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WEIGHTED EXTRAPOLATION IN GRAND MORREY SPACES BEYOND THE MUCKENHOUPT RANGE

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Abstract. Rubio de Francia's extrapolation in weighted grand Morrey spaces with weights beyond the Muckenhoupt range is established. Based on this result, the boundedness of maximal and Calderón-Zygmund operators, and commutators of singular integrals in weighted grand Morrey spaces for appropriate class of weights is obtained. The problems are studies for spaces and operators defined on quasi–metric measure spaces (spaces of homogeneous type) but the results are new even for particular cases of spaces of homogeneous type.

1. Introduction

Let (X,d,μ) be a quasi–metric measure space with quasi–metric d and measure μ satisfying the condition: there are positive constants c and N such that for all $x\in X$ and $r\in(0,d_X)$ the following two-sided inequality holds: $\frac{1}{c}r^N\le\mu(B(x,r))\le cr^N$, where d_X denotes diameter of X, and $B(x,r)$ is the open ball in X with center x and radius r. The constant N is called the dimension of X. In this case the measure μ satisfies the *doubling condition*: there is a positive constant D_μ such that for all $x\in X$, $r>0$, the inequality $\mu B(x,2r)\le D_\mu\mu B(x,r)$ holds.

A quasi–metric measure space with doubling measure μ is called space of homogeneous type (SHT briefly). Examples of SHT are: (a) domains Ω in $\mathbb{R}^n$ satisfying the condition: $|B(x.r)\cap\Omega|\ge Cr^n$, $x\in\Omega$, with positive constant C independent of x, and r, where $|E|$ denotes Lebesgue measure of a set E; (b) rectifiable regular (Carleson) curves in $\mathbb{C}$ with Euclidean metric and arc-length measure; (c) nilpotent Lie groups with Haar measure (homogeneous groups), etc (see, e.g., [2, 18, 27] for the definition and properties of an SHT). Let w be a weight on X, i.e., let w be an integrable function on X. We denote by $L^s_w(X)$, $1\le s<\infty$, the weighted Lebesgue space with weight w which is defined with respect to the norm $\|f\|_{L^s_w(X)}=\Big(\int\limits_X|f(x)|^s w(x)d\mu(x)\Big)^{1/s}$.

In [16] the weighted extrapolation in the weighted grand Morrey spaces $L^{p),r,\theta}_w(X)$ for $A_p(X)$ weights was derived, where $L^{p),r,\theta}_w(X)$ is the space defined by the norm:

$$\|f\|_{L^{p),r,\theta}_w(X)}:=\sup_{0<\varepsilon<p-1}\varepsilon^\theta\|f\|_{L^{p-\varepsilon,r}_w(X)}:=\sup_{0<\varepsilon<p-1}\varepsilon^\theta\bigg(\sup_B\frac{1}{\big(w(B)\big)^{\frac{1}{p-\varepsilon}+r}}\|f\|_{L^{p-\varepsilon}_w(B)}\bigg),$$

where $1<p<\infty$, $-1/p<r<0$ and $\theta>0$. In particular, it was shown that if the one-weight inequality $\|g\|_{L^{p_0}_w(X)}\le C_0C([w]_{A_{p_0}(X)})\|f\|_{L^{p_0}_w(X)}$, $(f,g)\in\mathcal{F}$, holds for some family of pairs of non-negative functions $\mathcal{F}$, some $1\le p_0<\infty$ and every $w\in A_{p_0}(X)$, where the positive constant C_0 is independent of pairs (f,g) and w, and $\cdot\mapsto C(\cdot)$ is a non-decreasing mapping, then for all $1<p<\infty$ and $w\in A_p(X)$, $\|g\|_{L^{p),r,\theta}_w(X)}\le C_0\|f\|_{L^{p),r,\theta}_w(X)}$, $(f,g)\in\mathcal{F}$.

The result of [16] was derived by using the ideas of [3], where the similar result in the classical weighted Morrey spaces was established (see also [25] for related topics). We refer to [12] for the weighted extrapolation in grand Lebesgue spaces.

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Our aim is to study the similar problem for different grand weighted Morrey space $\mathcal{L}_w^{p),\lambda,\theta}(\sigma, X)$. This space is defined on the base of the Morrey space $\mathcal{L}_w^{p,\lambda}(X)$ and given by the norm

$$\|f\|_{\mathcal{L}_w^{p),\lambda,\theta}(\sigma,X)} := \sup_{0<\varepsilon<\sigma} \varepsilon^\theta \sup_B \left(\frac{1}{\big(\mu(B)\big)^\lambda} \int\limits_B |f(y)|^{p-\varepsilon} w(y) d\mu(y) \right)^{1/(p-\varepsilon)} := \sup_{0<\varepsilon<\sigma} \varepsilon^\theta \|f\|_{\mathcal{L}_w^{p-\varepsilon,\lambda}(X)},$$

where $0 \le \lambda < 1$ and σ is a small constant satisfying such that $0 < \sigma \le p-1$.

If $\sigma = p-1$, then we denote

$$\mathcal{L}_w^{p),\lambda,\theta}(p-1, X) \equiv \mathcal{L}_w^{p),\lambda,\theta}(X).$$

This space for $w(\cdot) \equiv 1$, i.e. $\mathcal{L}^{p),\lambda,\theta}(X)$, was introduced and studied in [19]. Later, in [24] the author introduced generalized grand Morrey space $\mathcal{L}^{p),\lambda),\theta}(X)$ defined by the norm including the "grandification" taken not only with respect to p but also for λ. Grand Morrey spaces $\mathcal{L}^{p),\lambda,\theta}$ are generalizations of grand Lebesgue spaces $L^{p)}$ introduced in 1992 by T. Iwaniec and C. Sbordone [10] in their studies related with the integrability properties of the Jacobian in a bounded open set Ω. A generalized version of them, $L^{p),\theta}(\Omega)$, $\theta > 0$, appeared in L. Greco, T. Iwaniec and C. Sbordone [8], where the authors investigated the solvability of nonhomogeneous n- harmonic equation div $A(x, \nabla u) = \mu$. Associate space to $L^{p),\theta}$ is called small Lebesgue space (see [5]). Grand Lebesgue space is a Banach space which is non-separable and non-reflexive (see, e.g., [5]).

The space $\mathcal{L}_w^{p),\lambda,\theta}(X)$ for $\lambda = 0$ is the weighted grand Lebesgue space $L_w^{p),\theta}(X)$. In this space one–weight criteria under the A_p condition on weights were derived in [6, 11] and [20] for the Hardy–Littlewood maximal operator, Hilbert transform and fractional integrals, respectively (see also the monograph [13, Ch. 14] and references cited therein). The Rubio de Francia's extrapolation result in $L_w^{p),\theta}(X)$ space was proved in [12].

Weighted extrapolation for the classical weighted Morrey spaces $\mathcal{L}_w^{p,\lambda}(\mathbb{R}^n)$, $0 < \lambda < 1$, was established in [4] on Euclidean spaces.

Unlike $L_w^{p,r}$, the weighted Morrey space $\mathcal{L}_w^{p,\lambda}$ has the property that the operators of Harmonic Analysis are bounded in these space for weights beyond the Muckenhoupt range. In particular, in [26] the author proved the boundedness of the Hilbert transform in $\mathcal{L}_{|x|^\alpha}^{p,\lambda}(\mathbb{R})$ for $0 \le \lambda < 1$ and $\lambda - 1 < \alpha < \lambda + p - 1$. This range of values of α shows a shift with respect to the corresponding the $A_p(\mathbb{R})$ class which is $-1 < \alpha < p-1$. The boundedness of the Hardy–Littlewood maximal operator M in $\mathcal{L}_{|x|^\alpha}^{p,\lambda}(\mathbb{R}^n)$ for the sharp range $\lambda - n \le \alpha < \lambda + n(p-1)$. Later, the similar result was established in [22] for the Riesz transforms.

It is known that weighted grand Morrey spaces are Banach spaces (for further properties see, e.g., [14–16]).

Denote by $A_p(X)$, $1 < p < \infty$, the Muckenhoupt class of weights defined on X, i.e., this is the class of all integrable on X functions such that

$$\big[w\big]_{A_p} := \sup_B \left(\frac{1}{\mu(B)} \int\limits_B w(x) d\mu(x) \right) \left(\frac{1}{\mu(B)} \int\limits_B w^{1-p'}(x) d\mu(x) \right)^{p-1} < \infty, \; p' = \frac{p}{p-1}.$$

Further, a weight w belongs to $A_1(X)$ if $(Mw)(x) \le Cw(x)$ a.e., where $Mw(x)$ is the maximal function of w given by the formula:

$$(Mw)(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int\limits_B w(y) d\mu(y).$$

The weight function $d(x_0, x)^\alpha$ belongs to $A_p(X)$ if and only if $-N < \alpha < N(p-1)$, where N is a dimension of X (see e.g., [7, Ch. 7] for Euclidean space but the proof for an SHT is the same). Further, the class RH_η is a collection of weights for which the *reverse Hölder's inequality* holds, i.e., there is a positive constant C such that for all balls,

$$\left(\frac{1}{\mu(B)} \int\limits_B w^\eta(x) d\mu(x) \right)^{1/\eta} \le \frac{C}{\mu(B)} \int\limits_B w(x) d\mu(x).$$

It can be checked that (see also [27]) that if $s > r$, then $RH_s \subset RH_r$, and that $\bigcup\limits_{1<r<\infty} RH_r = \bigcup\limits_{p>1} A_p$.

It is known that the operators of Harmonic Analysis such as maximal and singular integral operators are bounded in the classical weighted Lebesgue space L^p_w under the Muckenhoupt's A_p condition (see, e.g., [7, Ch. 7] for Euclidean spaces, [27] for an SHT). This property remains valid for the weighted Morrey space $L^{p,r}_w$, $-1/p < r < 0$ (see [17]).

Suppose that x_0 is a point in X. Let w be a weight function on X. Denote by w_α the following function $w_\alpha(x) := w(x)d(x_0, x)^\alpha$. We proved the extrapolation statement in the following form: if $\|g\|_{L^{p_0}_w(X)} \le C_0\|f\|_{L^{p_0}_w(X)}$, $(f, g) \in \mathcal{F}$, for some $1 \le p_0 < \infty$ and every $w \in A_{p_0}(X)$, with positive constant C_0 independent of (f, g) and depending on $[w]_{A_p}$, then for all $1 < p < \infty$, $\theta > 0$, $w \in A_p(X) \cap RH_\eta$, we have appropriate weighted inequality for the grand weighted Morrey space $\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(X)$ with, generally speaking, not necessarily A_p weight function w_α.

Morrey spaces $\mathcal{L}^{p,\lambda}$ were introduced in 1938 by C. Morrey [21] in relation to regularity problems of solutions to PDEs, and provided a useful tool in the regularity theory of PDEs.

2. Main Results

Now we formulate the main results of this note:

Theorem 2.1. *Let $1 \le p_0 < \infty$, x_0 be a point in X. Suppose that $\mathcal{F}$ is a family of non-negative pairs of functions on X. Assume that for all $(f, g) \in \mathcal{F}$ and every $w \in A_{p_0}(X)$,*

$$\|g\|_{L^{p_0}_w(X)} \le C_0 C([w]_{A_{p_0}(X)})\|f\|_{L^{p_0}_w(X)}, \quad (f, g) \in \mathcal{F},$$

for some $1 \le p_0 < \infty$ and every $w \in A_{p_0}(X)$, where C_0 is a positive constant independent of (f, g) and w, and $\cdot \mapsto C(\cdot)$ is a non-decreasing mapping, then for all $1 < p < \infty$ and $w \in A_{p_0}(X)$, depending only on $[w]_{A_{p_0}(X)}$. Then

(a) *for every $1 < p < \infty$, $\theta > 0$, and $w \in A_p(X) \cap RH_\eta$, we have*

$$\|g\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(\sigma,X)} \le C_0\|f\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(\sigma,X)}, \quad (f, g) \in \mathcal{F}, \tag{$*$}$$

where, $w_\alpha(x) := w(x)d(x_0, x)^\alpha$, $0 \le \lambda < 1/\eta'$, $0 \le \alpha < \lambda N$, and σ is sufficiently small positive constant depending on parameters of the space;

(b) *for every $1 < p < \infty$ and $\theta > 0$, inequality $(*)$ holds, where $w_\alpha(x) = d(x_0, x)^\alpha$ with $N(\lambda - 1) < \alpha < N(\lambda + p - 1)$, and σ is sufficiently small positive constant depending on parameters of the space;*

(c) *for every $1 < p < \infty$, $\theta > 0$, and $w_\alpha(x) = d(x_0, x)^\alpha$ with $N(\lambda - 1) < \alpha < N(\lambda + p - 1)$, we have*

$$\|g\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(X)} \le C_0\|f\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(X)}, \quad (f, g) \in \mathcal{F}.$$

This statement enables us to formulate the boundedness results for operators of Harmonic Analysis such as maximal and Calderón–Zygmund singular operators; commutators of singular integrals; fractional integrals, etc.

Let K be the Calderón–Zygmund operator on X given by the formula

$$(Kf)(x) = \lim_{\varepsilon\to 0} \int\limits_{X\setminus B(x,\varepsilon)} k(x, y)f(y)d\mu(y), \quad x \in X,$$

where $k(x, y)$ is the Calderón-Zygmund kernel on $X \times X$ (see e.g., [1, 16] for the definition).

Denote by $\mathcal{D}(X)$ the class of all bounded functions on X.

Let us define the class of *bounded mean oscillation* functions $BMO(X)$. This is the set of all real-valued locally integrable functions on X such that

$$\|f\|_{BMO} = \sup_{\substack{x\in X\\ 0<r<\ell}} \frac{1}{\mu B(x, r)} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}|d\mu(y) < \infty,$$

where $f_{B(x,r)}$ is the integral average over the ball $B(x, r)$. $BMO(X)$ is a Banach space with respect to the norm $\|\cdot\|_{BMO}$ when we regard the space $BMO(X)$ as the class of equivalent functions modulo additive constants.

Let U be an operator and b a locally integrable function. We define the *commutator* $U_b f$ as

$$U_b f = bU(f) - U(bf).$$

Further, for $b \in BMO(X)$, let $K_b^m f(x) = \int\limits_X \big(b(x)-b(y)\big)^m k(x,y)f(y)d\mu(y)$, $m = 0,1,2,\dots$, be $m-th$ order commutator of singular integral, where $k(x,y)$ is the Calderón–Zygmund kernel. It is clear that $K_b^0 f$ is the CZ singular operator.

One-weight inequalities for commutators of singular integrals defined in the classical Lebesgue spaces $L^p(X)$ were established in [23].

Theorem 2.2. *Let $1 < p < \infty$ and let x_0 be a point in X. Suppose that $\theta > 0$ and that $w \in A_p(X) \cap RH_\eta$. Let $0 \le \lambda < 1/\eta'$, $0 \le \alpha < \lambda N$. We set $w_\alpha(x) := w(x)d(x_0,x)^\alpha$. Then*

(a) *the maximal operator M is bounded in $\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(\sigma, X)$ for some small positive constant σ;*

(b) *there is a positive constant C such that for all $f \in \mathcal{D}(X)$,*

$$\|Kf\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(\sigma,X)} \le C\|f\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(\sigma,X)}$$

for some small positive constant σ;

(c) *if $b \in BMO(X)$, then there is a positive constant C such that for all $f \in \mathcal{D}(X)$,*

$$\|K_b f\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(\sigma,X)} \le C\|b\|_{BMO}\|f\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(\sigma,X)}$$

for some small positive constant σ;

(d) *if $b \in BMO(X)$, then there is a positive constant C such that for all $f \in \mathcal{D}(X)$,*

$$\|K_b^m f\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(X)} \le C\|b\|^m_{BMO}\|M^{m+1}f\|_{\mathcal{L}^{p),\lambda,\theta}_{w_\alpha}(X)},$$

for some small positive constant σ, where M^{m+1} is $m+1$-th iterated maximal operator.

Further, for power-type weights we have:

Theorem 2.3. *Let $1 < p < \infty$ and let x_0 be a point in X. Suppose that $\theta > 0$ and $N(\lambda-1) < \beta < N(\lambda+p-1)$. Then the following statements hold:*

(a) *the maximal operator M is bounded in $\mathcal{L}^{p),\lambda,\theta}_{d(x_0,x)^\beta}(X)$;*

(b) *there is a positive constant C such that for all $f \in \mathcal{D}(X)$,*

$$\|Kf\|_{\mathcal{L}^{p),\lambda,\theta}_{d(x_0,x)^\beta}(X)} \le C\|f\|_{\mathcal{L}^{p),\lambda,\theta}_{d(x_0,x)^\beta}(X)},$$

(c) *if $b \in BMO(X)$, then there is a positive constant C such that for all $f \in \mathcal{D}(X)$,*

$$\|K_b f\|_{\mathcal{L}^{p),\lambda,\theta}_{d(x_0,x)^\beta}(X)} \le C\|b\|_{BMO}\|f\|_{\mathcal{L}^{p),\lambda,\theta}_{d(x_0,x)^\beta}(X)};$$

(d) *if $b \in BMO(X)$, then there is a positive constant C such that for all $f \in \mathcal{D}(X)$,*

$$\|K_b^m f\|_{\mathcal{L}^{p),\lambda,\theta}_{d(x_0,x)^\beta}(X)} \le C\|b\|^m_{BMO}\|M^{m+1}f\|_{\mathcal{L}^{p),\lambda,\theta}_{d(x_0,x)^\beta}(X)},$$

where M^{m+1} is the $m+1$-th iterated maximal operator.

Necessary conditions for maximal operators and Hilbert transform are also obtained.

Finally we mention that appropriate results for commutators of fractional integrals operators are derived.

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