Transactions of A. Razmadze
Mathematical Institute
Vol. **176** (2022), issue 2, 269–271

THE CARDINALITY NUMBER OF THE CERTAIN CLASSES OF MEASURES

MARIAM BERIASHVILI

Abstract. In the paper, several classes of measures and their cardinality number are discussed.

In the measure theory, the standard concept of measurability of sets and functions with respect to a fixed measure μ on E is well known. Now, we introduce a concept of measurability of sets and functions not with respect to a fixed measure μ, but with respect to the classes of measures, which are defined on different σ-algebras on a base space E.

The above-mentioned concepts are more useful for investigation of special point sets. A sufficiently developed methodology allowing one to study the measurability properties of sets and functions with respect to certain classes is available. In this paper, it is shown that there exist different classes of measures with different cardinality numbers. Here, we also consider some types of subsets of real numbers relying on the concept of measurability of real-valued functions with respect to certain classes of measures.

Throughout this article, we use the following standard notation:
$\mathbf{N}$ is the set of all natural numbers;
$\mathbf{Q}$ is the set of all rational numbers;
$\mathbf{R}$ is the set of all real numbers;
$\mathbf{R}^n$ is the n-dimensional Euclidean space ($n \neq 1$);
$\mathbf{c}$ is cardinality of the continuum;
ω is the cardinality of $\mathbf{N}$;
ω_1 is the first uncountable ordinal number;
$\mathrm{dom}(\mu)$ is the domain of a given measure μ;
$\mathrm{ran}(f)$ is the range of a given function f;
μ' is the completion of a given measure μ.

Let E be a set and M be a class of measures on E (in general, we do not assume that the measures belonging to M are defined on one and the same σ-algebra of a subset of E). The following definition due to Marczewski is essential.

Definition 1. We say that a function $f : E \to \mathbf{R}$ is *absolutely* (*or universally*) *measurable* with respect to M if f is measurable with respect to all measures from M.

We say that a function $f : E \to \mathbf{R}$ is *relatively measurable* with respect to M if there exists at least one measure $\mu \in M$ such that f is μ-measurable.

We say that a function $f : E \to \mathbf{R}$ is *absolutely non-measurable* with respect to M if there exists no one measure $\mu \in M$ such that f is μ-measurable.

The standard concept of measurability of sets and functions with respect to a fixed measure μ on E is a particular case of the above-mentioned definitions. In such a case, the M class contains only one element $\{\mu\}$.

Here, we consider one typical example of a universal set from the classical descriptive set theory.

Example 1. Let E be a Polish topological space and let M denote the class of the completions of all σ-finite Borel measures on E. Let $A(E)$ stand for the class of all Suslin subsets of E and let $\sigma(A(E))$ denote the σ-algebra generated by this class. It is well known that any set $X \in \sigma(A(E))$ is universally measurable with respect to M.

2020 *Mathematics Subject Classification.* 28A05, 28D05.
Key words and phrases. Extension of an invariant measure; Measure extension problem; Absolutely measurable functions; Cardinality of the class of measures.

For more details about this example, see [3, 8].

Let us show an example from the function theory, which is absolutely nonmeasurable.

Example 2. If E is an uncountable topological space, then we may introduce the class M'_E of completions of all nonzero σ-finite diffused Borel measures on E. Let us consider more thoroughly the particular case for $E = \mathbf{R}$. We know that a function $f : \mathbf{R} \to \mathbf{R}$ is relatively measurable with respect to $M'_{\mathbf{R}}$ if f admits a representation in the form

$$f = g \circ \phi,$$

where $g : \mathbf{R} \to \mathbf{R}$ is a Lebesgue measurable function and ϕ is a Borel isomorphism of $\mathbf{R}$ onto itself.

The algebraic sum of two relatively measurable functions with respect to $M'_{\mathbf{R}}$ may be an absolutely nonmeasurable function with respect to the same class. To show this fact, consider two nonzero σ-finite diffused Borel measures μ_1 and μ_2 on $\mathbf{R}$ such that

$$\mu_1([0, +\infty)) = 0, \mu_2(] - \infty, 0[) = 0.$$

Let B_1 denote a Bernstein subset of the half-line $[0, +\infty)$ and B_2 denote a Bernstein subset of the half-line $] - \infty, 0[$.

We define a function $f_1 : \mathbf{R} \to \mathbf{R}$ by putting

$f_1(x) = 1$ if $x \in B_1$ and $f_1(x) = 0$ if $x \in \mathbf{R} \setminus B_1$.

Also, we define a function $f_2 : \mathbf{R} \to \mathbf{R}$ by putting

$f_2(x) = 1$ if $x \in B_2$ and $f_2(x) = 0$ if $x \in \mathbf{R} \setminus B_2$.

It immediately follows from these definitions that the function f_1 is measurable with respect to the completion of μ_1 and the function f_2 is measurable with respect to the completion of μ_2. At the same time, the sum $f_1 + f_2$ is the characteristic function of the subset $B_1 \cup B_2$ of $\mathbf{R}$. But it is easy to see that $B_1 \cup B_2$ is a Bernstein subset of $\mathbf{R}$. Therefore $f_1 + f_2$ is absolutely nonmeasurable with respect to $M'_{\mathbf{R}}$.

Example 3. Let V be an equivalence relation on $\mathbf{R}$ whose all equivalence classes are the most countable. We say that $f : \mathbf{R} \to \mathbf{R}$ is a Vitali type function for V if $(r, f(r)) \in V$ for each $r \in \mathbf{R}$ and the set $\operatorname{ran}(f)$ is a selector of the partition of $\mathbf{R}$ determined by V. Let M_1 be the class of all translation invariant extensions of the Lebesgue measure λ on $\mathbf{R}$ and M_2 be the class of all translation quasi-invariant extensions of the Lebesgue measure λ on $\mathbf{R}$. Then there exists a Vitali type function which is relatively measurable with respect to the class M_2 and absolutely nonmeasurable with respect to the class M_1. For more details about this example, see [4–6].

When considering different classes of measures and measurability properties of sets and functions with respect the classes, there naturally arises the question: how large are the above-mentioned classes, or equivalently, what is the cardinality of the certain classes of measures?

Example 4. Let $M_{L_1} = \{\mu_j : j \in J\}$ be the class of invariant extensions of the Lebesgue measure λ on $\mathbf{R}$. The well known fact is that

$$\operatorname{card}(J) = 2^{2^{\mathbf{c}}}.$$

Definition 2. We recall that a subset X of $\mathbf{R^2}$ is λ_2-thick (or λ_2 massive) in $\mathbf{R^2}$ if for each λ_2-measurable set $Z \subseteq \mathbf{R^2}$ with $\lambda_2(Z) > 0$, we have

$$X \cap Z \neq \emptyset.$$

In other words, X is λ_2-thick in $\mathbf{R^2}$ if and only if the equality

$$(\lambda_2)_*(\mathbf{R^2} \setminus \mathbf{X}) = \mathbf{0}$$

is satisfied.

Example 5. Let $M_{L_2} = \{\mu_j : j \in J_1\}$ be the class of quasi-invariant extensions of the Lebesgue measure λ on $\mathbf{R}$, then

$$\operatorname{card}(J_1) = \mathbf{c},$$

that is, *there exists a continuum set of measures ν on $\mathbf{R}$ extending λ, which are quasi-invariant under the group of all translations of $\mathbf{R}$, and satisfy the relation $H \in \operatorname{dom}(\nu)$.*

For our further purposes, we need several auxiliary notions and propositions.

Let E be an uncountable set, $\mathcal{S}$ be a σ-algebra of subsets of E and μ be a σ-finite measure on E with $\mathrm{dom}(\mu) = \mathcal{S}$. For any two sets $X \in \mathcal{S}$ and $Y \in \mathcal{S}$ satisfying the relations $\mu(X) < +\infty$ and $\mu(Y) < +\infty$, we can put

$$d(X, Y) = \mu(X \triangle Y).$$

The function d is quasi-metric (pseudo-metric) on S which after appropriate factorization, makes the metric space canonically associated with μ. The topological weight of this metric space is called the weight of μ.

A measure μ is called non-separable if the above-mentioned metric space is non-separable (i.e., the weight of μ is strictly greater than the first infinite cardinal ω).

Let E be a nonempty set, G be a group of transformations of E and μ_1 be a σ-finite G-invariant measure defined on some σ-algebra of subsets of E. We say that the measure μ_1 has the uniqueness property if for any σ-finite G-invariant measure μ_2 defined on $\mathrm{dom}(\mu_1)$, there exists a coefficient $t \in \mathbf{R}$ (certainly, depending on μ_2) such that $\mu_2 = t \cdot \mu_1$ (in other words, μ_1 and μ_2 are proportional measures).

Proposition 1. *The cardinal number of the class of all invariant, non-separable measures on the space R^N, extending the Borel measure ξ and possessing the uniqueness property, is equal to 2^{2^c}.*

For the detailed proof of the above-mentioned statement, see [1].

Proposition 2. *On every infinite-dimensional Polish linear space X there is a non-zero σ-finite Borel measure which is invariant with respect to some dense linear subspace.*

For the detailed proof of the above-mentioned statement, see [2].

Theorem 1. *In an infinite-dimensional Polish linear space there exists a class of non-zero σ-finite Borel measure of cardinality, equal 2^{2^c}.*

Acknowledgement

This work was supported by the Shota Rustaveli National Science Foundation of Georgia (SRNSFG), YS-21-1667.

References

1. M. Beriashvili, A. Kirtadze, On the uniqueness property of non-separable extensions of invariant Borel measures and relative measurability of real-valued functions. *Georgian Math. J.* **21** (2014), no. 1, 49–56.
2. T. Gill, A. Kirtadze, G. Pantsulaia, A. Plichko, Existence and uniqueness of translation invariant measures in separable Banach spaces. *Funct. Approx. Comment. Math.* **50** (2014), no. 2, 401–419.
3. K. Kakutani, J. Oxtoby, Construction of a non-separable invariant extension of the Lebesgue measure space. *Ann. of Math. (2)* **52** (1950), 580–590.
4. A. B. Kharazishvili, A certain nonseparable extension of the Lebesgue measure. (Russian) *Dokl. Akad. Nauk SSSR* **226** (1976), no. 1, 69–72.
5. A. B. Kharazishvili, *Topics in Measure Theory and Real Analysis.* Atlantis Studies in Mathematics, 2. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2009.
6. A. B. Kharazishvili, A. P. Kirtadze, On the measurability of functions with respect to certain classes of measures. *Georgian Math. J.* **11** (2004), no. 3, 489–494.
7. A. P. Kirtadze, Nonseparable extensions of invariant measures and the uniqueness property. (Russian) *Soobshch. Akad. Nauk Gruzii* **142** (1991), no. 2, 261–264.
8. K. Kodaira, S. Kakutani, A non-separable translation invariant extension of the Lebesgue measure space. *Ann. of Math. (2)* **52** (1950), 574–579.

(Received 12.06.2022)

I. Vekua Institute of Applied Mathematics, 2 University Str., Tbilisi 0186, Georgia

Georgian Technical University, 77 Kostava Str., Tbilisi 0175, Georgia
E-mail address: mariam_beriashvili@yahoo.com