Transactions of A. Razmadze
Mathematical Institute
Vol. **176** (2022), issue 1, 99–113

ON SOME NEW SEQUENCE SPACES DEFINED BY q-PASCAL MATRIX

TAJA YAYİNG[1], BİPAN HAZARİKA[2] AND FEYZİ BAŞAR[3*]

Abstract. In this study, we construct the q-analog $P(q)$ of Pascal matrix and study the sequence spaces $c(P(q))$ and $c_0(P(q))$ defined as the domain of q-Pascal matrix $P(q)$ in the spaces c and c_0, respectively. We investigate certain topological properties, determine Schauder bases and compute Köthe duals of the spaces $c_0(P(q))$ and $c(P(q))$. We state and prove the theorems characterizing the classes of matrix mappings from the space $c(P(q))$ to the spaces ℓ_∞ of bounded sequences and f of almost convergent sequences. Additionally, we also derive the characterizations of some classes of infinite matrices as a direct consequence of the results about the classes $(c(P(q)), \ell_\infty)$ and $(c(P(q)), f)$. Finally, we obtain the necessary and sufficient conditions for a matrix operator to be compact from the space $c_0(P(q))$ to anyone of the spaces $\ell_\infty, c, c_0, \ell_1, cs_0, cs, bs$.

1. Introduction and Preliminaries

The q-analog of a mathematical expression means the generalization of that expression using the parameter q. The generalized expression returns the original expression when q approaches 1. The study of q-calculus dates back to the time of Euler. It is a wide and an interesting area of research in recent times. Several researchers are engaged in the field of q-calculus due to its vast applications in mathematics, physics and engineering sciences. In the field of mathematics, it is widely used by researchers in approximation theory, combinatorics, hypergeometric functions, operator theory, special functions, quantum algebras, etc.

Let $0 < q < 1$. Then the q-number $r(q)$ is defined by

$$r(q) = \begin{cases} \sum\limits_{v=0}^{r-1} q^v, & r = 1, 2, 3, \ldots, \\ 0, & r = 0. \end{cases}$$

One can notice that $r(q) = r$ whenever $q \to 1$.

The q-analog $\binom{r}{s}_q$ of the binomial coefficient $\binom{r}{s}$ is defined by

$$\binom{r}{s}_q = \begin{cases} \dfrac{r(q)!}{(r-s)(q)!s(q)!}, & r \geq s, \\ 0, & s > r, \end{cases}$$

where q-factorial $r(q)!$ of r is given by

$$r(q)! = r(q)(r-1)(q) \cdots 2(q)1(q).$$

Also, $\binom{0}{0}_q = \binom{r}{0}_q = \binom{r}{r}_q = 1$. Further, $\binom{r}{r-s}_q = \binom{r}{s}_q$ which is a natural q-analog of its ordinary version $\binom{r}{r-s} = \binom{r}{s}$. We strictly refer to [21] for detailed studies in q-calculus.

1.1. **Sequence space.** A linear subspace of ω, the space of all real- or complex-valued sequences, is called a sequence space. Few examples of classical sequence spaces are the space ℓ_k of k-absolutely summable sequences, $1 \leq k < \infty$, the space ℓ_∞ of bounded sequences, the space c of convergent sequences, the space c_0 of null sequences, etc. Further, the spaces of all bounded, convergent and null series are denoted by bs, cs and cs_0, respectively. A Banach sequence space having continuous

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2020 *Mathematics Subject Classification.* 46A45, 46B45, 47B37, 47B10.

Key words and phrases. Sequence space; q-Pascal matrix; Schauder basis; Köthe duals; Matrix mappings; Compact operator.

*Corresponding author.

coordinates is called a BK-space. The spaces c_0 and c are BK-spaces endowed with the supremum norm $\|x\|_\infty = \sup_{r\in\mathbb{N}_0} |x_r|$, where $\mathbb{N}_0$ is the set of non-negative integers.

It is well known that the matrix mappings between BK-spaces are continuous. Because of this celebrated property, the theory of matrix mappings takes an important place in the study of sequence spaces. Let X and Y be two sequence spaces and $\Phi = (\phi_{rs})$ be an infinite matrix of real or complex entries. Further, let Φ_r denote the r^{th} row of the matrix Φ, i.e., $\Phi_r = (\phi_{rs})_{s\in\mathbb{N}_0}$ for all $r \in \mathbb{N}_0$. The sequence $\Phi x = \{(\Phi x)_r\} = \left\{\sum_{s=0}^{\infty} \phi_{rs}x_s\right\}$ is called Φ-transform of the sequence $x = (x_s)$, provided that the series $\sum_{s=0}^{\infty} \phi_{rs}x_s$ converges for each $r \in \mathbb{N}_0$. Further, if $\Phi x \in Y$ for every sequence $x \in X$, then the matrix Φ is said to define a matrix mapping from X to Y. The notation (X, Y) represents the family of all matrices that map from X to Y. Furthermore, the matrix $\Phi = (\phi_{rs})$ is called a triangle if $\phi_{rr} \neq 0$ and $\phi_{rs} = 0$ for $r < s$.

The matrix domain X_Φ of the matrix Φ in the space X is defined by

$$X_\Phi = \{x \in \omega : \Phi x \in X\}. \tag{1.1}$$

The set X_Φ itself is a sequence space. This property plays a significant role in constructing new sequence spaces. Additionally, if Φ is a triangle and X is a BK-space, then the sequence space X_Φ is also a BK-space equipped with the norm $\|x\|_{X_\Phi} = \|\Phi x\|_X$. Several authors applied this celebrated theory in the past to construct new Banach (or BK) sequence spaces using some special triangles. For relevant literature, we refer to the papers [23, 34, 39, 41–45] and textbooks [9, 27, 40].

1.2. **Compact operators and Hausdorff measure of non-compactness (Hmnc).** Let X and Y be two Banach spaces. By $B(X, Y)$ we denote the set of all bounded linear operators from the space X into the space Y which is again a Banach space equipped with the norm $\|L\| = \sup_{x\in B_X} \|Lx\|$, where $L \in B(X, Y)$ and B_X denotes the open ball in X. Further, we denote $\|z\|_{X^\dagger} = \sup_{x\in B_X} \left|\sum_{s=0}^{\infty} z_s x_s\right|$. In this case, we observe that $x = (x_s) \in X^\beta$ provided that the supremum exists.

Now, we recall the definitions of compact operator and **Hmnc** of a bounded set.

Definition 1.1. An operator $L : X \to Y$ is said to be compact if the domain of L is all of X and for every bounded sequence (x_r) in X, the sequence $(L(x_r))$ has a convergent subsequence in Y.

Definition 1.2. The **Hmnc** of a bounded set H in a metric space X is defined by

$$\chi(H) = \inf\left\{\varepsilon > 0 : H \subset \cup_{s=0}^{r} B(x_s, a_s),\ x_s \in X,\ a_s < \varepsilon\ (s = 0, 1, 2, \ldots, r),\ r \in \mathbb{N}_0\right\},$$

where $B(x_s, a_s)$ is the open ball centered at x_s and of radius a_s for each $s = 0, 1, 2, \ldots, r$.

The compact operator and **Hmnc** are closely related. An operator $L : X \to Y$ is compact if and only if $\|L\|_\chi = 0$, where $\|L\|_\chi$ denotes **Hmnc** of the operator L and is defined by $\|L\|_\chi = \chi(L(B_X))$. Using **Hmnc**, several authors obtained the necessary and sufficient conditions for matrix operators to be compact between BK-spaces. For relevant literature, we refer to [12, 13, 28–31].

1.3. **Pascal matrix and related sequence spaces.** The Pascal matrix $P = (p_{rs})$ is defined by

$$p_{rs} = \begin{cases} \binom{r}{r-s}, & 0 \le s \le r, \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}_0$, (see [8, 24]). The domains $c(P)$ and $c_0(P)$ of the matrix P in the spaces c and c_0, respectively are studied by Polat [33]. Aydın and Polat [8] further extended these domains to difference spaces by introducing the difference spaces $c(P\nabla) = c_{P\nabla}$ and $c_0(P\nabla) = (c_0)_{P\nabla}$, where ∇ denotes the first order backward difference operator.

Let $0 < q < 1$. Then the q-analog of $P(q) = \left(p_{rs}^q\right)$ of Pascal matrix is defined by

$$p_{rs}^q = \begin{cases} \binom{r}{s}_q, & 0 \le s \le r, \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}_0$, (cf. [38]). We refer to [19,38] for some publications dealing with q-Pascal matrices.

The construction of sequence spaces using q-analog $C(q)$ of Cesàro matrix has been studied recently by Demiriz and Şahin [18], where $C(q) = \left(c^q_{rs}\right)$ is defined by

$$c^q_{rs} = \begin{cases} \dfrac{q^s}{(r+1)(q)}, & 0 \le s \le r, \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}_0$. The authors studied the domains $X_0(q) = (c_0)_{C(q)}$ and $X_c(q) = c_{C(q)}$. More recently, Yaying et al. [46] studied the Banach spaces $X^q_k = (\ell_k)_{C(q)}$ and $X^q_\infty = (\ell_\infty)_{C(q)}$, and the associated operator ideals. For studies in q-Hausdorff matrices, we refer to [1,2,15,35].

1.4. **The spaces f and f_0.** We begin with giving a short survey on the concept of *almost convergence*. The *shift operator* P is defined on ω by $P_s(x) = x_{s+1}$ for all $s \in \mathbb{N}_0$. A Banach limit L is defined on ℓ_∞, as a non-negative linear functional such that $L(Px) = L(x)$ and $L(e) = 1$, where $e = (1, 1, 1, \ldots)$. A sequence $x = (x_s) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x coincide, are equal to α [25] and is denoted by $f - \lim x_s = \alpha$. Let P^k be the composition of P with itself k times and for a sequence $x = (x_s)$, we write

$$t_{rs}(x) = \frac{1}{r+1} \sum_{k=0}^{r} P^k_s(x) \quad \text{for all} \quad r, s \in \mathbb{N}_0.$$

Lorentz [25] proved that $f - \lim x_s = \alpha$ if and only if $\lim\limits_{r\to\infty} t_{rs}(x) = \alpha$, uniformly in s. It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. For more detail on the Banach limit, the reader may refer to Çolak and Çakar [16], and Das [17]. Now, we can define the spaces f_0 and f of almost null and almost convergent sequences by

$$f_0 := \left\{ x = (x_s) \in \ell_\infty : \lim_{r\to\infty} \sum_{k=0}^{r} \frac{x_{s+k}}{r+1} = 0 \text{ uniformly in } s \right\},$$

$$f := \left\{ x = (x_s) \in \ell_\infty : \lim_{r\to\infty} \sum_{k=0}^{r} \frac{x_{s+k}}{r+1} = \alpha \text{ uniformly in } s \text{ for some } \alpha \in \mathbb{C} \right\}.$$

It is known that the α-, β- and γ-duals of the spaces f_0 and f are the space ℓ_1, (see [14, Part (d) of Theorem 7.1.11] and [11, Proposition 4.5]).

Inspired by the above studies, we construct the BK-spaces $c_0(P(q))$ and $c(P(q))$ generated by the q-analog $P(q)$ of the matrix P. We exhibit some topological properties and determine the bases for the spaces $c_0(P(q))$ and $c(P(q))$. In Section 3, we compute Köthe duals (α-, β- and γ-duals) of the spaces $c_0(P(q))$ and $c(P(q))$. In Section 4, we state and prove theorems characterizing the classes of matrix mappings from the space $c(P(q))$ to the spaces ℓ_∞ of bounded sequences and f of almost convergent sequences. In the final section, we determine the necessary and sufficient conditions for a matrix operator to be compact from the space $c_0(P(q))$ to the space $Y \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$.

2. The Sequence Spaces $c_0(P(q))$ and $c(P(q))$

The q-Pascal matrix $P(q)$ can be expressed in the explicit form as

$$P(q) = \begin{bmatrix} \binom{0}{0}_q & 0 & 0 & 0 & \cdots \\ \binom{1}{0}_q & \binom{1}{1}_q & 0 & 0 & \cdots \\ \binom{2}{0}_q & \binom{2}{1}_q & \binom{2}{2}_q & 0 & \cdots \\ \binom{3}{0}_q & \binom{3}{1}_q & \binom{3}{2}_q & \binom{3}{3}_q & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Clearly, the q-Pascal matrix $P(q)$ reduces to P when q approaches 1. Also, we observe that the sum of the elements of the r^{th} row is $\sum\limits_{s=0}^{r} p^q_{rs} = G_r(q)$, where $G_r(q)$ is called the r^{th} Galois number and is defined by the recurrence relation $G_{r+1}(q) = 2G_r(q) + (q^r - 1)G_{r-1}(q)$ with $G_0(q) = 1$ and $G_1(q) = 2$. If we consider the ordinary binomial coefficient in place of q-binomial coefficient, then the r^{th} Galois number reduces to 2^r. The Galois number plays significant role in determining the number of subspaces of a finite field. For more details, we refer to [21]. Further, we also point out here that the sequence in each column of the matrix $P(q)$ converges unlike the sequence in the columns of the ordinary Pascal matrix. More specifically, $p^q_{rs} \to 1/[s(q)!(1-q)^s]$, as $r \to \infty$.

Now we define the q-Pascal sequence spaces $c_0(P(q))$ and $c(P(q))$ by

$$c_0(P(q)) := \left\{ x = (x_s) \in \omega : \lim_{r\to\infty} \sum_{s=0}^{r} \binom{r}{s}_q x_s = 0 \right\},$$

$$c(P(q)) := \left\{ x = (x_s) \in \omega : \lim_{r\to\infty} \sum_{s=0}^{r} \binom{r}{s}_q x_s \text{ exists} \right\}.$$

We emphasize that when q tends to 1 the spaces $c_0(P(q))$ and $c(P(q))$ are reduced to the Pascal sequence spaces $c_0(P)$ and $c(P)$, respectively, introduced by Polat [33]. With the notation of (1.1), we redefine these sequence spaces by

$$c_0(P(q)) = (c_0)_{P(q)} \quad \text{and} \quad c(P(q)) = c_{P(q)}.$$

The sequence $y = (y_r)$ is defined as the $P(q)$-transform of the sequence $x = (x_s)$. That is,

$$y_r = (P(q)x)_r = \sum_{s=0}^{r} \binom{r}{s}_q x_s \tag{2.1}$$

for each $r \in \mathbb{N}_0$. We suppose throughout that the sequences x and y are connected with the relation in (2.1). Further, we observe by using (2.1) that

$$x_s = \sum_{v=0}^{s} (-1)^{s-v} q^{\binom{s-v}{2}} \binom{s}{v}_q y_v \tag{2.2}$$

for each $s \in \mathbb{N}_0$.

Now, we state our first result.

Theorem 2.1. *$c_0(P(q))$ and $c(P(q))$ are BK-spaces endowed with the norm defined by*

$$\|x\|_{c_0(P(q))} = \|x\|_{c(P(q))} = \sup_{r\in\mathbb{N}_0} \left| \sum_{s=0}^{r} \binom{r}{s}_q x_s \right|.$$

Proof. This is a routine verification. So, we omit details. □

Theorem 2.2. *$c_0(P(q)) \cong c_0$ and $c(P(q)) \cong c$.*

Proof. We provide the proof for the space $c_0(P(q))$. The proof for the space $c(P(q))$ can be obtained in a similar way. Define the mapping $\pi : c_0(P(q)) \to c_0$ by $\pi x = y = P(q)x$ for all $x \in c_0(P(q))$. Clearly, π is linear and one-one. Let $y = (y_r)$ be any sequence in c_0 and $x = (x_s)$ be defined as in (2.2). Then, since $y \in c_0$, we have

$$\begin{aligned} \lim_{r\to\infty} \sum_{s=0}^{r} \binom{r}{s}_q x_s &= \lim_{r\to\infty} \sum_{s=0}^{r} \binom{r}{s}_q \left[\sum_{t=0}^{s} (-1)^{s-t} q^{\binom{s-t}{2}} \binom{s}{t}_q y_t \right] \\ &= \lim_{r\to\infty} y_r = 0. \end{aligned}$$

Thus we realize that x is a sequence in $c_0(P(q))$ and the mapping π is onto, and norm preserving. Hence, $c_0(P(q)) \cong c_0$. □

To end this section, we construct the bases for the spaces $c_0(P(q))$ and $c(P(q))$. We recall that the domain X_Φ of a triangle Φ in the space X has a basis if and only if X has a basis, (see Jarrah and Malkowsky [20, Theorem 2.3]). This statement together with Theorem 2.2 leads us to the following result.

Theorem 2.3. *For every fixed $s \in \mathbb{N}_0$, define the sequence $b^{(s)}(q) = \left(b_r^{(s)}(q)\right)$ of the elements of the space $c_0(P(q))$ by*

$$b_r^{(s)}(q) = \begin{cases} (-1)^{r-s} q^{\binom{r-s}{2}} \binom{r}{s}_q, & s \leq r, \\ 0, & s > r. \end{cases}$$

Then the following statements hold:

(a) *The set $\left\{b^{(0)}(q), b^{(1)}(q), b^{(2)}(q), \ldots\right\}$ forms the basis for the space $c_0(P(q))$ and every $x \in c_0(P(q))$ has a unique representation $x = \sum_{s=0}^{\infty} y_s b^{(s)}(q)$.*

(b) *The set $\left\{e, b^{(0)}(q), b^{(1)}(q), b^{(2)}(q), \ldots\right\}$ forms the basis for the space $c(P(q))$ and every $x \in c(P(q))$ can be uniquely expressed in the form $x = ze + \sum_{s=0}^{\infty}(y_s - z) b^{(s)}(q)$, where $y_s = (P(q)x)_s \to z$, as $s \to \infty$.*

3. Köthe Duals

In this section, we determine Köthe duals (α-, β-, γ-duals) of the spaces $c(P(q))$ and $c_0(P(q))$. Since the computation of duals is similar for both spaces, we omit the proof for the space $c(P(q))$. Before proceeding, we recall the definitions of Köthe duals.

Definition 3.1. The Köthe-Toeplitz duals or α-, β- and γ-duals X^α, X^β and X^γ of a sequence space X are defined by

$$\begin{aligned} X^\alpha &:= \{u = (u_s) \in \omega : ux = (u_s x_s) \in \ell_1 \text{ for all } x \in X\}, \\ X^\beta &:= \{u = (u_s) \in \omega : ux = (u_s x_s) \in cs \text{ for all } x \in X\}, \\ X^\gamma &:= \{u = (u_s) \in \omega : ux = (u_s x_s) \in bs \text{ for all } x \in X\}, \end{aligned}$$

respectively.

The following lemma is essential to determine the dual spaces. Throughout the paper, we denote the collection of all finite subsets of $\mathbb{N}_0$ by $\mathcal{N}$.

Lemma 3.2 ([37])**.** *The following statements hold:*

(i) $\Phi = (\phi_{rs}) \in (c_0, \ell_1) = (c, \ell_1)$ *if and only if*

$$\sup_{K \in \mathcal{N}} \sum_{r=0}^{\infty} \left| \sum_{s \in K} \phi_{rs} \right| < \infty; \tag{3.1}$$

(ii) $\Phi = (\phi_{rs}) \in (c_0, c)$ *if and only if*

$$\sup_{r \in \mathbb{N}_0} \sum_{s=0}^{\infty} |\phi_{rs}| < \infty \tag{3.2}$$

$$\exists \alpha_s \in \mathbb{C} \ni \lim_{r \to \infty} \phi_{rs} = \alpha_s \quad \text{for each} \quad s \in \mathbb{N}_0; \tag{3.3}$$

(iii) $\Phi = (\phi_{rs}) \in (c_0, \ell_\infty) = (c, \ell_\infty)$ *if and only if* (3.2) *holds.*

Theorem 3.3. *The α-dual of the spaces $c_0(P(q))$ and $c(P(q))$ is the set $\delta_1(q)$ which is defined by*

$$\delta_1(q) := \left\{ z = (z_r) \in \omega : \sup_{R \in \mathcal{N}} \sum_{s=0}^{\infty} \left| \sum_{r \in R} (-1)^{r-s} q^{\binom{r-s}{2}} \binom{r}{s}_q z_r \right| < \infty \right\}$$

Proof. Consider the following equality:

$$z_r x_r = \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \binom{r}{s}_q z_r y_s = (A(q)y)_r \tag{3.4}$$

for all $r \in \mathbb{N}_0$, where the matrix $A(q) = \left(a_{rs}^q\right)$ is defined by

$$a_{rs}^q = \begin{cases} (-1)^{r-s} q^{\binom{r-s}{2}} \binom{r}{s}_q z_r, & 0 \leq s \leq r, \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}_0$. We realize by using (3.4) that $zx = (z_r x_r) \in \ell_1$ whenever $x \in c_0(P(q))$ if and only if $A(q)y \in \ell_1$ whenever $y \in c_0$. Thus we deduce that $z = (z_r)$ is a sequence in the α-dual of $c_0(P(q))$ if and only if the matrix $A(q)$ belongs to the class (c_0, ℓ_1). Thus we conclude from Part (i) of Lemma 3.2 that $[c_0(P(q))]^\alpha = \delta_1(q)$. □

Theorem 3.4. *Define the sets $\delta_2(q)$, $\delta_3(q)$ and $\delta_4(q)$ by*

$$\begin{aligned} \delta_2(q) :=& \left\{ z = (z_r) \in \omega : \sum_{r=s}^{\infty} (-1)^{r-s} q^{\binom{r-s}{2}} \binom{r}{s}_q z_r \text{ exists for each } s \in \mathbb{N}_0 \right\}, \\ \delta_3(q) :=& \left\{ z = (z_r) \in \omega : \sup_{r \in \mathbb{N}_0} \sum_{s=0}^{r} \left| \sum_{t=s}^{r} (-1)^{t-s} q^{\binom{t-s}{2}} \binom{t}{s}_q z_t \right| < \infty \right\}, \\ \delta_4(q) :=& \left\{ z = (z_r) \in \omega : \lim_{r \to \infty} \sum_{s=0}^{r} \sum_{t=s}^{r} (-1)^{t-s} q^{\binom{t-s}{2}} \binom{t}{s}_q z_t \text{ exists} \right\}. \end{aligned}$$

Then $[c_0(P(q))]^\beta = \delta_2(q) \cap \delta_3(q)$ and $[c(P(q))]^\beta = \delta_2(q) \cap \delta_3(q) \cap \delta_4(q)$.

Proof. Consider the following equality

$$\begin{aligned} \sum_{s=0}^{r} z_s x_s =& \sum_{s=0}^{r} \left[\sum_{t=0}^{s} (-1)^{s-t} q^{\binom{s-t}{2}} \binom{s}{t}_q y_t \right] z_s \\ =& \sum_{s=0}^{r} \left[\sum_{t=s}^{r} (-1)^{t-s} q^{\binom{t-s}{2}} \binom{t}{s}_q z_t \right] y_s \\ =& (B(q)y)_r \end{aligned} \tag{3.5}$$

for each $r \in \mathbb{N}_0$, where the matrix $B(q) = (b_{rs}^q)$ is defined by

$$b_{rs}^q = \begin{cases} \sum\limits_{t=s}^{r} (-1)^{t-s} q^{\binom{t-s}{2}} \binom{t}{s}_q z_t, & 0 \leq s \leq r, \\ 0, & s > r \end{cases}$$

for all $r, s \in \mathbb{N}_0$. Thus, on using (3.5), we realize that $zx = (z_r x_r) \in cs$ whenever $x = (x_r) \in c_0(P(q))$ if and only if $B(q)y \in c$ whenever $y = (y_s) \in c_0$. This yields that $z = (z_r)$ is a sequence in β-dual of $c_0(P(q))$ if and only the matrix $B(q)$ belongs to the class (c_0, c). This in turn implies by using Part (ii) of Lemma 3.2 that

$$\sup_{r \in \mathbb{N}_0} \sum_{s=0}^{r} |b_{rs}^q| < \infty \text{ and } \lim_{r \to \infty} b_{rs}^q \text{ exists for each } s \in \mathbb{N}_0.$$

Thus $[c_0(P(q))]^\beta = \delta_2(q) \cap \delta_3(q)$. □

Theorem 3.5. *The γ-dual of the spaces $c_0(P(q))$ and $c(P(q))$ is the set $\delta_3(q)$.*

Proof. This is similar to the proof of Theorem 3.4 with Part (iii) instead of Part (ii) of Lemma 3.2. To avoid the repetition of the similar statements, we omit details. □

4. Matrix Mappings

In the present section, we essentially determine the necessary and sufficient conditions for a matrix to define the mapping from the space $c(P(q))$ to the spaces ℓ_∞ and f, firstly. Later, we give the characterizations from the space $c(P(q))$ to any of the spaces c, f_0, c_0, bs, fs, fs_0, cs, cs_0.

Now, define the matrix $\Theta = (\theta_{rs})$ via the matrix $\Phi = (\phi_{rs})$ by

$$\theta_{rs} = \sum_{t=s}^{\infty} (-1)^{t-s} q^{\binom{t-s}{2}} \binom{t}{s}_q \phi_{rt} \quad \text{for all} \quad r, s \in \mathbb{N}_0. \tag{4.1}$$

Throughout the text, we suppose that the elements of the matrices $\Theta = (\theta_{rs})$ and $\Phi = (\phi_{rs})$ are connected with the relation (4.1).

Theorem 4.1. $\Phi = (\phi_{rs}) \in (c(P(q)), \ell_\infty) = (c_0(P(q)), \ell_\infty)$ *if and only if*

$$\sup_{r \in \mathbb{N}_0} \sum_{s=0}^{\infty} |\theta_{rs}| < \infty. \tag{4.2}$$

Proof. Suppose that (4.2) holds and take any $x = (x_s) \in c(P(q))$. Then $\Phi_r = (\phi_{rs})_{s \in \mathbb{N}_0} \in [c(P(q))]^\beta$ for each $r \in \mathbb{N}_0$. This implies the existence of Φx. Let $r \in \mathbb{N}_0$ be fixed. Let us consider the following equality obtained from the j^{th} partial sum of the series $\sum_{s=0}^{\infty} \phi_{rs} x_s$ with (2.2):

$$\begin{aligned} \sum_{s=0}^{j} \phi_{rs} x_s &= \sum_{s=0}^{j} \phi_{rs} \left[\sum_{t=0}^{s} (-1)^{s-t} q^{\binom{s-t}{2}} \binom{s}{t}_q y_t \right] \\ &= \sum_{s=0}^{j} \left[\sum_{t=s}^{j} (-1)^{t-s} q^{\binom{t-s}{2}} \binom{t}{s}_q \phi_{rt} \right] y_s \end{aligned} \tag{4.3}$$

for all $j, r \in \mathbb{N}_0$. Then, by letting $j \to \infty$ in (4.3), we observe that

$$(\Phi x)_r = \sum_{s=0}^{\infty} \theta_{rs} y_s = (\Theta y)_r \tag{4.4}$$

for all $r \in \mathbb{N}_0$. Therefore the condition in (3.2) of Part (iii) of Lemma 3.2 is satisfied by the matrix Θ. This leads to the fact that $\Theta y = \Phi x \in \ell_\infty$. Hence the condition in (4.2) is sufficient.

Conversely, suppose that $\Phi = (\phi_{rs}) \in (c(P(q)), \ell_\infty)$. Then Φx exists and belongs to the space ℓ_∞ for every $x = (x_s) \in c(P(q))$. Then $\Phi_r = (\phi_{rs})_{s \in \mathbb{N}_0} \in [c(P(q))]^\beta$ for each $r \in \mathbb{N}_0$. This means that the condition in (4.2) is necessary. □

Prior to giving the theorem characterizing the class of matrix transformations from the space $c(P(q))$ to the space f, we quote the following lemma due to King [22] which yields the necessary and sufficient conditions of the class (c, f) of almost conservative matrices.

Lemma 4.2. $\Phi = (\phi_{rs}) \in (c, f)$ *if and only if*

$$\sup_{r \in \mathbb{N}_0} \sum_{s=0}^{\infty} |\phi_{rs}| < \infty, \tag{4.5}$$

$$\exists (\alpha_s) \in \omega \quad \textit{such that} \quad f - \lim\ \phi_{rs} = \alpha_s \quad \textit{for each} \quad s \in \mathbb{N}_0, \tag{4.6}$$

$$\exists \alpha \in \mathbb{C} \quad \textit{such that} \quad f - \lim\ \sum_{s=0}^{\infty} \phi_{rs} = \alpha. \tag{4.7}$$

Theorem 4.3. $\Phi = (\phi_{rs}) \in (c(P(q)), f)$ *if and only if* (4.2) *holds, and*

$$\exists (\alpha_s) \in \omega \quad \textit{such that} \quad f - \lim\ \theta_{rs} = \alpha_s \quad \textit{for each} \quad s \in \mathbb{N}_0, \tag{4.8}$$

$$\exists \alpha \in \mathbb{C} \quad \textit{such that } f - \lim\ \sum_{s=0}^{\infty} \theta_{rs} = \alpha. \tag{4.9}$$

Proof. Suppose that the conditions in (4.2), (4.8) and (4.9) hold, and take any $x = (x_s) \in c(P(q))$. Then, since $\Phi_r = (\phi_{rs})_{s\in\mathbb{N}_0} \in [c(P(q))]^\beta$ for each $r \in \mathbb{N}_0$, Φx exists. Therefore, bearing in mind the relation (4.4), one can see that the conditions in (4.2), (4.8) and (4.9) correspond to the conditions in (4.5), (4.6) and (4.7) of Lemma 4.2 with θ_{rs} instead of ϕ_{rs}, respectively. That is to say that $\Theta y \in f$ which leads by (4.4) to the desired result that $\Phi \in (c(P(q)), f)$.

Conversely, let $\Phi = (\phi_{rs}) \in (c(P(q)), f)$. Then Φx exists and is in the space f for all $x = (x_s) \in c(P(q))$. Since the inclusion $f \subset \ell_\infty$ holds, the necessity of the condition in (4.2) follows from Theorem 4.1. In this situation, since we again have (4.4), one can easily see by passing to f-limit that $\Theta y \in f$. Hence $\Theta \in (c, f)$. Therefore the conditions in (4.6) and (4.7) are satisfied by the matrix Θ which correspond to the conditions in (4.8) and (4.9). □

If we replace the space f_0 by the space f, then Theorem 4.3 reduces to the following:

Corollary 4.4. $\Phi = (\phi_{rs}) \in (c(P(q)), f_0)$ *if and only if* (4.2) *holds, and*

$$f - \lim\ \theta_{rs} = 0 \quad \textit{for each} \quad s \in \mathbb{N}_0, \tag{4.10}$$

$$f - \lim\ \sum_{s=0}^{\infty} \theta_{rs} = 0. \tag{4.11}$$

If we replace the spaces c and c_0 by the spaces f and f_0, then Theorem 4.3 and Corollary 4.4 respectively reduce to the following

Corollary 4.5. $\Phi = (\phi_{rs}) \in (c(P(q)), c)$ *if and only if* (4.2) *holds, and*

$$\exists (\alpha_s) \in \omega \quad \textit{such that} \quad \lim_{r\to\infty} \theta_{rs} = \alpha_s \quad \textit{for each} \quad s \in \mathbb{N}_0, \tag{4.12}$$

$$\exists \alpha \in \mathbb{C} \quad \textit{such that} \ \lim_{r\to\infty} \sum_{s=0}^{\infty} \theta_{rs} = \alpha. \tag{4.13}$$

Corollary 4.6. $\Phi = (\phi_{rs}) \in (c(P(q)), c_0)$ *if and only if* (4.2) *holds, and*

$$\lim_{r\to\infty} \theta_{rs} = 0 \quad \textit{for each} \quad s \in \mathbb{N}_0, \tag{4.14}$$

$$\lim_{r\to\infty} \sum_{s=0}^{\infty} \theta_{rs} = 0. \tag{4.15}$$

Lemma 4.7 ([10, Lemma 5.3])**.** *Let X and Y be any two sequence spaces, A be an infinite matrix, and B be a triangle matrix. Then $A \in (X, Y_B)$ if and only if $BA \in (X, Y)$.*

It is trivial that combining Theorems 4.1, 4.3 and Corollaries 4.4, 4.5 and 4.6 with Lemma 4.7, one can derive the following results.

Corollary 4.8. *Let $\Phi = (\phi_{rs})$ be an infinite matrix over the complex field. Then the following statements hold:*

(i) $\Phi \in (c(P(q)), bs)$ *if and only if* (4.2) *holds with* $\sum_{j=0}^{r} \theta_{js}$ *instead of* θ_{rs} *for all* $r, s \in \mathbb{N}_0$.

(ii) $\Phi \in (c(P(q)), bv_\infty)$ *if and only if* (4.2) *holds with* $\theta_{rs} - \theta_{r-1,s}$ *instead of* θ_{rs}*, where* bv_∞ *denotes the space of all sequences* $x = (x_s)$ *such that* $(x_s - x_{s-1}) \in \ell_\infty$*,* (cf. Başar and Altay [10]).

(iii) $\Phi \in (c(P(q)), X_\infty)$ *if and only if* (4.2) *holds with* $\sum_{j=0}^{r} \theta_{js}/(r+1)$ *instead of* θ_{rs} *for all* $r, s \in \mathbb{N}_0$*, where* X_∞ *denotes the space of all sequences* $x = (x_s)$ *such that* $\left\{ \sum_{s=0}^{r} x_s/(r+1) \right\} \in \ell_\infty$*,* (cf. Ng and Lee [32]).

(iv) $\Phi \in (c(P(q)), g^t_\infty)$ *if and only if* (4.2) *holds with* $\sum_{j=0}^{r} t_j\theta_{js}/T_r$ *instead of* θ_{rs} *for all* $r, s \in \mathbb{N}_0$*, where* $T_r = t_0 + t_1 + \cdots + t_r$ *and* g^t_∞ *denotes the space of all sequences* $x = (x_s)$ *such that* $\left(\sum_{s=0}^{r} t_s x_s/T_r \right) \in \ell_\infty$*,* (cf., Altay and Başar [4]).

(v) $\Phi \in (c(P(q)), a^t_\infty)$ *if and only if* (4.2) *holds with* $\sum_{j=0}^{r}(1+t^j)\theta_{js}/(1+r)$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$*, where* a^t_∞ *denotes the space of all sequences* $x=(x_s)$ *such that* $\left\{\sum_{s=0}^{r}(1+t^s)x_s/(1+r)\right\} \in \ell_\infty$, (cf., Aydın and Başar [7]).

(vi) $\Phi \in (c(P(q)), e^t_\infty)$ *if and only if* (4.2) *holds with* $\sum_{j=0}^{r}\binom{r}{j}(1-t)^{r-j}t^j\theta_{js}$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$*, where* e^t_∞ *denotes the space of all sequences* $x=(x_s)$ *such that* $\left\{\sum_{s=0}^{r}\binom{r}{s}(1-t)^{r-s}t^s x_s\right\} \in \ell_\infty$, (cf., Altay et al. [5]).

Corollary 4.9. *Let* $\Phi = (\phi_{rs})$ *be an infinite matrix over the complex field. Then the following statements hold:*

(i) $\Phi \in (c(P(q)), fs)$ *if and only if* (4.2), (4.8) *and* (4.9) *hold with* $\sum_{j=0}^{r}\theta_{js}$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$*, where* fs *denotes the space of all series whose sequence of partial sums are in the space* f.

(ii) $\Phi \in (c(P(q)), cs)$ *if and only if* (4.2), (4.12) *and* (4.13) *hold with* $\sum_{j=0}^{r}\theta_{js}$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$.

(iii) $\Phi \in (c(P(q)), c(\Delta))$ *if and only if* (4.2), (4.12) *and* (4.13) *hold with* $\theta_{rs}-\theta_{r-1,s}$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$*, where* $c(\Delta)$ *denotes the space of all sequences* $x=(x_s)$ *such that* $(x_s - x_{s-1}) \in c$, (cf., Başar [9]).

(iv) $\Phi \in (c(P(q)), \widetilde{c})$ *if and only if* (4.2), (4.12) *and* (4.13) *hold with* $\sum_{j=0}^{r}\theta_{js}/(r+1)$ *instead of* θ_{rs} *for all* $k,n \in \mathbb{N}_0$*, where* $\widetilde{c}$ *denotes the space of all sequences* $x=(x_s)$ *such that* $\left\{\sum_{s=0}^{r}x_s/(r+1)\right\} \in c$, (cf., Şengönül and Başar [36]).

(v) $\Phi \in (c(P(q)), g^t_c)$ *if and only if* (4.2), (4.12) *and* (4.13) *hold with* $\sum_{j=0}^{r}t_j\theta_{js}/T_r$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$*, where* g^t_c *denotes the space of all sequences* $x=(x_s)$ *such that* $\left(\sum_{s=0}^{r}t_s x_s/T_r\right) \in c$, (cf., Altay and Başar [4]).

(vi) $\Phi \in (c(P(q)), a^t_c)$ *if and only if* (4.2), (4.12) *and* (4.13) *hold with* $\sum_{j=0}^{r}(1+t^j)\theta_{js}/(1+r)$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$*, where* a^t_c *denotes the space of all sequences* $x=(x_s)$ *such that* $\left\{\sum_{s=0}^{r}(1+t^s)x_s/(1+r)\right\} \in c$, (cf., Aydın and Başar [6]).

(vii) $\Phi \in (c(P(q)), e^t_c)$ *if and only if* (4.2), (4.12) *and* (4.13) *hold with* $\sum_{j=0}^{r}\binom{r}{j}(1-t)^{r-j}t^j\theta_{js}$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$*, where* e^t_c *denotes the space of all sequences* $x=(x_s)$ *such that* $\left\{\sum_{s=0}^{r}\binom{r}{s}(1-t)^{r-s}t^s x_s\right\} \in c$, (cf., Altay and Başar [3]).

Corollary 4.10. *Let* $\Phi = (\phi_{rs})$ *be an infinite matrix over the complex field. Then the following statements hold:*

(i) $\Phi \in (c(P(q)), fs_0)$ *if and only if the conditions in* (4.2), (4.10) *and* (4.11) *hold with* $\sum_{j=0}^{r}\theta_{js}$ *instead of* θ_{rs} *for all* $r,s \in \mathbb{N}_0$*, where* fs_0 *denotes the space of all series whose sequence of partial sums are in the space* f_0.

(ii) $\Phi \in (c(P(q)), cs_0)$ *if and only if the conditions in* (4.2), (4.14) *and* (4.15) *hold with* $\sum_{j=0}^{r} \theta_{js}$ *instead of* θ_{rs} *for all* $r, s \in \mathbb{N}_0$.

(iii) $\Phi \in (c(P(q)), c_0(\Delta))$ *if and only if the conditions in* (4.2), (4.14) *and* (4.15) *hold with* $\theta_{rs} - \theta_{r-1,s}$ *instead of* θ_{rs} *for all* $r, s \in \mathbb{N}_0$, *where* $c_0(\Delta)$ *denotes the space of all sequences* $x = (x_s)$ *such that* $(x_s - x_{s-1}) \in c_0$, (cf., Başar [9]).

(iv) $\Phi \in (c(P(q)), \widetilde{c}_0)$ *if and only if the conditions in* (4.2), (4.14) *and* (4.15) *hold with* $\sum_{j=0}^{r} \theta_{js}/(r+1)$ *instead of* θ_{rs} *for all* $k, n \in \mathbb{N}_0$, *where* $\widetilde{c}_0$ *denotes the space of all sequences* $x = (x_s)$ *such that* $\left\{ \sum_{s=0}^{r} x_s/(r+1) \right\} \in c_0$, (cf., Şengönül and Başar [36]).

(v) $\Phi \in (c(P(q)), g_0^t)$ *if and only if the conditions in* (4.2), (4.14) *and* (4.15) *hold with* $\sum_{j=0}^{r} t_j \theta_{js}/T_r$ *instead of* θ_{rs} *for all* $r, s \in \mathbb{N}_0$, *where* g_0^t *denotes the space of all sequences* $x = (x_s)$ *such that* $\left(\sum_{s=0}^{r} t_s x_s/T_r \right) \in c_0$, (cf., Altay and Başar [4]).

(vi) $\Phi \in (c(P(q)), a_0^r)$ *if and only if the conditions in* (4.2), (4.14) *and* (4.15) *hold with* $\sum_{j=0}^{r}(1 + t^j)\theta_{js}/(1 + r)$ *instead of* θ_{rs} *for all* $r, s \in \mathbb{N}_0$, *where* a_0^t *denotes the space of all sequences* $x = (x_s)$ *such that* $\left\{ \sum_{s=0}^{r} (1 + t^s)x_s/(1 + r) \right\} \in c_0$, (cf., Aydın and Başar [6]).

(vii) $\Phi \in (c(P(q)), e_0^t)$ *if and only if the conditions in* (4.2), (4.14) *and* (4.15) *hold with* $\sum_{j=0}^{r} \binom{r}{j}(1-t)^{r-j} t^j \theta_{js}$ *instead of* θ_{rs} *for all* $r, s \in \mathbb{N}_0$, *where* e_0^t *denotes the space of all sequences* $x = (x_s)$ *such that* $\left\{ \sum_{s=0}^{r} \binom{r}{s}(1-t)^{r-s} t^s x_s \right\} \in c_0$, (cf., Altay and Başar [3]).

5. Compactness by Hmnc

It is known from Part (a) of Theorem 3.2.4 of [27] that if X and Y are any two BK-spaces, then every matrix $\Phi \in (X, Y)$ defines a linear operator $L_\Phi \in B(X, Y)$, where $L_\Phi x = \Phi x$ for all $x \in X$. Moreover, if $X \supset \sigma$ is a BK-space and $\Phi \in (X, Y)$, then $\|L_\Phi\| = \|\Phi\|_{(X,Y)} = \sup_{r \in \mathbb{N}_0} \|\|\Phi_r\|_{X^\dagger} < \infty$ (see [26, Theorem 1.23]), where σ represents the set of all sequences that terminate in zeroes. The following lemmas are essential for our investigation.

Lemma 5.1. $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$. *Further, if* $X \in \{\ell_\infty, c, c_0\}$, *then* $\|x\|_{X^\dagger} = \|x\|_{\ell_1}$.

Lemma 5.2 ([26, Theorem 2.15])**.** *Let* H *be a bounded subset in* c_0 *and define the operator* $\pi_r : c_0 \to c_0$ *by* $\pi_r(x_0, x_1, x_2, \ldots) = (x_0, x_1, x_2, \ldots, x_r, 0, 0, \ldots)$ *for all* $x = (x_r) \in c_0$, *then*

$$\chi(H) = \lim_{r \to \infty} \left[\sup_{r \in H} \|(I - \pi_r)(x)\| \right],$$

where I *is the identity operator on* c_0.

Lemma 5.3 ([29, Theorem 3.7])**.** *Let* $X \supset \sigma$ *be a* BK*-space. Then the following statements hold:*

(a) *If* $\Phi \in (X, c_0)$, *then* $\|L_\Phi\|_\chi = \limsup_{r \to \infty} \|\Phi_r\|_{X^\dagger}$ *and* L_Φ *is compact if and only if* $\lim_{r \to \infty} \|\Phi_r\|_{X^\dagger} = 0$.

(b) *If* X *has* AK *and* $\Phi \in (X, c)$, *then*

$$\frac{1}{2} \limsup_{r \to \infty} \|\Phi_r - \phi\|_{X^\dagger} \leq \|L_\Phi\|_\chi \leq \limsup_{r \to \infty} \|\Phi_r - \phi\|_{X^\dagger}$$

and L_Φ *is compact if and only if* $\lim_{r \to \infty} \|\Phi_r - \phi\|_{X^\dagger} = 0$, *where* $\phi = (\phi_s)$ *with* $\phi_{rs} \to \phi_s$, *as* $r \to \infty$, *for all* $s \in \mathbb{N}_0$.

(c) *If* $\Phi \in (X, \ell_\infty)$, *then* $0 \leq \|L_\Phi\|_\chi \leq \limsup_{r \to \infty} \|\Phi_r\|_{X^\dagger}$ *and* L_Φ *is compact if* $\lim_{r \to \infty} \|\Phi_r\|_{X^\dagger} = 0$.

In the rest of the paper, $\mathcal{R}_m$ is the subcollection of $\mathcal{R}$ consisting of subsets of $\mathbb{N}_0$ with elements that are greater than m.

Lemma 5.4 ([29, Theorem 3.11]). *Let $X \supset \sigma$ be a BK-space. If $\Phi \in (X, \ell_1)$, then*

$$\lim_{m\to\infty}\left(\sup_{R\in\mathcal{R}_m}\left\|\sum_{r\in R}\Phi_r\right\|_{X^\dagger}\right) \le \|L_\Phi\|_\chi \le 4\cdot \lim_{m\to\infty}\left(\sup_{R\in\mathcal{R}_m}\left\|\sum_{r\in R}\Phi_r\right\|_{X^\dagger}\right)$$

and L_Φ is compact if and only if $\lim\limits_{m\to\infty}\left(\sup\limits_{R\in\mathcal{R}_m}\left\|\sum\limits_{r\in R}\Phi_r\right\|_{X^\dagger}\right) = 0.$

Lemma 5.5 ([29, Theorem 4.4, Corollary 4.5]). *Let $X \supset \sigma$ be a BK-space and let*

$$\|\Phi\|_{bs}^{[r]} = \left\|\sum_{s=0}^{r}\Phi_s\right\|_{X^\dagger}.$$

Then the following statements hold:

(a) *If $\Phi \in (X, cs_0)$, then $\|L_\Phi\|_\chi = \limsup\limits_{r\to\infty}\|\Phi\|_{(X,bs)}^{[r]}$ and L_Φ is compact if and only if $\lim\limits_{r\to\infty}\|\Phi\|_{(X,bs)}^{[r]} = 0$.*

(b) *If X has AK and $\Phi \in (X, cs)$, then*

$$\frac{1}{2}\limsup_{r\to\infty}\left\|\sum_{s=0}^{r}\Phi_s - \tilde{\phi}\right\|_{X^\dagger} \le \|L_\Phi\|_\chi \le \limsup_{r\to\infty}\left\|\sum_{s=0}^{r}\Phi_s - \tilde{\phi}\right\|_{X^\dagger}$$

and L_Φ is compact if and only if $\limsup\limits_{r\to\infty}\left\|\sum\limits_{s=0}^{r}\Phi_s - \tilde{\phi}\right\|_{X^\dagger} = 0$, where $\tilde{\phi} = (\tilde{\phi}_s)$ with $\tilde{\phi}_s = \lim\limits_{r\to\infty}\sum\limits_{m=0}^{r}\phi_{ms}$ for all $s \in \mathbb{N}_0$.

(c) *If $\Phi \in (X, bs)$, then $0 \le \|L_\Phi\|_\chi \le \limsup\limits_{r\to\infty}\|\Phi\|_{(X,bs)}^{[r]}$ and L_Φ is compact if $\lim\limits_{r\to\infty}\|\Phi\|_{(X,bs)}^{[r]} = 0$.*

Lemma 5.6. *Let X be a sequence space and $\Phi = (\phi_{rs})$ be an infinite matrix. If $\Phi \in (c_0(P(q)), X)$, then $\Theta \in (c_0, X)$ and $\Phi x = \Theta y$ for all $x \in c_0(P(q))$.*

Proof. Let $\Phi \in (c_0(P(q)), X)$ and $x \in c_0(P(q))$. Then $\Phi_r = (\phi_{rs})_{s\in\mathbb{N}_0} \in [c_0(P(q))]^\beta$ for all $r \in \mathbb{N}_0$. Consider the following equality:

$$\begin{aligned}(\Theta y)_r &= \sum_{s=0}^{\infty}\theta_{rs}y_s\\ &= \sum_{s=0}^{\infty}\sum_{t=s}^{\infty}(-1)^{t-s}q^{\binom{t-s}{2}}\binom{t}{s}_q\phi_{rt}\left[\sum_{v=0}^{s}\binom{s}{v}_q x_v\right]\\ &= \sum_{s=0}^{\infty}\phi_{rs}x_s = (\Phi x)_r\end{aligned}$$

for all $r \in \mathbb{N}_0$. Thus we realize that Θ_r is absolutely summable for each $r \in \mathbb{N}_0$ and $\Theta y \in X$. This yields the desired consequence $\Theta \in (c_0, X)$. □

Theorem 5.7. *The following statements hold:*

(a) *If $\Phi \in (c_0(P(q)), c_0)$, then $\|L_\Phi\|_\chi = \limsup\limits_{r\to\infty}\sum\limits_{s=0}^{\infty}|\theta_{rs}|$.*

(b) *If $\Phi \in (c_0(P(q)), c)$, then*

$$\frac{1}{2}\limsup_{r\to\infty}\sum_{s=0}^{\infty}|\theta_{rs} - \theta| \le \|L_\Phi\|_\chi \le \limsup_{r\to\infty}\sum_{s=0}^{\infty}|\theta_{rs} - \theta|,$$

where $\theta = (\theta_s)$ and $\theta_s = \lim\limits_{r\to\infty}\theta_{rs}$ for each $s \in \mathbb{N}_0$.

(c) *If $\Phi \in (c_0(P(q)), \ell_\infty)$, then $0 \le \|L_\Phi\|_\chi \le \limsup\limits_{r\to\infty}\sum\limits_{s=0}^{\infty}|\theta_{rs}|$.*

(d) *If* $\Phi \in (c_0(P(q)), \ell_1)$, *then*

$$\lim_{m\to\infty} \|\Phi\|^{[m]}_{(c_0(P(q)),\ell_1)} \leq \|L_\Phi\|_\chi \leq 4 \lim_{m\to\infty} \|\Phi\|^{[m]}_{(c_0(P(q)),\ell_1)},$$

where $\|\Phi\|^{[m]}_{(c_0(P(q)),\ell_1)} = \sup_{R\in\mathcal{R}_m} \sum_{s=0}^{\infty} \left| \sum_{r\in R} \theta_{rs} \right|$.

(e) *If* $\Phi \in (c_0(P(q)), cs_0)$, *then* $\|L_\Phi\|_\chi = \limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} \right| \right)$.

(f) *If* $\Phi \in (c_0(P(q)), cs)$, *then*

$$\frac{1}{2} \limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} - \tilde{\theta} \right| \right) \leq \|L_\Phi\|_\chi \leq \limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} - \tilde{\theta} \right| \right),$$

where $\tilde{\theta} = (\tilde{\theta}_s)$ *with* $\tilde{\theta}_s = \lim_{r\to\infty} \sum_{t=0}^{r} \theta_{ts}$ *for each* $s \in \mathbb{N}_0$.

(g) *If* $\Phi \in (c_0(P(q)), bs)$, *then* $0 \leq \|L_\Phi\|_\chi \leq \limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} \right| \right)$.

Proof. (a) Let $\Phi \in (c_0(P(q)), c_0)$. We observe that

$$\|\Phi_r\|_{c_0(P(q))^\dagger} = \|\Theta_r\|_{c_0^\dagger} = \|\Theta_r\|_{\ell_1} = \sum_{s=0}^{\infty} |\theta_{rs}|$$

for $r \in \mathbb{N}_0$. We realize by using Part (a) of Lemma 5.3 that

$$\|L_\Phi\|_\chi = \limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} |\theta_{rs}| \right).$$

(b) Notice that

$$\|\Theta_r - \theta\|_{c_0^\dagger} = \|\Theta_r - \theta\|_{\ell_1} = \sum_{s=0}^{r} |\theta_{rs} - \theta_s| \tag{5.1}$$

for each $r \in \mathbb{N}_0$. Now, let $\Phi \in (c_0(P(q)), c)$. Then Lemma 5.6 implies that $\Theta \in (c_0, c)$. Employing Part (b) of Lemma 5.3, we deduce that

$$\frac{1}{2} \limsup_{r\to\infty} \|\Theta_r - \theta\|_{c_0^\dagger} \leq \|L_\Phi\|_\chi \leq \limsup_{r\to\infty} \|\Theta_r - \theta\|_{c_0^\dagger},$$

which yields in the light of (5.1) that

$$\frac{1}{2} \limsup_{r\to\infty} \sum_{s=0}^{\infty} |\theta_{rs} - \theta_s| \leq \|L_\Phi\|_\chi \leq \limsup_{r\to\infty} \sum_{s=0}^{\infty} |\theta_{rs} - \theta_s|$$

which is the desired result.

(c) The proof is analogous to that of Part (a). So, we omit details.

(d) We have

$$\left\| \sum_{r\in R} \Theta_r \right\|_{c_0^\dagger} = \left\| \sum_{r\in R} \Theta_r \right\|_{\ell_1} = \sum_{s=0}^{\infty} \left| \sum_{r\in R} \theta_{rs} \right|. \tag{5.2}$$

Let $\Phi \in (c_0(P(q)), \ell_1)$. Then Lemma 5.6 implies that $\Theta \in (c_0, \ell_1)$. Hence by using Lemma 5.4, we get

$$\lim_{m\to\infty} \left(\sup_{R\in\mathcal{R}_m} \left\| \sum_{r\in R} \Theta_r \right\|_{c_0^\dagger} \right) \leq \|L_\Phi\|_\chi \leq 4 \cdot \lim_{m\to\infty} \left(\sup_{R\in\mathcal{R}_m} \left\| \sum_{r\in R} \Theta_r \right\|_{c_0^\dagger} \right)$$

which is reduced by using (5.2) to

$$\lim_{m\to\infty} \|\Phi\|^{[m]}_{(c_0(P(q)),\ell_1)} \leq \|L_\Phi\|_\chi \leq 4 \lim_{m\to\infty} \|\Phi\|^{[m]}_{(c_0(P(q)),\ell_1)},$$

as desired.

(e) Notice that

$$\left\| \sum_{t=0}^{r} \Phi_t \right\|_{c_0(P(q))^{\dagger}} = \left\| \sum_{t=0}^{r} \Theta_t \right\|_{c_0^{\dagger}} = \left\| \sum_{t=0}^{r} \Theta_t \right\|_{\ell_1} = \sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} \right|,$$

which yields by using Part (a) of Lemma 5.5 that

$$\|L_\Phi\|_\chi = \limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} \right| \right).$$

(f) We have

$$\left\| \sum_{t=0}^{r} \Theta_t - \tilde{\theta} \right\|_{c_0^{\dagger}} = \left\| \sum_{t=0}^{r} \Theta_t - \tilde{\theta} \right\|_{\ell_1} = \sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} - \tilde{\theta}_s \right| \tag{5.3}$$

for each $r \in \mathbb{N}_0$. Let $\Phi \in (c_0(P(q)), cs)$. Then Lemma 5.6 implies that $\Theta \in (c_0, cs)$. Thus with the aid of Part (b) of Lemma 5.5, we deduce that

$$\frac{1}{2} \limsup_{r\to\infty} \left\| \sum_{t=0}^{r} \Theta_t - \tilde{\theta}_s \right\|_{c_0^{\dagger}} \le \|L_\Phi\|_\chi \le \limsup_{r\to\infty} \left\| \sum_{t=0}^{r} \Theta_t - \tilde{\theta}_t \right\|_{c_0^{\dagger}},$$

which yields us by using (5.3) that

$$\frac{1}{2} \limsup_{r\to\infty} \left(\sum_{s=0} \left| \sum_{t=0}^{r} \theta_{ts} - \tilde{\theta}_s \right| \right) \le \|L_\Phi\|_\chi \le \limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} - \tilde{\theta}_s \right| \right),$$

as desired.

(g) Since the proof is analogous to that of Part (e), we omit the details. □

Now, we have the following

Corollary 5.8. *The following statements hold:* (a) *Let $\Phi \in (c_0(P(q)), c_0)$. Then L_Φ is compact if and only if $\lim_{r\to\infty} \sum_{s=0}^{\infty} |\theta_{rs}| = 0$.*

(b) *Let $\Phi \in (c_0(P(q)), c)$. Then L_Φ is compact if and only if $\lim_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \tilde{\theta}_{rs} - \tilde{\theta}_s \right| \right) = 0$.*

(c) *Let $\Phi \in (c_0(P(q)), \ell_\infty)$. Then L_Φ is compact if $\lim_{r\to\infty} \sum_{s=0}^{\infty} |\theta_{rs}| = 0$.*

(d) *Let $\Phi \in (c_0(P(q)), \ell_1)$. Then L_Φ is compact if and only if $\lim_{m\to\infty} \left[\sup_{R\in\mathcal{R}_m} \left(\sum_{s=0}^{\infty} \left| \sum_{r\in R} \theta_{rs} \right| \right) \right] = 0$.*

(e) *Let $\Phi \in (c_0(P(q)), cs_0)$. Then L_Φ is compact if and only if $\limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} \right| \right) = 0$.*

(f) *Let $\Phi \in (c_0(P(q)), cs)$. Then L_Φ is compact if and only if $\limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} - \tilde{\theta} \right| \right) = 0$.*

(g) *Let $\Phi \in (c_0(P(q)), bs)$. Then L_Φ is compact if $\limsup_{r\to\infty} \left(\sum_{s=0}^{\infty} \left| \sum_{t=0}^{r} \theta_{ts} \right| \right) = 0$.*

Acknowldgement

The research of the first author (T. Yaying) is supported by SERB, DST, New Delhi, India under the grant number EEQ/2019/000082.

The authors have benefited much from the constructive report and suggestions of the reviewer. So, they are indebted to reviewer for his/her valuable suggestions and comments on the first draft of the manuscript which improved the presentation and readability.

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(Received 22.01.2021)

[1]Department of Mathematics, Dera Natung Government College, Itanagar 791113, India

[2]Department of Mathematics, Gauhati University, Gauhati 781014, India

[3]Department of Primary Mathematics Teacher Education, İnönü University, Malatya 44280, Turkey, Current address: Dumlupinar Mah. Hizirbey Cad. Binyil Apt. No: 179-181, D:1, 34730 – Kadıköy/İstanbul, Turkey

E-mail address: tajayaying20@gmail.com
E-mail address: bh_gu@gauhati.ac.in
E-mail address: feyzibasar@gmail.com; feyzi.basar@inonu.edu.tr