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THE PROBLEM OF FINDING AN EQUISTRONG CONTOUR FOR A VISCOELASTIC RECTANGULAR DOMAIN

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Abstract. The problem of finding an equistrong contour in a rectangular viscoelastic plate is considered by using the Kelvin–Voigt model. It is assumed that normal contractive forces with prescribed principal vectors (or with constant normal displacements) are applied to the rectangle sides by means of a linear absolutely rigid punch, while an unknown part of the boundary (an unknown equistrong contour) is free from external forces. The equistrength of an unknown contour lies in the fact that tangential normal stress at each point of the contour admits the same values. To solve the problem, we use the methods of conformal mappings and of boundary value problems of analytic functions. The equation of an unknown contour, as a function of a point and time, is constructed effectively (analytically).

Introduction

The problems of finding an equistrong contour in the plane theory of elasticity and viscoelasticity may be attributed to a wide class of problems of optimization of shapes of elastic and viscoelastic bodies (see [1]). In the theory of elasticity, the above-mentioned problems for doubly-connected polygonal domains are considered in [2–4].

As is known, the presence of a hole in a plate leads to a non-uniform stress distribution in the vicinity of the holes contour and to the appearance of the so-called plastic zones. In this process, the tangential normal stress is of importance. As the hole expands, the stress values increase, and for viscoelastic bodies, following from the process of stress relaxation, their values decrease over time. So it becomes interesting to adjust the hole shape and size in such a way that for each moment of time the above-mentioned stresses remain constant values. It is to this question that the present work based on the Kelvin–Voigt model [10] is devoted.

Statement of the problem. Let a middle surface of a viscoelastic isotropic plate on the plane z of a complex variable occupy a doubly-connected domain S_0 bounded by a rectangle and a smooth closed contour L_0. Suppose that the rectangle sides are under the action of rectilinear smooth punches with the known principal vectors of normal contractive forces (or with constant normal displacements), while the internal part of the boundary (an unknown equistrong contour) is free from external forces. The equistrength of an unknown contour lies in the fact that the acting tangential normal stress at every point of that contour admits a constant value, i.e., $\sigma_\vartheta(\sigma,t)=K_0^*=\text{const}$ (in a general case, the above-mentioned stresses depend both on a point and on time). The viscoelasticity of the domain S_0 is understood by the Kelvin–Voigt model.

Solution of the problem. By virtue of the symmetry, we restrict ourselves with consideration of equilibria quarter portion of the domain S_0, lying in the first quarter of the coordinate plane, which we denote by S.

Introduce the notation $L=L_0\bigcup L_1$, where $L_0=A_5A_1$, $L_1=\bigcup L_1^{(k)}$ ($L_1^{(k)}=A_kA_{k+1}$, $k=\overline{1,4}$).

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We cite here certain results of works [5,7,8,11], viz, the boundary conditions of the first and second basic problems for a viscoelastic plate S have, according to the Kelvin–Voigt model, the form

$$\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}=i\int\limits_{A_1}^{\sigma}(X_n+iY_n)ds,\quad \sigma\in L_1, \tag{1}$$

$$\int\limits_0^t\Big[\varkappa^*e^{k(\tau-t)}\varphi(\sigma,\tau)+e^{m(\tau-t)}\big(\varphi(\sigma,\tau)-\sigma\overline{\varphi'(\sigma,\tau)}-\overline{\psi(\sigma,\tau)}\big)\Big]d\tau=2\mu^*(u+iv),\quad \sigma\in L_1, \tag{2}$$

$$\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}=0,\quad \sigma\in L_0,$$

$$\int\limits_0^t\Big[\varkappa^*e^{k(\tau-t)}\varphi(\sigma,\tau)+e^{m(\tau-t)}\big(\varphi(\sigma,\tau)-\sigma\overline{\varphi'(\sigma,\tau)}-\overline{\psi(\sigma,\tau)}\big)\Big]d\tau=0,\quad \sigma\in L_0,$$

$$\operatorname{Re}\big[\varphi'(\sigma,t)\big]=\operatorname{Re}\big[\Phi(\sigma,t)\big]=\frac{K_0^*}{4}=K_0,\quad \sigma\in L_0,$$

where

$$\varkappa^*=\frac{2\mu^*}{\lambda^*+\mu^*}\,;\quad k=\frac{\lambda+\mu}{\lambda^*+\mu^*}\,;\quad m=\frac{\mu}{\mu^*}\,.$$

Here and in the sequel, under t we mean a time parameter.

From (1) and (2), we get the equality

$$\Gamma^*\big[\varphi(\sigma,t)\big]=M\bigg[i\int\limits_{A_1}^{\sigma}(X_n+iY_n)ds\bigg]+2\mu^*(u+iv),\quad \sigma\in L_1, \tag{3}$$

where Γ^* and M are the functions of time,

$$\begin{gathered}\Gamma^*\big[\varphi(\sigma,t)\big]=\int\limits_0^t\big[\varkappa^*e^{k(\tau-t)}+2e^{m(\tau-t)}\big]\varphi(\sigma,\tau)d\tau,\\ M\bigg[i\int\limits_{A_1}^{\sigma}(X_n+iY_n)ds\bigg]=\int\limits_0^t e^{m(\tau-t)}\bigg[i\int\limits_{A_1}^{\sigma}(X_n+iY_n)ds\bigg]d\tau.\end{gathered} \tag{4}$$

In view of the fact that in the case under consideration $T(\sigma,t)=0$, $\sigma\in L_1$; $N(\sigma,t)=T(\sigma,t)=0$, $\sigma\in L_0$, $\nu_n=\nu_n^{(k)}=\text{const}$ $(k=\overline{1,4})$, $\nu_s=0$, $\sigma\in L_1$; $\nu_n=\nu_s=0$, $\sigma\in L_0$, and taking into account the equalities $X_n+iY_n=(N+iT)e^{i\alpha(\sigma)}$, $u+i\nu=(\nu_n+i\nu_s)e^{i\alpha(\sigma)}$ ($\alpha(\sigma)$ is the angle made by the Ox - axis and the outer normal to the contour L_1), from (3), we obtain

$$\begin{gathered}\operatorname{Re}\Big[\Gamma^*\big[e^{-i\alpha(\sigma)}\varphi(\sigma,t)\big]\Big]=C(\sigma)F(t)+2\mu^*\nu_n(\sigma),\quad \sigma\in L_1;\\ \operatorname{Re}\big[\varphi'(\sigma,t)\big]=K_0,\qquad \sigma\in L_0,\end{gathered} \tag{5}$$

where

$$C(\sigma)=\operatorname{Re}\bigg[i\int\limits_{A_1}^{\sigma}N(s_0)e^{i[\alpha(s_0)-\alpha(s)]}ds_0\bigg]=\sum_{j=1}^{r}N(s_0)\sin(\alpha_j-\alpha_r)ds_0=C_r=\text{const},$$

$$\sigma\in L_1^{(r)},\qquad r=\overline{1,4},$$

$$F(t)=\frac{1}{m}\big[1-e^{-mt}\big].$$

Performing the mapping of the domain S onto a unit circle by means of the function $z=\omega^0(\zeta)$, and then differentiating (5) with respect to the arc coordinate s, due to the piecewise permanence (with respect to σ) of the right-hand side of (5), for the function

$$\Omega(z,t)=\Gamma^*\big[\varphi'(z,t)-K_0\big]=\Gamma^*\big[\Phi(z,t)-K_0\big]$$

we get the Riemann-Hilbert boundary value problem for the circle

$$\text{Im } \Omega(\eta,t)=0, \quad \eta\in\ell_1, \quad \text{Re } \Omega(\eta,t)=0, \quad \eta\in\ell_0. \tag{6}$$

where ℓ_1 and ℓ_0 are the arcs corresponding to the lines L_1 and L_0.

Problem (6) has only trivial solution and, hence, we have

$$\Gamma^*\big[\Phi(z,t)-K_0\big]=0. \tag{7}$$

It is not difficult to prove that equation (7) has only trivial solution and thus for the function $\varphi(z,t)$ we get the formula

$$\varphi(z,t)-K_0\cdot z \tag{8}$$

(an arbitrary constant is assumed to be zero).

Taking into account (8), the boundary conditions (1) and (2) can be written in the form

$$2K_0\sigma+\overline{\psi(\sigma,t)}=i\int\limits_{A_1}^{\sigma}(X_n+iY_n)ds, \quad \sigma\in L_1, \tag{9}$$

$$\Gamma\big[K_0\sigma\big]-M\big[\overline{\psi(\sigma,t)}\big]=2\mu^*(u+i\nu), \quad \sigma\in L_1, \tag{10}$$

$$2K_0\sigma+\overline{\psi(\sigma,t)}=0, \quad \sigma\in L_0, \tag{11}$$

$$\Gamma\big[K_0\sigma\big]-M\big[\overline{\psi(\sigma,t)}\big]=0, \quad \sigma\in L_0, \tag{12}$$

where Γ is the time function,

$$\Gamma\big[K_0\sigma\big]=\int\limits_0^t \varkappa^* K_0\sigma e^{k(\tau-t)}d\tau, \tag{13}$$

and M is defined by formula (4).

From the boundary conditions (9), (10) and (12), after certain transformations, for the function

$$\Phi_1(z,t)=\Gamma\big[K_0z\big]+M\big[\psi(z,t)\big] \tag{14}$$

we obtain the following boundary conditions

$$\begin{aligned}
&\operatorname{Im}\Phi_1(\sigma,t)=0, \quad \sigma\in L_0, \qquad \operatorname{Im}\Phi_1(\sigma,t)=0, \quad \sigma\in L_1^{(1)},\\
&\operatorname{Re}\Phi_1(\sigma,t)=\Gamma\big[2K_0a\big]+2\mu^*\nu_n^{(2)}, \quad \sigma\in L_2;\\
&\operatorname{Im}\Phi_1(\sigma,t)=2\mu^*\nu_n^{(3)}, \quad \sigma\in L_1^{(3)}, \qquad \operatorname{Re}\Phi_1(\sigma,t)=0, \quad \sigma\in L_1^{(4)}.
\end{aligned} \tag{15}$$

Moreover, for normal displacements $\nu_n^{(2)}$ and $\nu_n^{(3)}$ we get the formulas

$$\begin{aligned}
2\mu^*\nu_n^{(1)}&=-\big[\Gamma[K_0a]+M[2K_0a+P/2]\big],\\
2\mu^*\nu_n^{(3)}&=-\big[\Gamma[K_0b]+M[2K_0b+Q/2]\big].
\end{aligned} \tag{16}$$

Analogously, from (9)–(12), for the function

$$\Phi_2(z,t)=i\big[\Gamma[K_0z]-M[\psi(z,t)]\big], \tag{17}$$

we have

$$\begin{aligned}
&\operatorname{Im}\Phi_2(\sigma,t)=0, \quad \sigma\in L_0, \qquad \operatorname{Re}\Phi_2(\sigma,t)=0, \quad \sigma\in L_1,\\
&\operatorname{Im}\Phi_2(\sigma,t)=\Gamma[K_0a]+M[2K_0a+P/2], \quad \sigma\in L_2,\\
&\operatorname{Re}\Phi_2(\sigma,t)=-\Gamma[K_0b]+M[2K_0b+Q/2], \quad \sigma\in L_3,\\
&\operatorname{Im}\Phi_2(\sigma,t)=0, \quad \sigma\in L_4,
\end{aligned} \tag{18}$$

Problems (15) and (18) are of the same type.

Let the function $z=\omega(\zeta)$ map conformally a unit semi-circle $D_0=\{|\zeta|<1;\ \operatorname{Im}\zeta>0\}$ onto the domain S. By a_k $(k=1,\dots,5)$ we denote preimages of the points A_k and assume that $a_1=1$, $a_3=i$, $a_5=-1$ (that is, the contour L_0 transforms into a segment $[-1;1]$). Consider the functions

$$W_j(\zeta,t)=\begin{cases}\Phi_j(\zeta,t), & \operatorname{Im}\zeta>0,\\ \Phi_{j^*}(\zeta,t), & \operatorname{Im}\zeta<0, \quad j=1,2,\end{cases}$$

where $\Phi_{j^*}(\zeta,t)=\overline{\Phi_j(\overline{\zeta},t)}$.

On the basis of (15) and (18), we can conclude that the functions $W_j(\zeta,t)$ $(j=1,2)$ are holomorphic in the circle $D=\{|\zeta|<1\}$, continuously extendable on the boundary $\ell=\{|\zeta|=1\}$ and satisfy the following boundary conditions:

$$\begin{aligned}&W_1(\omega,t)-W_1\Big(\frac{1}{\omega},t\Big)=0, \quad \omega\in\ell_1^{(1)}, && W_1(\omega,t)+W_1\Big(\frac{1}{\omega},t\Big)=2H_{11}, \quad \omega\in\ell_1^{(2)},\\ &W_1(\omega,t)-W_1\Big(\frac{1}{\omega},t\Big)=2iH_{12}, \quad \omega\in\ell_1^{(3)}, && W_1(\omega,t)+W_1\Big(\frac{1}{\omega},t\Big)=0, \quad \omega\in\ell_1^{(4)},\end{aligned} \tag{19}$$

$$\begin{aligned}&W_2(\omega,t)+W_2\Big(\frac{1}{\omega},t\Big)=0, \quad \omega\in\ell_1^{(1)}, && W_2(\omega,t)-W_2\Big(\frac{1}{\omega},t\Big)=2iH_{21}, \quad \omega\in\ell_1^{(2)},\\ &W_2(\omega,t)+W_2\Big(\frac{1}{\omega},t\Big)=2H_{22}, \quad \omega\in\ell_1^{(3)}, && W_2(\omega,t)-W_2\Big(\frac{1}{\omega},t\Big)=0, \quad \omega\in\ell_1^{(4)},\end{aligned} \tag{20}$$

where due to (16), we have

$$\begin{aligned}&H_{11}=\Gamma\big[K_0a\big]-M[2K_0a+P/2\big], \quad H_{12}=\Gamma\big[K_0b\big]+M\big[2K_0b+Q/2\big],\\ &H_{21}=\Gamma\big[K_0a\big]+M\big[2K_0a+P/2\big], \quad H_{22}=-\Gamma\big[K_0b\big]+M\big[2K_0b+Q/2\big],\end{aligned}$$

($\ell_1^{(k)}$ are the preimages of the lines $L_1^{(k)}$ $(k=\overline{1,4})$).

To solve problems (19) and (20), we use the method of conformal "sewing", provided that in the capacity of a "sewing" function we take Zhukovski's function $\xi=\zeta+\frac{1}{\zeta}$ (see [6]), mapping conformally the circle D onto the whole plane with a cut along the segment $I=[-2;2]$ of the real axis in such a way that the upper semi-circle ℓ^+ is mapped onto a lower edge of the segment, while the lower semi-circle ℓ^- is mapped onto an upper edge of the segment I. As a positive direction on I we take the direction of the real axis and consider an inverse function

$$\zeta(\xi)=\frac{1}{2}\big(\xi-\sqrt{\xi^2-4}\big),$$

where under the radical sign we mean its branch which is positive on the real axis autside the segment I. Then we have

$$\begin{aligned}\zeta^+(\eta)&=\frac{1}{2}\big(\eta-\sqrt{\eta^2-4}\big), \quad \omega\in\ell^+,\\ \zeta^-(\eta)&=\frac{1}{2}\big(\eta+\sqrt{\eta^2-4}\big), \quad \frac{1}{\omega}\in\ell^-.\end{aligned}$$

Consider the functions

$$W_{j0}(\xi,t)=W_j\big[\zeta(\xi),t\big]=W_j\Big[\big(\xi-\sqrt{\xi^2-4}\big)/2,t\Big] \quad (j=1,2).$$

We have

$$\begin{aligned}W_j(\omega,t)&=W_j\Big[\frac{1}{2}\big(\eta-\sqrt{\eta^2-4}\big),t\Big]=W_{j0}^+(\eta,t), \quad \omega\in\ell^+,\\ W_j\Big(\frac{1}{\omega},t\Big)&=W_j\Big[\frac{1}{2}\big(\eta+\sqrt{\eta^2-4}\big),t\Big]=W_{j0}^-(\eta,t), \quad \frac{1}{\omega}\in\ell^-, \quad j=1,2,\end{aligned}$$

and conditions (19) and (20) can be written in the form

$$\begin{aligned}
&W_{10}^{+}(\eta,t)-W_{10}^{-}(\eta,t)=0, \quad \eta\in[\delta;2], && W_{10}^{+}(\eta,t)+W_{10}^{-}(\eta,t)=2H_{11}, \quad \eta\in[0;\delta],\\
&W_{10}^{+}(\eta,t)-W_{10}^{-}(\eta,t)=2iH_{12}, \ \eta\in[-\delta_0;0], && W_{10}^{+}(\eta,t)+W_{10}^{-}(\eta,t)=0, \quad \eta\in[-2;-\delta_0],
\end{aligned} \tag{21}$$

$$\begin{aligned}
&W_{20}^{+}(\eta,t)+W_{20}^{-}(\eta,t)=0, \quad \eta\in[\delta;2], && W_{20}^{+}(\eta,t)-W_{20}^{-}(\eta,t)=2iH_{21}, \quad \eta\in[0;\delta],\\
&W_{20}^{+}(\eta,t)+W_{20}^{-}(\eta,t)=2H_{22}, \ \eta\in[-\delta_0;0], && W_{20}^{+}(\eta,t)-W_{20}^{-}(\eta,t)=0, \quad \eta\in[-2;-\delta_0],
\end{aligned} \tag{22}$$

where -2, $-\delta_0$, 0, δ, 2 are the points of the segment I, corresponding to the points a_k $(k=\overline{1,5})$, under the mapping $\xi=\zeta+\dfrac{1}{\zeta}$.

We are looking for bounded at infinity solutions of problems (21) and (22) of the class $h(-2;-\delta_0;0;\delta;2)$, (for this class see [8]), satisfying the condition

$$W_{j0}(\xi,t)=\overline{W_{j0}(\overline{\xi},t)} \quad (j=1,2). \tag{23}$$

Indices of the problems of the class mentioned above are equal to -2.

The necessary and sufficient conditions for the existence of a bounded at infinity solution of problems (21) and (22) of the class $h(-2;-\delta_0;0;\delta;2)$, have respectively the form

$$iH_{12}\int_{-\delta_0}^{0}\frac{d\eta}{\chi_1(\eta)}+H_{11}\int_{0}^{\delta}\frac{d\eta}{\chi_1(\eta)}=0, \tag{24}$$

$$H_{22}\int_{-\delta_0}^{0}\frac{d\eta}{\chi_2(\eta)}+iH_{21}\int_{0}^{\delta}\frac{d\eta}{\chi_2(\eta)}=0, \tag{25}$$

where

$$\chi_1(\xi)=\sqrt{(\xi+2)(\xi+\delta_0)\xi(\xi-\delta)}; \quad \chi_2(\xi)=\sqrt{(\xi+\delta_0)\xi(\xi-\delta)(\xi-2)}. \tag{26}$$

If conditions (24) and (25) are satisfied, then a solution of problems (21) and (22) is given by the formulas

$$W_{10}(\xi,t)=\frac{\chi_1(\xi)}{\pi i}\left[iH_{12}\int_{-\delta_0}^{0}\frac{d\eta}{\chi_1(\eta)(\eta-\xi)}+H_{11}\int_{0}^{\delta}\frac{d\eta}{\chi_1(\eta)(\eta-\xi)}\right], \tag{27}$$

$$W_{20}(\xi,t)=\frac{\chi_2(\xi)}{\pi i}\left[H_{22}\int_{-\delta_0}^{0}\frac{d\eta}{\chi_2(\eta)(\eta-\xi)}+iH_{21}\int_{0}^{\delta}\frac{d\eta}{\chi_2(\eta)(\eta-\xi)}\right]. \tag{28}$$

It can be easily verified that the functions $W_{j0}(\xi,t)=W_j[\zeta(\xi),t]$ $(j=1,2)$ satisfy condition (23).

Integrals appearing in formulas (24)–(28) are expressed by the first and third kind elliptic integrals, viz (see [9]),

$$\int_{-\delta_0}^{0}\frac{d\eta}{\chi_1(\eta)}=\frac{2}{\sqrt{2(\delta+\delta_0)}}\,F\left[\frac{\pi}{2};\sqrt{\frac{\delta_0(\delta+2)}{2(\delta+\delta_0)}}\right],$$

$$\int_{0}^{\delta}\frac{d\eta}{\chi_1(\eta)}=-\frac{2i}{\sqrt{2(\delta+\delta_0)}}\,F\left[\frac{\pi}{2};\sqrt{\frac{\delta(2-\delta_0)}{2(\delta+\delta_0)}}\right],$$

$$\int_{-\delta_0}^{0}\frac{d\eta}{\chi_1(\eta)(\eta-\xi)}=-\frac{2}{\xi(\xi-\delta)\sqrt{2(\delta+\delta_0)}}$$
$$\times\Big\{-\delta\prod\Big[\frac{\pi}{2};\frac{\delta_0(\xi-\delta)}{\xi(\delta+\delta_0)};\sqrt{\frac{\delta_0(\delta+2)}{2(\delta+\delta_0)}}\Big]+(\xi+\delta_0)F\Big[\frac{\pi}{2};\sqrt{\frac{\delta_0(\delta+2)}{2(\delta+\delta_0)}}\Big]\Big\},$$
$$\int_{0}^{\delta}\frac{d\eta}{\chi_1(\eta)(\eta-\xi)}=\frac{2i}{\xi(\xi+\delta_0)\sqrt{2(\delta+\delta_0)}}$$
$$\times\Big\{\delta_0\prod\Big[\frac{\pi}{2};\frac{\delta(\xi+\delta_0)}{\xi(\delta+\delta_0)};\sqrt{\frac{\delta(2-\delta_0)}{2(\delta+\delta_0)}}\Big]+\xi F\Big[\frac{\pi}{2};\sqrt{\frac{\delta(2-\delta_0)}{2(\delta+\delta_0)}}\Big]\Big\},$$

$$\int_{-\delta_0}^{0}\frac{d\eta}{\chi_2(\eta)}=\frac{-2i}{\sqrt{2(\delta+\delta_0)}}\,F\Big[\frac{\pi}{2};\sqrt{\frac{(2-\delta)\delta_0}{2(\delta+\delta_0)}}\Big],$$
$$\int_{0}^{\delta}\frac{d\eta}{\chi_2(\eta)}=\frac{2}{\sqrt{2(\delta+\delta_0)}}\,F\Big[\frac{\pi}{2};\sqrt{\frac{\delta(2+\delta_0)}{2(\delta+\delta_0)}}\Big],$$

$$\int_{-\delta_0}^{0}\frac{d\eta}{\chi_2(\eta)(\eta-\xi)}=\frac{2i}{\xi(\xi-\delta)\sqrt{2(\delta+\delta_0)}}$$
$$\times\Big\{-\delta\prod\Big[\frac{\pi}{2};\frac{\delta_0(\xi-\delta)}{\xi(\delta-\delta_0)};\sqrt{\frac{(2-\delta)\delta_0}{2(\delta+\delta_0)}}\Big]+\xi F\Big[\frac{\pi}{2};\sqrt{\frac{(2-\delta)\delta_0}{2(\delta+\delta_0)}}\Big\},$$
$$\int_{0}^{\delta}\frac{d\eta}{\chi_2(\eta)(\eta-\xi)}=\frac{-2}{\xi(\xi+\delta_0)\sqrt{2(\delta+\delta_0)}}$$
$$\times\Big\{\delta_0\prod\Big[\frac{\pi}{2};\frac{\delta(\xi+\delta_0)}{\xi(\delta+\delta_0)};\sqrt{\frac{\delta(2+\delta_0)}{2(\delta+\delta_0)}}\Big]+(\xi-\delta)F\Big[\frac{\pi}{2};\sqrt{\frac{\delta(2+\delta_0)}{2(\delta+\delta_0)}}\Big]\Big\},$$

where

$$F[\varphi;k]=\int_0^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}\,;\quad \prod(\varphi,n,k)=\int_0^{\varphi}\frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}$$

are the first and third kind elliptic integrals, respectively.

If we are satisfied with the approximations

$$F\Big[\frac{\pi}{2};k\Big]\approx\frac{\pi}{2}\Big(1+\frac{k^2}{4}\Big);\quad \prod\Big[\frac{\pi}{2};n;k\Big]\approx\frac{\pi}{2}\Big(1+\frac{k^2}{4}+\frac{n}{2}\Big),$$

then conditions (24) and (25) can be written as follows:

$$H_{12}\Big[1+\frac{\delta_0(\delta+2)}{8(\delta+\delta_0)}\Big]-H_{11}\Big[1+\frac{\delta(2-\delta_0)}{8(\delta+\delta_0)}\Big]=0,$$
$$H_{22}\Big[1+\frac{\delta_0(2-\delta)}{8(\delta+\delta_0)}\Big]-H_{21}\Big[1+\frac{\delta(2+\delta_0)}{8(\delta+\delta_0)}\Big]=0.$$

If the above conditions are satisfied, then the functions $W_{10}(\xi,t)$ and $W_{20}(\xi,t)$ are given by the formulas

$$W_{10}(\xi,t) = \frac{\delta_0\,\delta\,\chi_1(\xi)}{\big[2(\delta+\delta_0)\big]^{3/2}\xi^2}\,(H_{12}+H_{11}),$$
$$W_{20}(\xi,t) = -\frac{\delta_0\,\delta\,\chi_2(\xi)}{\big[2(\delta+\delta_0)\big]^{3/2}\xi^2}\,(H_{22}+H_{21}).$$

Having found the functions $W_{j0}(\xi,t)$ $(j=1,2)$, for determining conformally mapping function $z=\omega[\zeta(\xi),t]=\omega_0(\xi,t)$, relying on (13), (14) and (16), we get the integral equation

$$\varkappa^* \int\limits_0^t e^{k\tau}\omega(\xi,\tau)d\tau = e^{kt}N(\xi,t), \tag{29}$$

where

$$N(\xi,t) = \frac{1}{2K_0} = \big[W_{10}(\xi,t) - W_{20}(\xi,t)\big]. \tag{30}$$

From (29), differentiating with respect to t , we obtain

$$\omega_0(\xi,t) = \frac{1}{\varkappa^*}\big[kN(\xi,t) + \dot N(\xi,t)\big],$$

where $N(\xi,t)$ is defined by formula (30) and $\dot N(\xi,t)$ denotes differentiation with respect to t.

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