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SIERPIŃSKI–ZYGMUND FUNCTIONS AND ω-POWERS

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Abstract. It is shown that, by using some method of extending the topology of a Baire space without isolated points, the extended topology preserves Baire's property but does not have Blumberg's property.

All topological spaces $(E, \mathcal{T})$ considered throughout this note are assumed to satisfy the T_1-separation axiom.

As usual, we denote by $\mathcal{B}(E, \mathcal{T})$ the Borel σ-algebra of E. If E is infinite, then $\mathrm{card}(\mathcal{B}(E, \mathcal{T})) \geq (\mathrm{card}(E))^{\omega}$, where ω stands for the least infinite cardinal (ordinal) number.

Let $\mathbf{R}$ denote the real line. By definition, $(E, \mathcal{T})$ has Blumberg's property (or E is a Blumberg space) if, for any function $f : E \to \mathbf{R}$, there exists an everywhere dense set $X \subset E$ such that the restriction $f|X$ is continuous.

This concept is motivated by the classical theorem of Blumberg [1], according to which $E = \mathbf{R}$, equipped with its standard topology, is a Blumberg space.

It can easily be verified that every Blumberg space is a Baire space. As is well-known, the converse assertion does not hold (cf. Theorem 2 of this note). However, as was proved in [2], if E is a metrizable Baire space, then E has Blumberg's property. Any discrete topological space is a Blumberg space, and this trivial fact implies that a bijective continuous image of a Blumberg space is not, in general, a Blumberg space. The topological sum of any family of Blumberg spaces is a Blumberg space. At the same time, there exists a metrizable Baire space whose topological square is not a Baire space (see [3, 5, 6, 8]). Thus, Blumberg's property is not preserved under finite topological products.

In this note, we show that Blumberg's property of $(E, \mathcal{T})$ is rather delicate and can be destroyed by a fairly standard method of extending the original topology $\mathcal{T}$.

We need the following auxiliary proposition which generalizes the classical result of Sierpiński and Zygmund [7] on the existence of a totally discontinuous function acting from $\mathbf{R}$ into $\mathbf{R}$ (cf. also [4, Chapter 11, Lemma 3]).

Lemma 1. *Let $\mathbf{a}$ be an infinite cardinal number and let $(E, \mathcal{T})$ be a topological space such that*

$$\mathrm{card}(E) = \mathrm{card}(\mathcal{B}(E, \mathcal{T})) = \mathbf{a}.$$

Then there exists a function $f : E \to \mathbf{R}$ having the following property: for every set $X \subset E$ with $\mathrm{card}(X) = \mathbf{a}$, the restriction $f|X$ is not continuous.

Proof. First, observe that the relation $\mathrm{card}(E) = \mathrm{card}(\mathcal{B}(E, \mathcal{T}))$ implies at once that $\mathbf{a}^{\omega} = \mathbf{a}$. In other words, $\mathbf{a}$ is an ω-power.

Denote by $\mathcal{G}$ the family of all real-valued functions g such that g is defined on a Borel subset of E, is continuous and $\mathrm{card}(\mathrm{dom}(g)) = \mathbf{a}$.

In view of the assumption of this lemma, we have the equality

$$\mathrm{card}(\mathcal{G}) = \mathbf{a}^{\omega} = \mathbf{a}.$$

Using the standard diagonal argument (cf. [7]), we are able to construct a function $f : E \to \mathbf{R}$ satisfying the following condition: for any function $g \in \mathcal{G}$, the cardinality of the set

$$X_g = \{x \in \mathrm{dom}(g) : g(x) = f(x)\}$$

is strictly less than $\mathbf{a}$.

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Let us check that f has the property indicated in the formulation of the lemma. For this purpose, take an arbitrary set $X \subset E$ with $\mathrm{card}(X) = \mathbf{a}$ and suppose to the contrary that $h = f|X$ is continuous. Let $\mathrm{cl}(X)$ denote the closure of X and let

$$A = \{x \in E : \mathrm{osc}_h(x) = 0\},$$

where the symbol $\mathrm{osc}_h(x)$ stands for the oscillation of h at x (in general, x does not belong to X). The set A is of type G_δ, so the set

$$B = \mathrm{cl}(X) \cap A$$

is Borel in E and includes X (because h is continuous on X).

Take any point $x \in B$ and consider the family of closed sets

$$\{\mathrm{cl}(h(U)) : U \text{ is an open neighborhood of } x\}.$$

This family is centered and contains members with arbitrarily small diameters. Consequently, the intersection of this family is a singleton, say $\{y\}$. We define $g(x) = y$. Now, it is not difficult to check that the obtained function

$$g : B \to \mathbf{R}$$

is a continuous extension of $h = f|X$. In particular, we have

$$\mathrm{card}(\{x \in \mathrm{dom}(g) : g(x) = f(x)\}) \geq \mathrm{card}(X) = \mathbf{a},$$

which yields a contradiction and finishes the proof. □

Remark 1. Assume the Continuum Hypothesis and let E be a Sierpiński subset of $\mathbf{R}$. Denote by $\mathcal{T}_d$ the topology on E induced by the density topology of $\mathbf{R}$. Then $(E, \mathcal{T}_d)$ is a Baire space satisfying the conditions of Lemma 1, so there exists a Sierpiński–Zygmund type function on $(E, \mathcal{T}_d)$. This circumstance implies that $(E, \mathcal{T}_d)$ is not a Blumberg space. Analogously, if S is a Suslin line, then S is a Baire space and also satisfies the conditions of Lemma 1. In view of this lemma, there exists a Sierpiński–Zygmund type function on S. Consequently, S does not have Blumberg's property.

Recall that a topological space E is isodyne if $\mathrm{card}(E) = \mathrm{card}(U)$ for every nonempty open set $U \subset E$.

Let $\mathbf{b}$ be an infinite cardinal number.

We say that a topological space $(E, \mathcal{T})$ is $\mathbf{b}$-Lindelöf if every open covering of E contains a subcovering whose cardinality is strictly less than $\mathbf{b}$.

For example, if $\mathbf{b} = \omega$, then $\mathbf{b}$-Lindelöf spaces are precisely quasi-compact spaces; if $\mathbf{b} = \omega_1$, then $\mathbf{b}$-Lindelöf spaces are precisely Lindelöf spaces in the usual sense.

We say that a topological space $(E, \mathcal{T})$ is $\mathbf{b}$-Baire if no nonempty open subset U of E can be represented as the union of a family $\mathcal{F}$ of nowhere dense sets in E, where $\mathrm{card}(\mathcal{F}) < \mathbf{b}$.

For example, if $\mathbf{b} = \omega_1$, then $\mathbf{b}$-Baire spaces are precisely Baire spaces in the usual sense.

Lemma 2. *Let $\mathbf{b}$ be an infinite regular cardinal and let $(E, \mathcal{T})$ be a hereditarily $\mathbf{b}$-Lindelöf topological space with $\mathrm{card}(E) = \mathbf{b}$.*

Then the family of sets

$$\mathcal{T}^* = \{U \setminus D \ : \ U \in \mathcal{T}, \ D \subset E, \ \mathrm{card}(D) < \mathbf{b}\}$$

is a topology on E extending $\mathcal{T}$ and $(E, \mathcal{T}^*)$ is also a hereditarily $\mathbf{b}$-Lindelöf space.

In addition, if $(E, \mathcal{T})$ is a $\mathbf{b}$-Baire space without isolated points (respectively, an isodyne $\mathbf{b}$-Baire space), then $(E, \mathcal{T}^*)$ is also a $\mathbf{b}$-Baire space without isolated points (respectively, an isodyne $\mathbf{b}$-Baire space).

Theorem 1. *Let $(E, \mathcal{T})$ be an uncountable hereditarily $\mathrm{card}(E)$-Lindelöf space such that:*

(1) $\mathrm{card}(\mathcal{T}) = \mathrm{card}(E)$*;*

(2) $\mathrm{card}(E^\kappa) = \mathrm{card}(E)$ *for every nonzero cardinal $\kappa < \mathrm{card}(E)$.*

Let $\mathcal{T}^*$ denote the topology on E described in Lemma 2.

Then $(E, \mathcal{T}^*)$ is not a Blumberg space.

Proof. It follows from the assumption (2) of the theorem that card(E) is a regular cardinal number. Consequently, $\mathcal{T}^*$ is well-defined and

$$\mathrm{card}(\mathcal{B}(E,\mathcal{T}^*)) = \mathrm{card}(E).$$

Applying Lemma 1 to $(E,\mathcal{T}^*)$, we can find a function $f : E \to \mathbf{R}$ such that, for any set $X \subset E$ with $\mathrm{card}(X) = \mathrm{card}(E)$, the restriction $f|X$ is not continuous on X with respect to the topology $\mathcal{T}^*|X$.

It remains to show that there exists no everywhere dense subset Z of $(E,\mathcal{T}^*)$ such that $f|Z$ is continuous with respect to $\mathcal{T}^*|Z$. But the latter fact is almost trivial, because any everywhere dense subset Z of $(E,\mathcal{T}^*)$ must be equinumerous with E and, by virtue of the definition of f, the partial function $f|Z$ is not continuous on Z with respect to $\mathcal{T}^*|Z$. □

Let $\mathbf{c}$ denote the cardinality of the continuum and let $(E,\mathcal{T})$ be an uncountable isodyne Polish space. If $\mathbf{c}$ is a regular cardinal, then we may consider the topology $\mathcal{T}^*$ described in Lemma 2 (as is known, for $E = \mathbf{R}$, the topology $\mathcal{T}^*$ was introduced by Sierpiński). Theorem 1 implies the following statement.

Theorem 2. *Under Martin's Axiom, the isodyne* $\mathbf{c}$*-Baire hereditarily* $\mathbf{c}$*-Lindelöf space* $(E,\mathcal{T}^*)$ *does not have Blumberg's property.*

In fact, there exists a Sierpiński–Zygmund function for $(E,\mathcal{T}^*)$ *such that its restriction to any second category subset of* $(E,\mathcal{T}^*)$ *is not continuous.*

Remark 2. Suppose that $\mathbf{c} = 2^{\omega_1} = \omega_2$ (this assumption is consistent with Martin's Axiom). Let K be a set of cardinality ω_1 endowed with the discrete topology. Put $E = K^\omega$ and equip this E with the product topology. Clearly, E is an isodyne nonseparable complete metrizable space, hence E is a Blumberg space. Denote

$$\mathcal{T}^* = \{U \setminus D \ : \ U \text{ is open in } E,\ D \subset E,\ \mathrm{card}(D) < \mathbf{c}\}.$$

Then $(E,\mathcal{T}^*)$ is an isodyne Baire hereditarily $\mathbf{c}$-Lindelöf space. Using Lemma 1, we conclude that this space does not have Blumberg's property.

Lemma 3. *Let* $(E,\mathcal{T})$ *be a Blumberg space, let* $\mathcal{T}'$ *be a topology on* E *extending* $\mathcal{T}$*, and suppose that there exists a subfamily of* $\mathcal{T}$ *which is a pseudo-base of* $\mathcal{T}'$*.*

Then $(E,\mathcal{T}')$ *is also a Blumberg space.*

Remark 3. Let $\mathcal{T}$ be the standard topology on $\mathbf{R}$ and let $\mathcal{T}_s$ denote Sorgenfrey's topology on the same $\mathbf{R}$. As is known, $\mathcal{T}_s$ extends $\mathcal{T}$ and $(\mathbf{R},\mathcal{T}_s)$ is an isodyne hereditarily Lindelöf space. It follows from Lemma 3 that $(\mathbf{R},\mathcal{T}_s)$ is a Blumberg space. However, under Martin's Axiom, the isodyne $\mathbf{c}$-Baire hereditarily $\mathbf{c}$-Lindelöf space $(\mathbf{R},\mathcal{T}_s^*)$ does not have Blumberg's property.

There are other similar examples of extensions of a Baire space topology, which do not possess Blumberg's property. It should be noticed that, in the works $[2,3,5,8,9]$ concerned with Blumberg's property, the transfinite construction of Sierpiński–Zygmund type functions is not mentioned at all.

Remark 4. If $(E,\mathcal{T})$ is a Blumberg space, then each open subset of E is a Blumberg space, so the family of all isodyne open Blumberg subspaces of E forms a pseudo-base of E. Under the Generalized Continuum Hypothesis, the cardinality of any uncountable Hausdorff isodyne space of second category (hence the cardinality of any uncountable Hausdorff isodyne Blumberg space) is an ω-power. Conversely, if $\mathbf{a}$ is an ω-power, then there exists an isodyne complete metric space F of cardinality $\mathbf{a}$ and, consequently, in view of [2], F is a Blumberg space.

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