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AN APPLICATION ON AN ABSOLUTE MATRIX SUMMABILITY METHOD

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Abstract. The aim of this paper is to generalize a main theorem concerning absolute weighted arithmetic mean summability factors of infinite series and Fourier series to the $|A, p_n; \delta|_k$ summability method by using a quasi-f-power increasing sequence.

1. Introduction

Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, that is (see [9]),

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \quad (t_n^1 = t_n),$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0.$$

The series $\sum a_n$ is said to be summable $|C, \alpha; \delta|_k$, $k \geq 1$, and $\delta \geq 0$ if (see [12])

$$\sum_{n=1}^{\infty} n^{\delta k + k - 1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} n^{\delta k - 1} |t_n^\alpha|^k < \infty.$$

If we take $\delta = 0$, then we get the $|C, \alpha|_k$ summability (see [11]).

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence (w_n) of the weighted arithmetic mean or, simply, the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [13]). The series $\sum a_n$ is said to be summable $|\bar{N}, p_n; \delta|_k$, $k \geq 1$, and $\delta \geq 0$ if (see [3])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} | w_n - w_{n-1} |^k < \infty.$$

If we take $\delta = 0$, then we get the $\left|\bar{N}, p_n\right|_k$ summability (see [2]) and if we take $p_n = 1$ for all n, then we have the $|C, 1; \delta|_k$ summability.

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2. Known Result

A positive sequence (b_n) is said to be an almost increasing sequence if there exist a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $K f_n X_n \geq f_m X_m$ for all $n \geq m \geq 1$, where $f = \{f_n(\sigma, \beta)\} = \left\{n^\sigma (logn)^\beta, \beta \geq 0, 0 < \sigma < 1\right\}$ (see [20]).

If we take $\beta = 0$, then we have a quasi-σ-power increasing sequence (see [14]). Every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$.

For any sequence (λ_n) we write that $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$ (see [18]).

For the papers related to absolute summability factors we refer the reader to [4–7, 17, 21, 23–26]. From these papers, Mazhar has obtained a result dealing with the Riesz summability by taking (X_n) as an almost increasing sequence (see [15]) and then Bor has proved a new theorem by taking (X_n) as a quasi -σ-power increasing sequence (see [7]). Also, the following theorem dealing with the $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series including a quasi-f-power increasing sequence, is known.

Theorem 2.1 ([8]). *Let (X_n) be a quasi-f-power increasing sequence. If the sequences (X_n), (λ_n) and (p_n) satisfy the conditions*

$$|\lambda_m| X_m = O(1) \quad as \quad m \to \infty, \tag{1}$$

$$\sum_{n=1}^{m} n X_n |\Delta^2 \lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \quad as \quad m \to \infty,$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty,$$

then the series $\sum a_n \lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

3. The Main Result

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix with nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, \dots .$$

The series $\sum a_n$ is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$, and $\delta \geq 0$ if (see [16])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

In the special case, if we take $a_{nv} = \frac{p_v}{P_n}$, then the $|A, p_n; \delta|_k$ summability reduces to the $\left|\bar{N}, p_n; \delta\right|_k$ summability. If we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then the $|A, p_n; \delta|_k$ summability reduces to the $\left|\bar{N}, p_n\right|_k$ summability. Also if we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all n, then $|A, p_n; \delta|_k$ summability reduces to $|C, 1; \delta|_k$ summability. Moreover, if we take $\delta = 0$, the $|A, p_n; \delta|_k$ summability is the same as the $|A, p_n|_k$ summability (see [19]). Finally, if we take $\delta = 0$ and $p_n = 1$ for all n, then the $|A, p_n; \delta|_k$ summability is the same as the $|A|_k$ summability (see [22]).

The aim of this paper is to generalize Theorem 2.1 to the $|A, p_n; \delta|_k$ summability method for infinite series and Fourier series by taking a quasi-f-power increasing sequence.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \ldots$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \ldots .$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v.$$

Using this notation, we have the following

Theorem 3.1. *Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \ n = 0, 1, \ldots,$$
$$a_{n-1,v} \geq a_{nv}, \ \text{for } n \geq v + 1,$$
$$a_{nn} = O\left(\frac{p_n}{P_n}\right)$$
$$\sum_{v=1}^{n-1} \frac{1}{v} |\hat{a}_{n,v+1}| = O(a_{nn}).$$

Let (X_n) be a quasi-f-power increasing sequence. If the sequences (X_n), (λ_n) and (p_n) satisfy the conditions (1), (2) of Theorem 2.1 and the conditions

$$\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k - 1} \frac{|t_v|^k}{X_v^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty,$$
$$\sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k} \frac{|t_v|^k}{v X_v^{k-1}} = O(X_m) \quad \text{as} \quad m \to \infty,$$
$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\hat{a}_{nv})| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k - 1}\right\} \quad \text{as} \quad m \to \infty,$$
$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\hat{a}_{n,v+1}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\} \quad \text{as} \quad m \to \infty.$$

are satisfied, then the series $\sum a_n \lambda_n$ is $|A, p_n; \delta|_k$ summable.

To prove our theorem, we need the following

Lemma ([4]). *Under the conditions* (1) *and* (2) *of Theorem* 2.1, *we have*

$$\sum_{n=1}^{\infty} X_n |\Delta \lambda_n| < \infty,$$
$$n X_n |\Delta \lambda_n| = O(1) \quad as \quad n \to \infty.$$

4. Proof of Theorem 3.1

Let (I_n) denote the A-transform of the series $\sum a_n\lambda_n$. Then we have

$$\bar{\Delta} I_n = \sum_{v=0}^{n} \hat{a}_{nv} a_v \lambda_v .$$

Applying Abel's transformation to this sum, we have

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{v=1}^{n} v a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big)(v+1)t_v + \hat{a}_{nn}\lambda_n \frac{n+1}{n} t_n \\
&= \sum_{v=1}^{n-1} \frac{v+1}{v} \Delta_v(\hat{a}_{nv})\lambda_v t_v + \sum_{v=1}^{n-1} \frac{v+1}{v} \hat{a}_{n,v+1} \Delta\lambda_v t_v + \sum_{v=1}^{n-1} \hat{a}_{n,v+1}\lambda_{v+1} \frac{t_v}{v} + a_{nn}\lambda_n t_n \frac{n+1}{n} \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it suffices to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} | I_{n,r} |^k < \infty, \quad \text{for} \quad r = 1,2,3,4.$$

Firstly, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, and using the fact that $\Delta_v(\hat{a}_{nv}) = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v}$, and then $\sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \leq a_{nn}$, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} | I_{n,1} |^k &\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left\{ \sum_{v=1}^{n-1} |\frac{v+1}{v}|\, |\Delta_v(\hat{a}_{nv})|\, |\lambda_v||t_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\, |\lambda_v|^k |t_v|^k \times \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1} \left\{ \sum_{v=1}^{n-1} |\Delta_v(\hat{a}_{nv})|\, |\lambda_v|^k |t_v|^k \right\} \\
&= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v||t_v|^k \sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v(\hat{a}_{nv})| \\
&= O(1) \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k-1} \frac{1}{X_v^{k-1}} |\lambda_v||t_v|^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} \left(\frac{P_r}{p_r}\right)^{\delta k-1} \frac{|t_r|^k}{X_r^{k-1}} + O(1)|\lambda_m| \sum_{v=1}^{m} \left(\frac{P_v}{p_v}\right)^{\delta k-1} \frac{|t_v|^k}{X_v^{k-1}} \\
&= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)|\lambda_m| X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma. Also, we have

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} | I_{n,2} |^k &\leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left\{ \sum_{v=1}^{n-1} |\frac{v+1}{v}||\hat{a}_{n,v+1}||\Delta\lambda_v||t_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left\{ \sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v||t_v| \right\}^k
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\frac{1}{X_v^{k-1}}|t_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|X_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\frac{1}{X_v^{k-1}}|t_v|^k\right\}\times\left\{\sum_{v=1}^{n-1}|\Delta\lambda_v|X_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\frac{1}{X_v^{k-1}}|t_v|^k\right\}\\
&= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|\frac{1}{X_v^{k-1}}|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{1}{vX_v^{k-1}}|t_v|^k\left(\frac{P_v}{p_v}\right)^{\delta k}\\
&= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|t_r|^k}{rX_r^{k-1}}+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v|+O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v|+O(1)m|\Delta\lambda_m|X_m\\
&= O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma. Furthermore, we have

$$
\begin{aligned}
&\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|\ I_{n,3}\ |^k\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\right\}^k\\
&\leq\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{|t_v|^k}{v}\right\}\times\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{|t_v|^k}{v}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\frac{|t_v|^k}{v}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}|\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{v}\frac{1}{X_v^{k-1}}|\lambda_{v+1}|\\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_{v+1}|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k}\frac{|t_r|^k}{rX_r^{k-1}}+O(1)|\lambda_{m+1}|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{|t_v|^k}{vX_v^{k-1}}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_{v+1}|X_{v+1}+O(1)|\lambda_{m+1}|X_{m+1}\\
&= O(1)\sum_{v=2}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_{m+1}|X_{m+1}\\
&= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_{m+1}|X_{m+1}\\
&= O(1)\quad\text{as}\quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Theorem 3.1 and Lemma. Again, as in $I_{n,1}$, we have

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^k|\lambda_n|^k|t_n|^k\\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}\frac{1}{X_n^{k-1}}|\lambda_n||t_n|^k\\ &= O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma. This completes the proof of Theorem 3.1.

In a special case where $\delta=0$ and $a_{nv}=\frac{p_v}{P_n}$, we have Theorem 2.1. If we take $\delta=0$, then we have a result dealing with the $|A,p_n|_k$ summability (see [25]). Also, if we take $\delta=0$ and $p_n=1$ for all n, then we have a result for the $|A|_k$ summability.

An Application of absolute matrix summability to the Fourier Series

Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi,\pi)$. The trigonometric Fourier series of f is defined as

$$f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}C_n(x),$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)dx,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)dx.$$

We write

$$\phi(t)=\frac{1}{2}\left\{f(x+t)+f(x-t)\right\},$$

$$\phi_\alpha(t)=\frac{\alpha}{t^\alpha}\int_0^t(t-u)^{\alpha-1}\phi(u)\,du,\quad(\alpha>0).$$

It is well known that if $\phi_1(t)\in\mathcal{BV}(0,\pi)$, then $t_n(x)=O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nC_n(x))$ (see [10]).

In [8], Bor has also obtained a new theorem including the trigonometric Fourier series about the $|\bar{N},p_n|_k$ summability.

Theorem 4.1 ([8])**.** *Let (X_n) be a quasi-f-power increasing sequence. If $\phi_1(t)\in\mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n) and (X_n) satisfy the conditions of Theorem* 2.1, then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.

Now, we generalize Theorem 4.1 to Theorem 4.2 for the $|A,p_n;\delta|_k$ summability method.

Theorem 4.2. *Let A be a positive normal matrix as in Theorem* 3.1. *Let (X_n) be a quasi-f-power increasing sequence. If $\phi_1(t)\in\mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n) and (X_n) satisfy all the conditions of Theorem* 3.1, then the series $\sum C_n(x)\lambda_n$ is summable $|A,p_n;\delta|_k$, $k\geq 1$ and $0\leq\delta<1/k$.

5. Applications

It is noted that if we take $\delta=0$ and $a_{nv}=\frac{p_v}{P_n}$ in Theorem 4.2, then we get Theorem 4.1, and also, if we take $\beta=0$, then we have new theorem on a quasi-σ-power increasing sequence. If we take $a_{nv}=\frac{p_v}{P_n}$ and $p_n=1$ for all n, then we have a theorem on the $|C,1;\delta|_k$ summability factors of Fourier series. If we take $\delta=0$, then we have a result dealing with the $|A,p_n|_k$ summability factors of Fourier series (see [25]). Finally, if we take $\delta=0$ and $p_n=1$ for all n, then we obtain a theorem on the $|A|_k$ summability factors of Fourier series.

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