inside and outside the unit circle $|\zeta| = 1$ and, by virtue of equation (3.9), will satisfy, on a part of the circle $|\zeta| = 1$, the boundary condition

$$F^+(\sigma) = F^-(\sigma), \quad \sigma \in L_2. \tag{3.11}$$

If we take into consideration equalities (3.7), (3.8) and (3.10), then for the analytic function $F(\zeta)$ in the plane cut along the line L_1, we obtain the boundary conditions

$$\operatorname{Im} \frac{F^+(\sigma)}{\sigma} e^{i\alpha} = 0, \qquad \sigma \in L_1, \tag{3.12}$$

$$\operatorname{Im} \frac{F^-(\sigma)}{\sigma} e^{i\alpha} = 0, \qquad \sigma \in L_1. \tag{3.13}$$

In the case under consideration, the term $e^{-2i\alpha}$ on the contour L_1 gets the values equal to 1 or -1. Thus, if we multiplying equality (3.13) by $e^{-2i\alpha}$, then for the analytic function $F(\zeta)$ in the complex plane ζ, cut along the line L_1, we obtain the following boundary conditions:

$$\operatorname{Im} \frac{F^+(\sigma)}{\sigma} e^{i\alpha} = 0, \qquad \sigma \in L_1, \tag{3.14}$$

$$\operatorname{Im} \frac{F^-(\sigma)}{\sigma} e^{-i\alpha} = 0, \qquad \sigma \in L_1. \tag{3.15}$$

The obtained equalities can be rewritten as follows:

$$\frac{F^+(\sigma)}{\sigma} \cdot e^{i\alpha} = \sigma \cdot \overline{F^+(\sigma)} \cdot e^{-i\alpha}, \qquad \sigma \in L_1, \tag{3.16}$$

$$\frac{F^-(\sigma)}{\sigma} \cdot e^{-i\alpha} = \sigma \cdot \overline{F^-(\sigma)} \cdot e^{i\alpha}, \qquad \sigma \in L_1. \tag{3.17}$$

On the contour $|\zeta|=1$, the positive direction is chosen so that when moving along this direction, the domain $|\zeta|<1$ remains on the left side.

Let us consider the function $F_*(\zeta)$ given as follows

$$F_*(\zeta)=\overline{F\Big(\frac{1}{\overline{\zeta}}\Big)}=\begin{cases}\dfrac{\overline{i\omega'\big(\frac{1}{\overline{\zeta}}\big)}}{2\zeta^2}\cdot\overline{\sqrt{\dfrac{\frac{1}{\overline{\zeta}}-m}{\omega\big(\frac{1}{\overline{\zeta}}\big)}}}\cdot\overline{\Psi_0\big(\frac{1}{\overline{\zeta}}\big)}\cdot\sqrt{\zeta-m}, & |\zeta|<1,\\ \dfrac{-bi\omega'(\zeta)}{2}\cdot\sqrt{\dfrac{\zeta-m}{\omega(\zeta)}}\cdot\overline{\sqrt{\frac{1}{\overline{\zeta}}-m}}, & |\zeta|>1,\end{cases} \tag{3.18}$$

and also consider the functions $W(\zeta)$ and $W_*(\zeta)$ defined by the equalities

$$W(\zeta)=\frac{1}{\zeta}F(\zeta), \tag{3.19}$$

$$W_*(\zeta)=\overline{W\Big(\frac{1}{\overline{\zeta}}\Big)}. \tag{3.20}$$

Further, we introduce the function $\Omega(\zeta)$,

$$\Omega(\zeta)=W(\zeta)+W_*(\zeta). \tag{3.21}$$

The boundary values of the function $\Omega(\zeta)$ are written in the form

$$\Omega^+(\sigma)=W^+(\sigma)+W_*^+(\sigma)=\frac{1}{\sigma}F^+(\sigma)+\sigma\overline{F^-(\sigma)}, \tag{3.22}$$

$$\Omega^-(\sigma)=W^-(\sigma)+W_*^-(\sigma)=\frac{1}{\sigma}F^-(\sigma)+\sigma\overline{F^+(\sigma)}. \tag{3.23}$$

Using equalities (3.11), (3.22) and (3.23), we have

$$\Omega^+(\sigma)=\Omega^-(\sigma),\quad \sigma\in L_2. \tag{3.24}$$

For the boundary values of the function $\Omega(\zeta)$ on the internal and the external sides of the contour L_1, by virtue of conditions (3.16), (3.17) and equalities (3.22), (3.23), we obtain the equality

$$\Omega^+(\sigma)=e^{-2i\alpha}\Omega^-(\sigma),\quad \sigma\in L_1. \tag{3.25}$$

Let us introduce the function $T(\zeta)$ defined by the equality

$$T(\zeta)=W(\zeta)-W_*(\zeta). \tag{3.26}$$

The boundary values of the function $T(\zeta)$ are written in the form

$$T^+(\sigma)=W^+(\sigma)-W_*^+(\sigma)=\frac{1}{\sigma}F^+(\sigma)-\sigma\overline{F^-(\sigma)}, \tag{3.27}$$

$$T^-(\sigma)=W^-(\sigma)-W_*^-(\sigma)=\frac{1}{\sigma}F^-(\sigma)-\sigma\overline{F^+(\sigma)}. \tag{3.28}$$

In view of equalities (3.11), (3.27) and (3.28), for the boundary values of the function $T(\zeta)$, we have

$$T^+(\sigma)=T^-(\sigma),\quad \sigma\in L_2. \tag{3.29}$$

For the boundary values of the function $T(\zeta)$ on the internal and the external sides of the contour L_1, by virtue of conditions (3.16), (3.17) and equalities (3.27), (3.28), we obtain the equality

$$T^+(\sigma)=-e^{-2i\alpha}\cdot T^-(\sigma),\quad \sigma\in L_1. \tag{3.30}$$

The multiplier $e^{-2i\alpha}$ on the contour L_1 gets the values

$$e^{-2i\alpha}=\begin{cases}1, & \sigma\in a_1a_2\cup a_5a_6,\\ -1, & \sigma\in a_3a_4\cup a_7a_8.\end{cases} \tag{3.31}$$

With (3.31) taken into account, the boundary equalities (3.25) and (3.30) can be rewritten as follows:

$$\begin{cases} \Omega^+(\sigma) = -\Omega^-(\sigma), & \sigma \in a_3a_4 \cup a_7a_8, \\ \Omega^+(\sigma) = \Omega^-(\sigma), & \sigma \in a_1a_2 \cup a_5a_6. \end{cases} \tag{3.32}$$

$$\begin{cases} T^+(\sigma) = -T^-(\sigma), & \sigma \in a_1a_2 \cup a_5a_6, \\ T^+(\sigma) = T^-(\sigma), & \sigma \in a_3a_4 \cup a_7a_8. \end{cases} \tag{3.33}$$

Equalities (3.32), (3.33) imply that for the function $\Omega(\zeta)$, the part of the contour $L_1(a_1a_2 \cup a_5a_6)$ and the curve L_2 is not the jump line. For the function $T(\zeta)$, the part of the contour $L_1(a_3a_4 \cup a_7a_8)$ and the curve L_2 is not the jump line.

The problem is thus reduced to a problem of finding analytic functions $\Omega(\zeta)$ and $T(\zeta)$ in the complex plane ζ, cut along a part of the contour L_1 (the plane is cut along the lines $a_3a_4 \cup a_7a_8$ for the function $\Omega(\zeta)$, and along the lines $a_1a_2 \cup a_5a_6$ for the function $T(\zeta)$) using the conditions

$$\Omega^+(\sigma) = -\Omega^-(\sigma), \quad \sigma \in a_3a_4 \cup a_7a_8, \tag{3.34}$$

$$T^+(\sigma) = -T^-(\sigma), \quad \sigma \in a_1a_2 \cup a_5a_6. \tag{3.35}$$

By virtue of equalities (3.10), (3.18), (3.19), (3.20), (3.21) and (3.26), we may conclude that the sought functions $\Omega(\zeta)$ and $T(\zeta)$ must satisfy the following additional conditions:

$$\Omega(\zeta) = \overline{\Omega\Big(\frac{1}{\overline{\overline{\zeta}}}\Big)}, \tag{3.36}$$

$$T(\zeta) = -\overline{T\Big(\frac{1}{\overline{\overline{\zeta}}}\Big)}. \tag{3.37}$$

Problems (3.34), (3.35) are the particular cases of a linear conjugation problem, where the boundary consists of separately lying smooth contours. In particular, the coefficient of the problem is $G(\sigma) = -1$.

We will seek unbounded solutions near the points a_k (unlimited less than the first order) or, which is the same, solutions of the class h_0 [2].

A general solution of problem (3.34) has the form

$$\Omega(\zeta) = \chi_1(\zeta) \cdot P_1(\zeta), \tag{3.38}$$

where $P_1(\zeta)$ is a polynomial, the function $\chi_1(\zeta)$ is a canonical solution of the same problem that in the general case has the form

$$\chi(\zeta) = e^{\gamma(\zeta)} \prod_{k=1}^{n} (\zeta - a_k)^{\lambda_k}. \tag{3.39}$$

In our case, this formula takes the form

$$\begin{aligned} \chi_1(\zeta) =& e^{\gamma(\zeta)}(\zeta - a_3)^{\lambda_3} \cdot (\zeta - a_4)^{\lambda_4} \cdot (\zeta - a_7)^{\lambda_7} \cdot (\zeta - a_8)^{\lambda_8}, \\ \gamma(\zeta) =& \frac{1}{2\pi i} \int\limits_{a_3a_4} \frac{\pi i d\sigma}{\sigma - \zeta} + \frac{1}{2\pi i} \int\limits_{a_7a_8} \frac{\pi i d\sigma}{\sigma - \zeta} = \frac{1}{2} \ln \frac{\zeta - a_4}{\zeta - a_3} + \frac{1}{2} \ln \frac{\zeta - a_8}{\zeta - a_7}, \\ e^{\gamma(\zeta)} =& \Big(\frac{\zeta - a_4}{\zeta - a_3}\Big)^{\frac{1}{2}} \cdot \Big(\frac{\zeta - a_8}{\zeta - a_7}\Big)^{\frac{1}{2}}. \end{aligned}$$

Here, under the expressions $\left(\frac{\zeta - a_4}{\zeta - a_3}\right)^{\frac{1}{2}}$ and $\left(\frac{\zeta - a_8}{\zeta - a_7}\right)^{\frac{1}{2}}$ we mean the holomorphic branches in the plane cut along the arcs a_3a_4 and a_7a_8 which at the point at infinity are equal to unity,

$$\lambda_3 = \lambda_7 = 0, \quad \lambda_4 = \lambda_8 = -1.$$

For the index of the problem, we obtain the equality

$$\delta_1 = -(\lambda_4 + \lambda_8) = 2. \tag{3.40}$$

For a canonical solution of the class h_0, we eventually obtain the expression

$$\chi_1(\zeta) = \frac{C_1^*}{\sqrt{R_1(\zeta)}}, \tag{3.41}$$

where C_1^* is any fixed constant, different from zero,

$$R_1(\zeta) = (\zeta - a_3) \cdot (\zeta - a_4) \cdot (\zeta - a_7) \cdot (\zeta - a_8). \tag{3.42}$$

Under $\frac{1}{\sqrt{R_1(\zeta)}}$ we mean the holomorphic branch in the plane, cut along the arcs a_3a_4 and a_7a_8. The decomposition of this function into decreasing degrees of a variable ζ, near an infinitely distant point, has the form

$$\frac{1}{\sqrt{R_1(\zeta)}} = \zeta^{-2} + B_1'\zeta^{-3} + B_2'\zeta^{-4} + \cdots . \tag{3.43}$$

From equalities (3.10), (3.18), (3.19) and (3.21) we see that the function $\Omega(\zeta)$ at the points $\zeta = 0$ and $\zeta = \infty$ has a first order pole. Since the order of the canonical function $\chi_1(\zeta)$ is equal to $-\delta_1$ at the point at infinity, applying the above argumentation and equality (3.40), for the function $\Omega(\zeta)$, we obtain

$$\Omega(\zeta) = \chi_1(\zeta) \cdot \Big(\frac{c_0'}{\zeta} + c_1' + c_2'\zeta + c_3'\zeta^2 + c_4'\zeta^3\Big). \tag{3.44}$$

In view of equality (3.36), we may conclude that the constants c_0', c_1', c_2', c_3', c_4' satisfy the conditions

$$c_0' = \overline{c_4'}, \quad c_1' = \overline{c_3'}, \quad c_2' = \overline{c_2'}. \tag{3.45}$$

By an analogous reasoning, for problem (3.35), we obtain

$$\chi_2(\zeta) = \frac{C_2^*}{\sqrt{R_2(\zeta)}}, \tag{3.46}$$

where C_2^* is any fixed constant different from zero,

$$R_2(\zeta) = (\zeta - a_1) \cdot (\zeta - a_2) \cdot (\zeta - a_5) \cdot (\zeta - a_6). \tag{3.47}$$

In this case, too, under $\frac{1}{\sqrt{R_2(\zeta)}}$ we mean that holomorphic branch on the plane, cut along the arcs a_1a_2 and a_5a_6, the expansion of which near the point at infinity has the form

$$\frac{1}{\sqrt{R_2(\zeta)}} = \zeta^{-2} + B_1''\zeta^{-3} + B_2''\zeta^{-4} + \cdots , \tag{3.48}$$

$$\delta_2 = 2. \tag{3.49}$$

For the sought function $T(\zeta)$, we finally obtain

$$T(\zeta) = \chi_2(\zeta) \cdot \Big(\frac{c_0''}{\zeta} + c_1'' + c_2''\zeta + c_3''\zeta^2 + c_4''\zeta^3\Big), \tag{3.50}$$

where the constants $c_0'', c_1'', \dots, c_4''$ satisfy the conditions

$$c_0'' = \overline{c_4''}, \quad c_1'' = \overline{c_3''}, \quad c_2'' = \overline{c_2''}. \tag{3.51}$$

The constants c_0', c_1', c_2', c_3', c_4' and c_0'', c_1'', c_2'', c_3'', c_4'', in (3.44) and (3.50) for the functions $\Omega(\zeta)$ and $T(\zeta)$ can be found if we use the known lengths of the linear parts of the plate boundary and fix some angular point.

After that, knowing the functions $\Omega(\zeta)$ and $T(\zeta)$, by virtue of equalities (3.10), (3.19), (3.21) and (3.26), we define the function $F(\zeta)$. Knowing the function $F(\zeta)$ and using equalities (3.10) and (3.18) we find the functions $f'(\zeta)$ (thereby an unknown part of the boundary) and $\Psi_0(\zeta)$ ($z = f(\zeta) = -i\sqrt{\omega(\zeta)}$).

So, we have defined $\Psi_0(\zeta)$ and at the same time the function $\Psi(z)$ which together with the function $\Phi(z)$, by virtue of equality (2.3) describe a plate deflection.

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(Received 23.01.2022)

Department of Mathematics, Georgian Technical University, 77 Kostava Str., Tbilisi 0175, Georgia
E-mail address: zura.g.abashidze@gmail.com