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POTENTIAL METHOD IN THE COUPLED LINEAR THEORY OF ELASTICITY FOR MATERIALS WITH TRIPLE POROSITY

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Abstract. In the present paper, the coupled linear model of elastic triple-porosity materials is considered in which the coupled phenomenon of the concepts of Darcy's law and the volume fractions of three levels of pores (macro-, meso- and micropores) is proposed. The basic boundary value problems of steady vibrations are investigated by means of the potential method (boundary integral equation method) and the theory of singular integral equations. In particular, the fundamental solution of the system of steady vibration equations is constructed explicitly by means of elementary functions and Green's identities are obtained. The basic properties of the surface and volume potentials are established. On the basis of Green's identities and the properties of these potentials, the existence and uniqueness theorems for the classical solutions of the basic boundary value problems of the theory under consideration are proved.

1. Introduction

The presentation of mathematical models of multi-porosity continua and the intensive research of the problems connected with these models arise by extensive use of porous materials in civil and geotechnical engineering, hydrology, technology, etc. (see, for details, de Boer [8], Bear [4], Cheng [9], Coussy [10], Ichikawa and Selvadurai [13], Wang [34]). In addition, in recent years, very important area of application for multiple porosity elasticity is biomechanics (see Cowin [11], Park and Lakes [21] and references therein).

Moreover, since the publication of fundamental work by Truesdell and Toupin [33], the interest in formulations of theories of single- and multi-porosity solids has appreciably increased. The mathematical models of triple-porosity materials represent a new possibility to study the important problems of applied sciences. Nowadays, triple porosity mathematical models for solids with hierarchical macro-, meso-, and microporosity structure are based mostly on one of the following two phenomena: (i) the concept of Darcy's classical law and (ii) the concept of volume fraction.

Indeed, on the basis of Darcy's law the governing equations of poroelasticity for a single-porosity material is proposed by Biot [5]. A number of triple-porosity theories based on Darcy's extended law have been presented by several authors (see Abdassah and Ershaghi [1], Aguilera [2], Bai and Roegiers [3], Straughan [23], Svanadze [24], Wu et al. [35]). The basic equations of these theories of triple-porosity materials involve the displacement vector field and the changes of fluid pressures in the three levels of pores.

On the other hand, using the concept of volume fraction of pores, the theory of elastic materials with single voids is proposed by Nunziato and Cowin [12, 20]. This theory is extended and the mathematical models for deformable materials with double and triple voids are developed by Ieşan and Quintanilla [14] and Svanadze [25, 26], respectively. A wide information on the theories of multi-porosity media can be found in the books by Straughan [22] and Svanadze [27].

Furthermore, many engineering problems of a coupled physical nature need consideration of several coupled mechanical concepts simultaneously. A comprehensive literature is devoted to the thermo-hydro-mechanical-chemical coupling processes in fractured-porous media; such, for example, is the book of Kolditz et al. [15] in which several engineering problems are discussed in terms of numerical methods.

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Recently, Svanadze [28,29] introduced the linear models of elasticity and thermoelasticity for single-porosity materials in which the coupled phenomenon of the concepts of Darcy's law and the volume fraction of pore network is considered. The basic boundary value problems (BVPs) of steady vibrations in the quasi-static case of these models are studied by Bitsadze [6,7] and Mikelashvili [17,18]. More recently, this coupled phenomenon is extended to double-porosity elastic materials [30], viscoelastic solids with single and double porosities [31,32].

In the present paper, the coupled linear model of elastic triple-porosity materials is considered in which the coupled phenomenon of the concepts of Darcy's law and the volume fractions of three levels of pores (macro-, meso- and micropores) is proposed.

This work is organized as follows. In Section 2, the system of governing equations of steady vibrations in the coupled linear theory of elastic triple-porosity materials is presented. In Section 3, the fundamental solution of this system of equations is constructed explicitly by means of elementary functions, and its basic properties are established. In Section 4, the radiation conditions are given and the basic BVPs of steady vibrations are formulated. In Section 5, Green's identities are obtained and the basic properties of the surface (single-layer and double-layer) and volume potentials are established. Finally, in Sections 6 and 7, on the basis of Green's identities and the properties of these potentials, the uniqueness and existence theorems for the classical solutions of the BVPs of steady vibrations are proved, respectively.

2. Basic Equations

In what follows, we consider an isotropic and homogeneous elastic solid with macro-, meso- and microporosity (first, second and third porosity) structure that occupies the region Ω of the Euclidean three-dimensional space $\mathbb{R}^3$. Let $\mathbf{u} = (u_1, u_2, u_3)$ be the displacement vector of a solid elastic skeleton, $\varphi_l(\mathbf{x})$ and $p_l(\mathbf{x})$ be the changes of the volume fraction and the fluid pressure from the reference configuration for l-th pore network, respectively ($l = 1, 2, 3$ and $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$). We assume that repeated indices are summed over the range (1,2,3) and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinates.

Following Svanadze [30], the set of field equations of steady vibrations in the coupled linear theory of elastic triple-porosity materials consists of the following equations:

1. *Equations of steady vibrations*

$$
\begin{gathered}
t_{lj,j} + \rho\omega^2 u_l = -\rho F_l^{(1)}, \\
\sigma_{j,j}^{(l)} + \xi^{(l)} + \rho_l \omega^2 \varphi_l = -\rho F_l^{(2)} \quad \text{(no sum by } l\text{)};
\end{gathered} \tag{1}
$$

2. *Constitutive equations*

$$
t_{lj} = 2\mu e_{lj} + \lambda e_{rr}\delta_{lj} + (b_r\varphi_r - \beta_r p_r)\delta_{lj}, \quad \sigma_j^{(l)} = a_{lr}\varphi_{r,j}; \tag{2}
$$

3. *Equations of fluid mass conservation*

$$
v_{j,j}^{(l)} - i\omega(\zeta_l + \beta_l e_{jj}) + d_l = 0; \tag{3}
$$

4. *Darcy's extended law*

$$
\mathbf{v}^{(l)} = -\frac{\kappa_{lj}}{\mu'}\,\nabla p_j - \hat{\mathbf{s}}^{(l)}. \tag{4}
$$

Here, t_{lj} is the component of total stress tensor, $\mathbf{F}^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)})$ is the body force per unit mass, ρ is the reference mass density, $\rho > 0$; $\sigma_j^{(l)}$ and ρ_l are the component of the equilibrated stress and the coefficient of the equilibrated inertia for l-th pore network, respectively; $\rho_l > 0$, $\mathbf{F}^{(2)} = (F_1^{(2)}, F_2^{(2)}, F_3^{(2)})$ is the extrinsic equilibrated body force per unit mass associated with the pore networks; $\hat{\mathbf{s}}^{(l)} = \rho_{l+3}\,\mathbf{s}^{(l)}$ (no sum), ρ_{l+3} and $\mathbf{s}^{(l)}$ are the density of a fluid and the external forces (such as gravity) for the l-th pore phase, respectively; μ' is the fluid viscosity and κ_{lj} is the macroscopic intrinsic permeability associated with the three pore systems; e_{lj} is the component of the strain tensor,

$$
e_{lj} = \frac{1}{2}(u_{l,j} + u_{j,l}), \tag{5}
$$

the function $\xi^{(l)}$ is the intrinsic equilibrated body force for the l-th pore phase and defined by

$$\xi^{(l)} = -b_l e_{rr} - c_{lj}\varphi_j + m_{lj}p_j; \tag{6}$$

$\mathbf{v}^{(l)} = (v_1^{(l)}, v_2^{(l)}, v_3^{(l)})$ is the fluid flux vector for the l-th pore network, β_l is the effective stress parameter,

$$d_l = \sum_{j=1;\, j\neq l}^{3} \gamma_{l+j-2}\,(p_l - p_j)\,, \tag{7}$$

γ_l is the internal transport coefficient (leakage parameter) corresponding to a fluid transfer rate respecting the intensity of flow between the pore phases, $\gamma_l \geq 0$; ζ_l is the increment of fluid (volumetric strain) in the l-th pore phase and defined by

$$\zeta_l = \alpha_{lj}\, p_j + m_{lj}\varphi_j, \tag{8}$$

α_{lj} is the cross-coupling compressibility for a fluid flow at the interface between the three pore systems and measures the compressibility of pore systems, λ and μ are the Lamé constants, δ_{lj} is the Kronecker delta, m_{lj} is the coefficient of cross-correlation term, $m_{lj} = m_{jl}$, ω is the oscillation frequency, $\omega > 0$, c_{lj} is the constitutive constant and $l, j = 1, 2, 3$.

Substituting equations (2), (4)–(8) into (1) and (3), we get the following system of equations of steady vibrations in the coupled linear theory of elastic triple-porosity materials expressed in terms of the displacement vector $\mathbf{u}$, the changes of volume fractions vector $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ and the changes of fluid pressures vector $\mathbf{p} = (p_1, p_2, p_3)$:

$$\begin{gathered} (\mu\Delta + \rho\omega^2)u_l + (\lambda+\mu)u_{j,lj} + b_j\varphi_{j,l} - \beta_j\, p_{j,l} = -\rho F_l^{(1)}, \\ (a_{lj}\Delta + c'_{lj})\varphi_j - b_l\, u_{j,j} + m_{lj}\, p_j = -\rho\, F_l^{(2)}, \\ (k_{lj}\Delta + \alpha'_{lj})\, p_j + \beta'_l\, u_{j,j} + m'_{lj}\varphi_j = -\operatorname{div}\hat{\mathbf{s}}^{(l)}, \end{gathered} \tag{9}$$

where Δ denotes the Laplace operator, $c'_{lj} = \rho_l\omega^2\delta_{lj} - c_{lj}$ (no sum), $\alpha'_{lj} = i\omega\alpha_{lj} - d_{lj}$, $\beta'_l = i\omega\beta_l$, $m'_{lj} = i\omega m_{lj}$, $l, j = 1, 2, 3$ and

$$\begin{gathered} d_{11} = \gamma_1 + \gamma_2, \quad d_{22} = \gamma_1 + \gamma_3, \quad d_{33} = \gamma_2 + \gamma_3, \\ d_{12} = d_{21} = -\gamma_1, \quad d_{13} = d_{31} = -\gamma_2, \quad d_{23} = d_{32} = -\gamma_3. \end{gathered}$$

Further, we introduce the matrix differential operator $\mathbf{A}(\mathbf{D_x}) = (A_{lj}(\mathbf{D_x}))_{9\times 9}$, where

$$\begin{gathered} A_{lj}(\mathbf{D_x}) = (\mu\,\Delta + \rho\omega^2)\delta_{lj} + (\lambda+\mu)\frac{\partial^2}{\partial x_l \partial x_j}, \quad A_{l;j+3}(\mathbf{D_x}) = b_j\frac{\partial}{\partial x_l}, \\ A_{l;j+6}(\mathbf{D_x}) = -\beta_j\frac{\partial}{\partial x_l}, \quad A_{l+3;j}(\mathbf{D_x}) = -b_l\frac{\partial}{\partial x_j}, \quad A_{l+3;j+3}(\mathbf{D_x}) = a_{lj}\Delta + c'_{lj}, \\ A_{l+3;j+6}(\mathbf{D_x}) = m_{lj}, \quad A_{l+6;j}(\mathbf{D_x}) = \beta'_l\frac{\partial}{\partial x_j}, \quad A_{l+6;j+3}(\mathbf{D_x}) = m'_{lj}, \\ A_{l+6;j+6}(\mathbf{D_x}) = k_{lj}\Delta + \alpha'_{lj}, \quad \mathbf{D_x} = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\Big), \quad l, j = 1, 2, 3. \end{gathered}$$

Then the system (9) can be written as

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{F}(\mathbf{x}), \tag{10}$$

where $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$ and $\mathbf{F} = (-\rho\mathbf{F}^{(1)}, -\rho\mathbf{F}^{(2)}, -\operatorname{div}\hat{\mathbf{s}}^{(1)}, -\operatorname{div}\hat{\mathbf{s}}^{(2)}, -\operatorname{div}\hat{\mathbf{s}}^{(3)})$ are nine-component vector functions.

The goal of this paper is to establish the existence and uniqueness theorems for classical solutions of the BVPs of system (9). For this reason, in the next sections, the fundamental solution of system (9) is constructed explicitly by means of elementary functions, Green's identities are obtained and the potentials are introduced.

3. Fundamental Solution

Let $\tau_1^2, \tau_2^2, \dots, \tau_7^2$ be the roots of algebraic equation $\Lambda_1(-\xi) = 0$ (with respect to ξ), where

$$\Lambda_1(\Delta) = \frac{1}{\mu_0 a_0 k_0} \det \mathbf{M}(\Delta) = \prod_{j=1}^{7} (\Delta + \tau_j^2),$$

$\mu_0 = \lambda + 2\mu$, $a_0 = \det(a_{lj})_{3\times 3}$, $k_0 = \det(k_{lj})_{3\times 3}$ and

$$\begin{gathered}\mathbf{M}(\Delta) = (M_{lj}(\Delta)_{7\times 7}, \quad M_{11}(\Delta) = \mu_0 \Delta + \rho\omega^2, \quad M_{1;j+1}(\Delta) = -b_j \Delta,\\ M_{1;j+4}(\Delta) = \beta_j' \Delta, \quad M_{l+1;1}(\Delta) = b_l, \quad M_{l+1;j+1}(\Delta) = a_{lj}\Delta + c_{lj}',\\ M_{l+1;j+4}(\Delta) = m_{lj}', \quad M_{l+4;1}(\Delta) = -\beta_l, \quad M_{l+4;j+1}(\Delta) = m_{lj},\\ M_{l+4;j+4}(\Delta) = k_{lj}\Delta + \alpha_{lj}', \quad l, j = 1, 2, 3.\end{gathered}$$

We assume that the values $\tau_1^2, \tau_2^2, \dots, \tau_8^2$ are distinct and different from zero, where $\tau_8^2 = \dfrac{\rho\omega^2}{\mu}$.

Let us introduce the following notation:

a)

$$\begin{gathered}\mathbf{N}(\mathbf{D_x}) = (N_{lj}(\mathbf{D_x}))_{9\times 9}, \quad N_{lj}(\mathbf{D_x}) = \frac{1}{\mu}\Lambda_1(\Delta)\,\delta_{lj} + n_{11}(\Delta)\frac{\partial^2}{\partial x_l \partial x_j},\\ N_{l;r+2}(\mathbf{D_x}) = n_{1r}(\Delta)\frac{\partial}{\partial x_l}, \quad N_{r+2;l}(\mathbf{D_x}) = n_{r1}(\Delta)\frac{\partial}{\partial x_l},\\ N_{r+2;s+2}(\mathbf{D_x}) = n_{rs}(\Delta), \quad l, j = 1, 2, 3, \quad r, s = 2, 3, \dots, 7,\end{gathered} \tag{11}$$

where

$$\begin{gathered}n_{j1}(\Delta) = -\frac{1}{\mu\mu_0 a_0 k_0}\left[(\lambda+\mu)M_{j1}^*(\Delta) - b_r M_{j;r+1}^*(\Delta) + \beta_r' M_{j;l+4}^*(\Delta)\right],\\ n_{jl}(\Delta) = \frac{1}{\mu_0 a_0 k_0} M_{jl}^*(\Delta), \quad l = 2, 3, \dots, 7, \quad j = 1, 2, \dots, 7,\end{gathered}$$

and M_{lj}^* is the cofactor of the element M_{lj} of matrix $\mathbf{M}$.

b)

$$\begin{gathered}\mathbf{\Phi}(\mathbf{x}) = (\Phi_{lr}(\mathbf{x}))_{9\times 9}, \quad \Phi_{11}(\mathbf{x}) = \Phi_{22}(\mathbf{x}) = \Phi_{33}(\mathbf{x}) = \sum_{j=1}^{8} \eta_{2j}\gamma^{(j)}(\mathbf{x}),\\ \Phi_{44}(\mathbf{x}) = \Phi_{55}(\mathbf{x}) = \cdots = \Phi_{99}(\mathbf{x}) = \sum_{j=1}^{7} \eta_{1j}\gamma^{(j)}(\mathbf{x}),\\ \Phi_{lr}(\mathbf{x}) = 0, \quad l \neq r, \quad l, r = 1, 2, \dots, 9,\end{gathered} \tag{12}$$

where

$$\gamma^{(j)}(\mathbf{x}) = -\frac{e^{i\tau_j|\mathbf{x}|}}{4\pi\,|\mathbf{x}|} \tag{13}$$

and

$$\begin{gathered}\eta_{1s} = \prod_{l=1,\, l\neq s}^{7} (\tau_l^2 - \tau_s^2)^{-1}, \quad \eta_{2m} = \prod_{l=1,\, l\neq m}^{8} (\tau_l^2 - \tau_m^2)^{-1},\\ \\ s = 1, 2, \dots, 7, \quad j, m = 1, 2, \dots, 8.\end{gathered}$$

By a direct calculation, we obtain the following results.

Theorem 1. *If*

$$k a_0 \mu \mu_0 \neq 0, \tag{14}$$

then the matrix $\mathbf{G}(\mathbf{x}) = (G_{lj}(\mathbf{x}))_{9\times 9}$ *defined by*

$$\mathbf{G}(\mathbf{x}) = \mathbf{N}(\mathbf{D_x})\mathbf{\Phi}(\mathbf{x}) \tag{15}$$

is the fundamental solution of system (9)*, where the matrices* $\mathbf{N}(\mathbf{D_x})$ *and* $\mathbf{\Phi}(\mathbf{x})$ *are given by* (11) *and* (12)*, respectively.*

Theorem 2. *If condition* (14) *is satisfied, then the fundamental solution of the system*

$$\mu\,\Delta u_l + (\lambda+\mu)\,u_{j,lj} = 0, \quad a_{lj}\Delta\varphi_j = 0, \quad k_{lj}\,\Delta p_j = 0, \quad l = 1,2,3$$

is the matrix $\mathbf{\Psi}\,(\mathbf{x}) = (\Psi_{lj}\,(\mathbf{x}))_{9\times 9}$*, where*

$$\Psi_{lj}\,(\mathbf{x}) = \lambda'\,\frac{\delta_{lj}}{|\mathbf{x}|} + \mu'\,\frac{x_l x_j}{|\mathbf{x}|^3}, \quad \Psi_{l+3;j+3}\,(\mathbf{x}) = \frac{a^*_{lj}}{a_0}\gamma^{(0)}(\mathbf{x}),$$

$$\Psi_{l+6;j+6}(\mathbf{x}) = \frac{k^*_{lj}}{k_0}\gamma^{(0)}(\mathbf{x}), \quad \Psi_{lm}\,(\mathbf{x}) = \Psi_{ml}\,(\mathbf{x}) = \Psi_{l+3;j+6}\,(\mathbf{x}) = \Psi_{l+6;j+3}\,(\mathbf{x}) = 0,$$

$$\gamma^{(0)}(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|}, \quad \lambda' = -\frac{\lambda+3\mu}{8\pi\mu\mu_0}, \quad \mu' = -\frac{\lambda+\mu}{8\pi\mu\mu_0},$$

$$l,j = 1,2,3, \quad m = 4,5,\dots,9;$$

a^*_{lj} *and* k^*_{lj} *are the cofactors of the elements* a_{lj} *and* k_{lj} *of matrixes* $(a_{lj})_{3\times 3}$ *and* $(k_{lj})_{3\times 3}$*, respectively.*

Theorem 3. *The relations*

$$\Psi_{lj}\,(\mathbf{x}) = O\big(\,|\mathbf{x}|^{-1}\,\big), \quad \Psi_{l+3;j+3}\,(\mathbf{x}) = O\big(\,|\mathbf{x}|^{-1}\,\big), \quad \Psi_{l+6;j+6}\,(\mathbf{x}) = O\big(\,|\mathbf{x}|^{-1}\,\big)$$

and

$$G_{lj}\,(\mathbf{x}) = O\big(\,|\mathbf{x}|^{-1}\,\big), \quad G_{l+3;j+3}\,(\mathbf{x}) = O\big(\,|\mathbf{x}|^{-1}\,\big),$$
$$G_{l+6;j+6}\,(\mathbf{x}) = O\big(\,|\mathbf{x}|^{-1}\,\big), \quad G_{lm}\,(\mathbf{x}) = O\,(1)\,,$$
$$G_{ml}\,(\mathbf{x}) = O\,(1)\,, \quad G_{l+3;j+6}\,(\mathbf{x}) = O\,(1)\,, \quad G_{l+6;j+3}\,(\mathbf{x}) = O\,(1)$$

hold in the neighborhood of the origin of $\mathbb{R}^3$*, where* $l,j = 1,2,3$*,* $m = 4,5,\dots,9$*.*

Theorem 4. *The matrix* $\mathbf{\Psi}\,(\mathbf{x})$ *is a singular part of the fundamental solution* $\mathbf{G}(\mathbf{x})$ *in the neighborhood of the origin of* $\mathbb{R}^3$*, i.e., the relations*

$$\Gamma_{lj}\,(\mathbf{x}) - \Psi_{lj}\,(\mathbf{x}) = \mathrm{const} + O\,(|\mathbf{x}|)$$

hold in the neighborhood of the origin of $\mathbb{R}^3$*, where* $l,j = 1,2,\dots,9$*.*

4. Boundary Value Problems

Henceforth, in this paper, we assume that $\mathrm{Im}\tau_j \geq 0$ $(j = 1,2,\dots,7)$, $\tau_8 > 0$, the constitutive coefficients satisfy the conditions: $(a_{lj})_{3\times 3}$, $(k_{lj})_{3\times 3}$, $(m_{lj})_{3\times 3}$, $(\alpha_{lj})_{3\times 3}$ are the positive definite matrices and

$$\mu > 0, \quad 3\lambda + 2\mu > 0. \tag{16}$$

Let S be the closed surface surrounding the finite domain Ω^+ in $\mathbb{R}^3$, $S \in C^{1,\nu}$, $0 < \nu \leq 1$, $\overline{\Omega^+} = \Omega^+ \cup S$, $\Omega^- = \mathbb{R}^3 \backslash \overline{\Omega^+}$; $\mathbf{n}(\mathbf{z})$ is the external unit normal vector to S at $\mathbf{z}$. The scalar product of two vectors $\mathbf{U} = (u_1, u_2, \dots, u_9)$ and $\mathbf{V} = (v_1, v_2, \dots, v_9)$ is denoted by $\mathbf{U}\cdot\mathbf{V} = \sum\limits_{j=1}^{9} u_j \bar{v}_j$, where $\bar{v}_j$ is the complex conjugate of v_j.

A vector function $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p}) = (U_1, U_2, \dots, U_9)$ is called regular in Ω^- (or Ω^+) if

(i)

$$U_l \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-}) \quad (\text{or } \ U_l \in C^2(\Omega^+) \cap C^1(\overline{\Omega^+})),$$

(ii)

$$U_l = \sum_{j=1}^{8} U_l^{(j)}, \quad U_l^{(j)} \in C^2(\Omega^-) \cap C^1(\bar{\Omega}^-),$$

(iii) $(\Delta+\tau_j^2)U_l^{(j)}(\mathbf{x})=0$ and

$$\Big(\frac{\partial}{\partial|\mathbf{x}|}-i\tau_j\Big)\,U_l^{(j)}(\mathbf{x})=e^{i\tau_j|\mathbf{x}|}o(|\mathbf{x}|^{-1})\quad\text{for}\quad|\mathbf{x}|\gg1,\tag{17}$$

where $U_m^{(8)}=0$, $j=1,2,\dots,8$, $l=1,2,\dots,9$, $m=4,5,\dots,9$.

Note that the relation (17) ensures

$$U_l^{(j)}(\mathbf{x})=e^{i\tau_j|\mathbf{x}|}O(|\mathbf{x}|^{-1})\quad\text{for}\quad|\mathbf{x}|\gg1,\tag{18}$$

where $j=1,2,\dots,8$, $l=1,2,\dots,9$.

The relations (17) and (18) are the radiation conditions in the coupled linear theory of elastic triple-porosity materials.

In the sequel, we use the matrix differential operator

$$\mathbf{P}(\mathbf{D_x},\mathbf{n})=(P_{lj}(\mathbf{D_x},\mathbf{n}))_{9\times9},$$

where

$$\begin{gathered}P_{lj}(\mathbf{D_x},\mathbf{n})=\mu\delta_{lj}\frac{\partial}{\partial\mathbf{n}}+\mu n_j\frac{\partial}{\partial x_l}+\lambda n_l\frac{\partial}{\partial x_j},\quad P_{l;j+3}(\mathbf{D_x},\mathbf{n})=b_j\,n_l,\\ P_{l;j+6}(\mathbf{D_x},\mathbf{n})=-\beta_j\,n_l,\quad P_{l+3;j+3}(\mathbf{D_x},\mathbf{n})=a_{lj}\frac{\partial}{\partial\mathbf{n}},\\ P_{l+6;j+6}(\mathbf{D_x},\mathbf{n})=k_{lj}\frac{\partial}{\partial\mathbf{n}},\quad P_{l+3;j}(\mathbf{D_x},\mathbf{n})=P_{l+3;j+6}(\mathbf{D_x},\mathbf{n})\\ =P_{l+6;m}(\mathbf{D_x},\mathbf{n})=0,\quad l,j=1,2,3,\quad m=1,2,\dots,6\end{gathered}\tag{19}$$

and $\dfrac{\partial}{\partial\mathbf{n}}$ is the derivative along the vector $\mathbf{n}$.

In the next sections, we will study the following basic internal and external BVPs of steady vibrations in the coupled linear theory of elastic triple-porosity materials.

Find a regular (classical) solution to (10) for $\mathbf{x}\in\Omega^+$ satisfying the boundary condition

$$\lim_{\Omega^+\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{U}(\mathbf{x})\equiv\{\mathbf{U}(\mathbf{z})\}^+=\mathbf{f}(\mathbf{z})\tag{20}$$

in the internal *Problem* $(I)^+_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^+\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{P}(\mathbf{D_x},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x})\equiv\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+=\mathbf{f}(\mathbf{z})\tag{21}$$

in the internal *Problem* $(II)^+_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are the prescribed nine-component vector functions.

Find a regular (classical) solution to (10) for $\mathbf{x}\in\Omega^-$ satisfying the boundary condition

$$\lim_{\Omega^-\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{U}(\mathbf{x})\equiv\{\mathbf{U}(\mathbf{z})\}^-=\mathbf{f}(\mathbf{z})$$

in the external *Problem* $(I)^-_{\mathbf{F},\mathbf{f}}$,

$$\lim_{\Omega^-\ni\mathbf{x}\to\mathbf{z}\in S}\mathbf{P}(\mathbf{D_x},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{x})\equiv\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^-=\mathbf{f}(\mathbf{z})\tag{22}$$

in the external *Problem* $(II)^-_{\mathbf{F},\mathbf{f}}$, where $\mathbf{F}$ and $\mathbf{f}$ are the prescribed nine-component vector functions, $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-.

5. Green's Identities and Potentials

In our further analysis we will need the following matrix differential operators:

$$\begin{gathered}\mathbf{A}^{(r)}(\mathbf{D_x})=\big(A_{lm}^{(r)}(\mathbf{D_x})\big)_{3\times9},\quad A_{lm}^{(r)}(\mathbf{D_x})=A_{lm}(\mathbf{D_x}),\\ \mathbf{P}^{(1)}(\mathbf{D_x},\mathbf{n})=\big(P_{lm}^{(1)}(\mathbf{D_x},\mathbf{n})\big)_{3\times9},\quad P_{lm}^{(1)}(\mathbf{D_x},\mathbf{n})=P_{lm}(\mathbf{D_x},\mathbf{n}),\\ \mathbf{P}^{(2)}(\mathbf{D_x},\mathbf{n})=\big(P_{lj}^{(2)}(\mathbf{D_x},\mathbf{n})\big)_{3\times3},\quad P_{lj}^{(2)}(\mathbf{D_x},\mathbf{n})=P_{l+3;j+3}(\mathbf{D_x},\mathbf{n}),\\ \mathbf{P}^{(3)}(\mathbf{D_x},\mathbf{n})=\big(P_{lj}^{(3)}(\mathbf{D_x},\mathbf{n})\big)_{3\times3},\quad P_{lj}^{(3)}(\mathbf{D_x},\mathbf{n})=P_{l+6;j+6}(\mathbf{D_x},\mathbf{n}),\end{gathered}$$

where $l,j=1,2,3$ and $m=1,2,\dots,9$.

Let $\mathbf{u}'=(u_1',u_2',u_3')$, $\boldsymbol{\varphi}'=(\varphi_1',\varphi_2',\varphi_3')$ and $\mathbf{p}'=(p_1',p_2',p_3')$ be three-component vector functions, $\mathbf{U}'=(\mathbf{u}',\boldsymbol{\varphi}',\mathbf{p}')$. In what follows, we use the following functions:

$$
\begin{aligned}
E^{(0)}(\mathbf{u},\mathbf{u}')&=\frac{1}{3}(3\lambda+2\mu)\operatorname{div}\mathbf{u}\operatorname{div}\overline{\mathbf{u}'}+\frac{\mu}{2}\sum_{l,j=1;\,l\neq j}^{3}\Big(\frac{\partial u_j}{\partial x_l}+\frac{\partial u_l}{\partial x_j}\Big)\Big(\frac{\partial\overline{u_j'}}{\partial x_l}+\frac{\partial\overline{u_l'}}{\partial x_j}\Big)\\
&\quad+\frac{\mu}{3}\sum_{l,j=1}^{3}\Big(\frac{\partial u_l}{\partial x_l}-\frac{\partial u_j}{\partial x_j}\Big)\Big(\frac{\partial\overline{u_l'}}{\partial x_l}-\frac{\partial\overline{u_j'}}{\partial x_j}\Big),
\end{aligned}\tag{23}
$$

$$
\begin{aligned}
E^{(1)}(\mathbf{U},\mathbf{u}')&=E^{(0)}(\mathbf{u},\mathbf{u}')-\rho\omega^2\mathbf{u}\cdot\mathbf{u}'+(b_l\varphi_l-\beta_l p_l)\operatorname{div}\overline{\mathbf{u}'},\\
E^{(2)}(\mathbf{U},\boldsymbol{\varphi}')&=a_{lj}\nabla\varphi_j\cdot\nabla\varphi_l'+(b_l\operatorname{div}\overline{\mathbf{u}}-c_{lj}'\varphi_j-m_{lj}p_j)\overline{\varphi_l'},\\
E^{(3)}(\mathbf{U},\mathbf{p}')&=k_{lj}\nabla p_j\cdot\nabla p_l'-(\beta_l'\operatorname{div}\mathbf{u}+m_{lj}'\varphi_j+\alpha_{lj}'p_j)\overline{p_l'},\\
E(\mathbf{U},\mathbf{U}')&=E^{(1)}(\mathbf{U},\mathbf{u}')+E^{(2)}(\mathbf{U},\boldsymbol{\varphi}')+E^{(3)}(\mathbf{U},\mathbf{p}').
\end{aligned}
$$

We have the following result.

Theorem 5. *If* $\mathbf{U}=(\mathbf{u},\mathbf{p})$ *is a regular vector in* Ω^+, $u_j',\varphi_j',\,p_j'\in C^1(\Omega^+)\cap C(\overline{\Omega^+})$, $j=1,2,3$, *then*

$$
\int_{\Omega^+}\big[\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x})+E(\mathbf{U},\mathbf{U}')\big]\,d\mathbf{x}=\int_S\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S,\tag{24}
$$

where $\mathbf{P}(\mathbf{D_z},\mathbf{n})$ *and* $E(\mathbf{U},\mathbf{U}')$ *are defined by* (19) *and* (23), *respectively.*

Proof. Using the identities

$$
\begin{aligned}
&\int_{\Omega^+}\Big[\mathbf{A}^{(1)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{u}'(\mathbf{x})+E^{(1)}(\mathbf{U},\mathbf{u}')\Big]\,d\mathbf{x}=\int_S\mathbf{P}^{(1)}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{u}'(\mathbf{z})\,d_{\mathbf{z}}S,\\
&\int_{\Omega^+}\Big[\mathbf{A}^{(2)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\boldsymbol{\varphi}'(\mathbf{x})+E^{(2)}(\mathbf{U},\boldsymbol{\varphi}')\Big]\,d\mathbf{x}=\int_S\mathbf{P}^{(2)}(\mathbf{D_z},\mathbf{n})\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}'(\mathbf{z})\,d_{\mathbf{z}}S,\\
&\int_{\Omega^+}\Big[\mathbf{A}^{(3)}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{p}'(\mathbf{x})+E^{(3)}(\mathbf{U},\mathbf{p}')\Big]\,d\mathbf{x}=\int_S\mathbf{P}^{(3)}(\mathbf{D_z},\mathbf{n})\mathbf{p}(\mathbf{z})\cdot\mathbf{p}'(\mathbf{z})\,d_{\mathbf{z}}S
\end{aligned}\tag{25}
$$

we get the relation (24). □

Theorem 5 and the radiation conditions (17) and (18) lead to the following consequence.

Theorem 6. *If* $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\mathbf{p})$ *and* $\mathbf{U}'=(\mathbf{u}',\boldsymbol{\varphi}',\mathbf{p}')$ *are regular vectors in* Ω^-, *then*

$$
\int_{\Omega^-}\big[\mathbf{A}(\mathbf{D_x})\,\mathbf{U}(\mathbf{x})\cdot\mathbf{U}'(\mathbf{x})+E(\mathbf{U},\mathbf{U}')\big]\,d\mathbf{x}=-\int_S\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}(\mathbf{z})\cdot\mathbf{U}'(\mathbf{z})\,d_{\mathbf{z}}S,\tag{26}
$$

where $\mathbf{P}(\mathbf{D_z},\mathbf{n})$ *and* $E(\mathbf{U},\mathbf{U}')$ *are defined by* (19) *and* (23), *respectively.*

Formulas (24) and (26) are Green's first identities in the considered theory for domains Ω^+ and Ω^-, respectively.

We introduce the following matrix differential operators:

1) $\tilde{\mathbf{A}}(\mathbf{D_x})=\mathbf{A}^\top(-\mathbf{D_x})$, where the superscript $\top$ denotes transposition and

2)

$$
\begin{gathered}
\tilde{\mathbf{P}}(\mathbf{D_x},\mathbf{n})=(\tilde{P}_{lj}(\mathbf{D_x},\mathbf{n}))_{9\times9},\quad \tilde{P}_{lr}(\mathbf{D_x},\mathbf{n})=P_{lr}(\mathbf{D_x},\mathbf{n}),\\
\tilde{P}_{l;j+6}(\mathbf{D_x},\mathbf{n})=-\beta_j'n_l,\quad \tilde{P}_{ms}(\mathbf{D_x},\mathbf{n})=P_{ms}(\mathbf{D_x},\mathbf{n}),\\
l,j=1,2,3,\quad r=1,2,\dots,6,\quad m=4,5,\dots,9,\quad s=1,2,\dots,9.
\end{gathered}\tag{27}
$$

Let $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\mathbf{p})=(U_1,U_2,\dots,U_9)$, the vector $\tilde{\mathbf{U}}_j$ be the j-th column of the matrix $\tilde{\mathbf{U}}=(\tilde{U}_{lj})_{9\times9}$, $\tilde{\mathbf{u}}_j=(\tilde{U}_{1j},\tilde{U}_{2j},\tilde{U}_{3j})^\top$, $\tilde{\boldsymbol{\varphi}}_j=(\tilde{U}_{4j},\tilde{U}_{5j},\tilde{U}_{6j})^\top$, $\tilde{\mathbf{p}}_j=(\tilde{U}_{7j},\tilde{U}_{8j},\tilde{U}_{9j})^\top$, $j=1,2,\dots,9$.

Now, it is not very difficult to prove the following theorems.

Theorem 7. *If U and $\tilde{U}_j$ $(j = 1, 2, \ldots, 9)$ are regular vectors in Ω^+, then*

$$\int_{\Omega^+} \Big\{ [\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) \Big\}\, d\mathbf{y}$$
$$= \int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}}S, \tag{28}$$

where the operator $\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})$ is defined by (27)*.*

Theorem 8. *If U and $\tilde{U}_j$ $(j = 1, 2, \ldots, 5)$ are regular vectors in Ω^-, then*

$$\int_{\Omega^-} \Big\{ [\tilde{\mathbf{A}}(\mathbf{D_y})\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{U}(\mathbf{y}) - [\tilde{\mathbf{U}}(\mathbf{y})]^\top \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) \Big\}\, d\mathbf{y}$$
$$= -\int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - [\tilde{\mathbf{U}}(\mathbf{z})]^\top \mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}}S. \tag{29}$$

Formulas (28) and (29) are Green's second identities in the considered theory for domains Ω^+ and Ω^-, respectively.

Let $\tilde{\mathbf{G}}(\mathbf{x})$ be the fundamental matrix of the operator $\tilde{\mathbf{A}}(\mathbf{D_x})$. Clearly, the matrix $\tilde{\mathbf{G}}(\mathbf{x})$ satisfies the following condition:

$$\tilde{\mathbf{G}}(\mathbf{x}) = \mathbf{G}^\top(-\mathbf{x}), \tag{30}$$

where $\mathbf{G}(\mathbf{x})$ is the fundamental solution of system (9).

Pick $\varepsilon > 0$ such that $\overline{B(\mathbf{x}, \varepsilon)} \subset \Omega^+$, where $B(\mathbf{x}, \varepsilon)$ is the (open) ball of radius ε and center $\mathbf{x}$ and $\mathbf{x} \in \Omega^+$. Applying formula (28) in the domain $\Omega^+ \backslash \overline{B(\mathbf{x}, \varepsilon)}$ with $\tilde{\mathbf{U}}(\mathbf{y}) = \tilde{\mathbf{G}}(\mathbf{y} - \mathbf{x})$, and letting $\varepsilon \to 0$, then using equation (30), we get the following results.

Theorem 9. *If $\mathbf{U}$ is a regular vector in Ω^+, then*

$$\mathbf{U}(\mathbf{x}) = \int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\mathbf{G}^\top(\mathbf{x} - \mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - \mathbf{G}(\mathbf{x} - \mathbf{z})\, \mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}}S$$
$$+ \int_{\Omega^+} \mathbf{G}(\mathbf{x} - \mathbf{y})\, \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) d\mathbf{y} \quad \textit{for} \quad \mathbf{x} \in \Omega^+. \tag{31}$$

Theorem 10. *If $\mathbf{U}$ is a regular vector in Ω^-, then*

$$\mathbf{U}(\mathbf{x}) = -\int_S \Big\{ [\tilde{\mathbf{P}}(\mathbf{D_z}, \mathbf{n})\mathbf{G}^\top(\mathbf{x} - \mathbf{z})]^\top \mathbf{U}(\mathbf{z}) - \mathbf{G}(\mathbf{x} - \mathbf{z})\, \mathbf{P}(\mathbf{D_z}, \mathbf{n})\mathbf{U}(\mathbf{z}) \Big\}\, d_{\mathbf{z}}S$$
$$+ \int_{\Omega^-} \mathbf{G}(\mathbf{x} - \mathbf{y})\, \mathbf{A}(\mathbf{D_y})\mathbf{U}(\mathbf{y}) d\mathbf{y} \quad \textit{for} \quad \mathbf{x} \in \Omega^-. \tag{32}$$

Formulas (31) and (32) are Green's third identities (integral representations of a regular vector) in the coupled linear theory of elastic triple porosity materials for domains Ω^+ and Ω^-, respectively.

We introduce the following notation:

(i) $\mathbf{W}^{(1)}(\mathbf{x}, \mathbf{g}) = \int_S \mathbf{G}(\mathbf{x} - \mathbf{y})\mathbf{g}(\mathbf{y}) d_{\mathbf{y}}S$ is the single-layer potential,

(ii) $\mathbf{W}^{(2)}(\mathbf{x}, \mathbf{g}) = \int_S [\tilde{\mathbf{P}}(\mathbf{D_y}, \mathbf{n}(\mathbf{y}))\mathbf{G}^\top(\mathbf{x} - \mathbf{y})]^\top \mathbf{g}(\mathbf{y}) d_{\mathbf{y}}S$ is the double-layer potential, and

(iii) $\mathbf{W}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^{\pm}) = \int\limits_{\Omega^{\pm}} \mathbf{G}(\mathbf{x}-\mathbf{y})\boldsymbol{\phi}(\mathbf{y})d\mathbf{y}$ is the volume potential, where $\mathbf{G}(\mathbf{x})$ is the fundamental matrix of the operator $\mathbf{A}(\mathbf{D_x})$ defined by (15), $\mathbf{g}$ and $\boldsymbol{\phi}$ are the nine-component vector functions.

Of course, by taking into account Green's third identities (31) and (32), the regular vector $\mathbf{U}$ in Ω^+ is represented by a sum of these potentials in the form

$$\mathbf{U}(\mathbf{x}) = \mathbf{W}^{(2)}(\mathbf{x},\mathbf{U}) - \mathbf{W}^{(1)}(\mathbf{x},\mathbf{PU}) + \mathbf{W}^{(3)}(\mathbf{x},\mathbf{AU},\Omega^+) \quad \text{for} \quad \mathbf{x}\in\Omega^+$$

and with a similar analysis we derive the following identity

$$\mathbf{U}(\mathbf{x}) = -\mathbf{W}^{(2)}(\mathbf{x},\mathbf{U}) + \mathbf{W}^{(1)}(\mathbf{x},\mathbf{PU}) + \mathbf{W}^{(3)}(\mathbf{x},\mathbf{AU},\Omega^-) \quad \text{for} \quad \mathbf{x}\in\Omega^-.$$

In view of the properties of matrix $\mathbf{G}(\mathbf{x})$ (see Section 3), we may derive the useful basic properties of the above potentials.

Theorem 11. *If $S\in C^{m+1,\nu}$, $\mathbf{g}\in C^{m,\nu'}(S)$, $0<\nu'<\nu\le 1$, and m is a non-negative integer, then:*

a)
$$\mathbf{W}^{(1)}(\cdot,\mathbf{g})\in C^{0,\nu'}(\mathbb{R}^3)\cap C^{m+1,\nu'}(\overline{\Omega^{\pm}})\cap C^{\infty}(\Omega^{\pm}),$$

b)
$$\mathbf{A}(\mathbf{D_x})\,\mathbf{W}^{(1)}(\mathbf{x},\mathbf{g}) = \mathbf{0},$$

c)
$$\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{W}^{(1)}(\mathbf{z},\mathbf{g})\}^{\pm} = \mp\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{W}^{(1)}(\mathbf{z},\mathbf{g}), \tag{33}$$

d)
$$\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{W}^{(1)}(\mathbf{z},\mathbf{g})$$

is a singular integral, where $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^{\pm}$.

Theorem 12. *If $S\in C^{m+1,\nu}$, $\mathbf{g}\in C^{m,\nu'}(S)$, $0<\nu'<\nu\le 1$, then:*

a)
$$\mathbf{W}^{(2)}(\cdot,\mathbf{g})\in C^{m,\nu'}(\overline{\Omega^{\pm}})\cap C^{\infty}(\Omega^{\pm}),$$

b)
$$\mathbf{A}(\mathbf{D_x})\,\mathbf{W}^{(2)}(\mathbf{x},\mathbf{g}) = \mathbf{0},$$

c)
$$\{\mathbf{W}^{(2)}(\mathbf{z},\mathbf{g})\}^{\pm} = \pm\frac{1}{2}\,\mathbf{g}(\mathbf{z}) + \mathbf{W}^{(2)}(\mathbf{z},\mathbf{g}) \tag{34}$$

for the non-negative integer m,

d) $\mathbf{W}^{(2)}(\mathbf{z},\mathbf{g})$ *is a singular integral, where $\mathbf{z}\in S$,*

e)
$$\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{W}^{(2)}(\mathbf{z},\mathbf{g})\}^{+} = \{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\,\mathbf{W}^{(2)}(\mathbf{z},\mathbf{g})\}^{-},$$

for the natural number m, where $\mathbf{z}\in S$, $\mathbf{x}\in\Omega^{\pm}$.

Theorem 13. *If $S\in C^{1,\nu}$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^+)$, $0<\nu'<\nu\le 1$, then:*

a)
$$\mathbf{W}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^+)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^{2}(\Omega^+)\cap C^{2,\nu'}(\overline{\Omega_0^+}),$$

b)
$$\mathbf{A}(\mathbf{D_x})\,\mathbf{W}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^+) = \boldsymbol{\phi}(\mathbf{x}),$$

where $\mathbf{x}\in\Omega^+$, Ω_0^+ is a domain in $\mathbb{R}^3$ and $\overline{\Omega_0^+}\subset\Omega^+$.

Theorem 14. *If $S\in C^{1,\nu}$, $\operatorname{supp}\boldsymbol{\phi}=\Omega\subset\Omega^-$, $\boldsymbol{\phi}\in C^{0,\nu'}(\Omega^-)$, $0<\nu'<\nu\le 1$, then:*

a)
$$\mathbf{W}^{(3)}(\cdot,\boldsymbol{\phi},\Omega^-)\in C^{1,\nu'}(\mathbb{R}^3)\cap C^{2}(\Omega^-)\cap C^{2,\nu'}(\overline{\Omega_0^-}),$$

b)
$$\mathbf{A}(\mathbf{D_x})\,\mathbf{W}^{(3)}(\mathbf{x},\boldsymbol{\phi},\Omega^-) = \boldsymbol{\phi}(\mathbf{x}),$$

where $\mathbf{x}\in\Omega^-$, Ω is a finite domain in $\mathbb{R}^3$ and $\overline{\Omega_0^-}\subset\Omega^-$.

We are now in a position to study the uniqueness and existence of classical solutions of the BVPs $(K)^+_{\mathbf{F},\mathbf{f}}$ and $(K)^-_{\mathbf{F},\mathbf{f}}$, where $K = I, II$.

6. Uniqueness Theorems

We have the following results.

Theorem 15. *Two regular solutions of the internal BVP $(I)^+_{\mathbf{F},\mathbf{f}}$ may differ only for an additive vector $\mathbf{U} = (\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p})$, where*

$$\mathbf{p}(\mathbf{x}) = \mathbf{0} \quad for \quad \mathbf{x} \in \Omega^+ \tag{35}$$

and the six-component vector $\mathbf{v} = (\mathbf{u}, \boldsymbol{\varphi})$ is a regular solution of the following system

$$\begin{gathered}(\mu\Delta + \rho\omega^2)\mathbf{u} + (\lambda+\mu)\nabla \mathrm{div}\mathbf{u} + b_j \nabla \varphi_j = \mathbf{0},\\ (a_{lj}\Delta + c'_{lj})\varphi_j - b_l \,\mathrm{div}\mathbf{u} = 0,\\ \beta_l \,\mathrm{div}\,\mathbf{u} + m_{lj}\,\varphi_j = 0, \quad l = 1,2,3\end{gathered} \tag{36}$$

satisfying the boundary condition

$$\{\mathbf{v}(\mathbf{z})\}^+ = \mathbf{0} \quad for \quad \mathbf{z} \in S. \tag{37}$$

In addition, problems $(I)^+_{\mathbf{0},\mathbf{0}}$ and (36), (37) *have the same eigenfrequencies.*

Proof. Suppose there are two regular solutions of problem $(I)^+_{\mathbf{F},\mathbf{f}}$. Then their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. We therefore see that $\mathbf{U}$ is a regular solution of the homogeneous system of equations

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}(\mathbf{x}) = \mathbf{0} \tag{38}$$

for $\mathbf{x} \in \Omega^+$ satisfying the homogeneous boundary condition

$$\{\mathbf{U}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S. \tag{39}$$

With the help of equations (38) and (39), from (25), we can deduce that

$$\int\limits_{\Omega^+} E^{(1)}(\mathbf{U}, \mathbf{u}) d\mathbf{x} = 0, \quad \int\limits_{\Omega^+} E^{(2)}(\mathbf{U}, \boldsymbol{\varphi}) d\mathbf{x} = 0, \quad \int\limits_{\Omega^+} E^{(j)}(\mathbf{U}, \mathbf{p}) d\mathbf{x} = 0. \tag{40}$$

Now employ (23) to see that

$$\begin{aligned}\mathrm{Im}\, E^{(1)}(\mathbf{U}, \mathbf{u}) =& \mathrm{Im}\left[(b_l \varphi_l - \beta_l p_l)\mathrm{div}\overline{\mathbf{u}}\right],\\ \mathrm{Im}\, E^{(2)}(\mathbf{U}, \boldsymbol{\varphi}) =& -\mathrm{Im}\,(b_l\varphi_l \mathrm{div}\overline{\mathbf{u}}) + \mathrm{Im}\,(m_{lj}\varphi_l \overline{p_j}),\\ \mathrm{Re}\, E^{(3)}(\mathbf{U}, \mathbf{p}) =& k_{lj}\nabla p_j \cdot \nabla p_l - \omega \mathrm{Im}\,(\beta_l p_l \mathrm{div}\overline{\mathbf{u}}) + \omega \mathrm{Im}\,(m_{lj}\theta \overline{p_l}\varphi_j)\\ &+\gamma_1|p_1 - p_2|^2 + \gamma_2|p_1 - p_3|^2 + \gamma_3|p_2 - p_3|^2.\end{aligned}$$

Combining these equations, we can obtain

$$\begin{gathered}\mathrm{Re}\, E^{(3)}(\mathbf{U}, \mathbf{p}) - \omega\left[\mathrm{Im}E^{(1)}(\mathbf{U}, \mathbf{u}) + \mathrm{Im}E^{(2)}(\mathbf{U}, \boldsymbol{\varphi})\right]\\ = k_{lj}\nabla p_j \cdot \nabla p_l + \gamma_1|p_1 - p_2|^2 + \gamma_2|p_1 - p_3|^2 + \gamma_3|p_2 - p_3|^2 \geq 0\end{gathered}$$

and from (40), it follows that

$$\int\limits_{\Omega^+} k_{lj}\nabla p_j(\mathbf{x}) \cdot \nabla p_l(\mathbf{x}) d\mathbf{x} = 0.$$

Hence $\nabla p_j(\mathbf{x}) \equiv \mathbf{0}$ in Ω^+ and we can derive

$$p_j(\mathbf{x}) = c_j = \text{const}, \quad j = 1,2,3 \quad \text{for} \quad \mathbf{x} \in \Omega^+. \tag{41}$$

Using the homogeneous boundary condition (39), from (41), we get Equation (35). On the basis of (35), from (38), we obtain system (36). Then, upon combining conditions (35) and (39), the six-component vector $\mathbf{v} = (\mathbf{u}, \boldsymbol{\varphi})$ satisfies the boundary condition (37).

Finally, we can easily verify that the homogeneous BVPs $(I)^+_{\mathbf{0},\mathbf{0}}$ and (36), (37) have the same eigenfrequencies. □

Let us introduce the matrix differential operator $\hat{\mathbf{P}}(\mathbf{D_z},\mathbf{n})$ by

$$\hat{\mathbf{P}}(\mathbf{D_z},\mathbf{n})=\big(\hat{P}_{lj}(\mathbf{D_z},\mathbf{n})\big)_{6\times 6},\quad \hat{P}_{lj}=P_{lj},\quad l,j=1,2,\dots,6,$$

where P_{lj} is again given by (19).

Theorem 16. *Two regular solutions of the internal BVP* $(II)^+_{\mathbf{F},\mathbf{f}}$*, may differ only for an additive vector* $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\mathbf{p})$*, where* $\mathbf{p}$ *satisfies condition* (35)*, the vector* $\mathbf{v}=(\mathbf{u},\boldsymbol{\varphi})$ *is a regular solution of system* (36) *satisfying the boundary condition*

$$\big\{\hat{\mathbf{P}}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{v}(\mathbf{z})\big\}^+=\mathbf{0}\quad \textit{for}\quad \mathbf{z}\in S. \tag{42}$$

In addition, problems $(II)^+_{\mathbf{0},\mathbf{0}}$ *and* (34)*,* (42) *have the same eigenfrequencies.*

Proof. Suppose that there are two regular solutions of problem $(II)^+_{\mathbf{F},\mathbf{f}}$. Their difference $\mathbf{U}$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. Then $\mathbf{U}$ is a regular solution of Equation (38) in Ω^+ satisfying the homogeneous boundary condition

$$\{\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{U}(\mathbf{z})\}^+=\mathbf{0}\quad \text{for}\quad \mathbf{z}\in S. \tag{43}$$

With a similar analysis as in Theorem 15, we obtain the relation (41). It follows from Equation (38) that

$$\Lambda_1(\Delta)\mathbf{p}(\mathbf{x})=\mathbf{0}. \tag{44}$$

Using (41) and the condition $\tau_j\neq 0\,(j=1,2,\dots,5)$, from (44), we get (35) and therefore, system (38) implies (36). Clearly, taking into account the relations (35) and (43), the vector $\mathbf{v}$ satisfies the boundary condition (42).

Finally, it is easy to see that the homogeneous BVPs $(II)^+_{\mathbf{0},\mathbf{0}}$ and (36), (42) have the same eigenfrequencies. □

Theorem 17. *The external BVP* $(K)^-_{\mathbf{F},\mathbf{f}}$ *has one regular solution, where* $K=I,II$*.*

Using the radiation conditions (17) and (18), Theorem 17 can be proved similarly to the previous two theorems.

7. Existence Theorems

In this section, the existence theorems for the classical solutions of the BVPs of steady vibrations are established in the coupled linear theory of elastic triple-porosity materials. These theorems are proved by using the potential method and the theory of singular integral equations.

Note that the definitions of a normal type singular integral operator, the symbol and the index of the operator, and also Fredholm's theorems for the singular integral equations are given in the books by Kupradze et al. [16] and Mikhlin [19]. An extensive review of the works and the basic results on the application of the potential method in the theories of porous media are presented in the book by Svanadze [27].

In our further analysis, we will need the following integral operators:

$$\begin{aligned}
\mathcal{L}^{(1)}\,\mathbf{g}(\mathbf{z})&\equiv\frac12\,\mathbf{g}(\mathbf{z})+\mathbf{W}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{L}^{(2)}\,\mathbf{g}(\mathbf{z})&\equiv-\frac12\,\mathbf{g}(\mathbf{z})+\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{W}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{L}^{(3)}\mathbf{g}(\mathbf{z})&\equiv-\frac12\,\mathbf{g}(\mathbf{z})+\mathbf{W}^{(2)}(\mathbf{z},\mathbf{g}),\\
\mathcal{L}^{(4)}\mathbf{g}(\mathbf{z})&\equiv\frac12\,\mathbf{g}(\mathbf{z})+\mathbf{P}(\mathbf{D_z},\mathbf{n}(\mathbf{z}))\mathbf{W}^{(1)}(\mathbf{z},\mathbf{g}),\\
\mathcal{L}_\chi\mathbf{g}(\mathbf{z})&\equiv\frac12\,\mathbf{g}(\mathbf{z})+\chi\,\mathbf{W}^{(2)}(\mathbf{z},\mathbf{g})
\end{aligned} \tag{45}$$

for $\mathbf{z} \in S$, where χ is a complex number. The symbol of the singular integral operator $\mathcal{L}^{(j)}$ $(j = 1,2,3,4)$ we denote by $\mathbf{\Theta}^{(j)} = (\Theta^{(j)}_{lm})_{9\times 9}$. Then by virtue of (16), from (45), it follows that

$$\begin{aligned}\det \mathbf{\Theta}^{(1)} &= -\det \mathbf{\Theta}^{(2)} = -\det \mathbf{\Theta}^{(3)} = \det \mathbf{\Theta}^{(4)} \\ &= -\frac{1}{512}\Big[1 - \frac{\mu^2}{(\lambda+2\mu)^2}\Big] = -\frac{(\lambda+\mu)(\lambda+3\mu)}{512(\lambda+2\mu)^2} < 0. \end{aligned} \tag{46}$$

Thus the operator $\mathcal{L}^{(j)}$ is of normal type, where $j = 1,2,3,4$.

Let $\mathbf{\Theta}_\chi$ and $\operatorname{ind}\mathcal{L}_\chi$ be the symbol and the index of the integral operator $\mathcal{L}_\chi$, respectively. We can easily verify that

$$\det \mathbf{\Theta}_\chi = -\frac{(\lambda+2\mu)^2 - \mu^2\chi^2}{512(\lambda+2\mu)^2}$$

and $\det \mathbf{\Theta}_\chi = 0$ only at two points χ_1 and χ_2 of the complex plane. In view of the relations (46) and $\det \mathbf{\Theta}_1 = \det \mathbf{\Theta}^{(1)}$, we can write $\chi_j \neq 1$ $(j = 1,2)$ and

$$\operatorname{ind}\mathcal{L}_1 = \operatorname{ind}\mathcal{L}^{(1)} = \operatorname{ind}\mathcal{L}_0 = 0.$$

With a similar analysis we derive $\operatorname{ind}\mathcal{L}^{(2)} = -\operatorname{ind}\mathcal{L}^{(3)} = 0$ and $\operatorname{ind}\mathcal{L}^{(4)} = -\operatorname{ind}\mathcal{L}^{(1)} = 0$.

Therefore the operator $\mathcal{L}^{(j)}$ $(j = 1,2,3,4)$ is of normal type with an index, equal to zero and, consequently, Fredholm's theorems are valid for the singular integral operator $\mathcal{L}^{(j)}$.

Moreover, in view of Theorems 13 and 14, the volume potential $\mathbf{W}^{(3)}(\mathbf{x},\mathbf{F},\Omega^\pm)$ is a regular solution of (10), where $\mathbf{F} \in C^{0,\nu'}(\Omega^\pm)$, $0 < \nu' \le 1$; $\operatorname{supp}\mathbf{F}$ is a finite domain in Ω^-. On this reason, further, we will consider problems $(K)^+_{\mathbf{0},\mathbf{f}}$ and $(K)^-_{\mathbf{0},\mathbf{f}}$ and prove the existence theorems of the classical solutions of these BVPs, where $K = I, II$.

Problem $(I)^+_{\mathbf{0},\mathbf{f}}$. We turn now to this problem and assume that ω is not an eigenfrequency of the homogeneous BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. We seek a regular solution to the BVP $(I)^+_{\mathbf{0},\mathbf{f}}$ in the form of the double-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{W}^{(2)}(\mathbf{x},\mathbf{g}) \quad \text{for} \quad \mathbf{x} \in \Omega^+, \tag{47}$$

where $\mathbf{g}$ is the required nine-component vector function.

By virtue of Theorem 12, the vector function $\mathbf{U}$ is a solution of (38) for $\mathbf{x} \in \Omega^+$. Using the boundary condition (20) and employing the identity (34), we can derive from (47) the following singular integral equation

$$\mathcal{L}^{(1)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S \tag{48}$$

allowing us to determine the unknown vector $\mathbf{g}$. We now prove that equation (48) is always solvable for an arbitrary vector $\mathbf{f}$.

Further, introduce the associate homogeneous equation by

$$\mathcal{L}^{(4)}\,\mathbf{h}(\mathbf{z}) = \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S, \tag{49}$$

where $\mathbf{h}$ is the required nine-component vector function. It will be sufficient to show that (49) has only the trivial solution. Indeed, let $\mathbf{h}_0$ be a solution of the homogeneous Equation (49). In view of Theorem 11 and equation (33), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{Q}^{(1)}(\mathbf{x},\mathbf{h}_0)$ is a regular solution of the BVP $(II)^-_{\mathbf{0},\mathbf{0}}$. Then, on the basis of Theorem 17, the problem $(II)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{50}$$

On the other hand, taking into account Theorem 11 and the relation (50), we deduce that

$$\{\mathbf{V}(\mathbf{z})\}^+ = \{\mathbf{V}(\mathbf{z})\}^- = \mathbf{0} \quad \text{for} \quad \mathbf{z} \in S,$$

i.e., by Theorem 11, the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of the homogeneous BVP $(I)^+_{\mathbf{0},\mathbf{0}}$. Using Theorem 15 and the assumption that ω is not an eigenfrequency of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$, the problem $(I)^+_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^+. \tag{51}$$

By virtue of the relations (50), (51) and (55) it follows that

$$\mathbf{h}_0(\mathbf{z}) = \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for} \quad \mathbf{z}\in S.$$

Thus the homogeneous Equation (49) has only the trivial solution and therefore on the basis of Fredholm's theorem, the integral Equation (48) is always solvable for an arbitrary vector $\mathbf{f}$. We have thereby proved the following result.

Theorem 18. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, and ω is not an eigenfrequency of the BVP $(I)^+_{\mathbf{0},\mathbf{0}}$, then a regular solution of the internal BVP $(I)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by double-layer potential* (47)*, where $\mathbf{g}$ is a solution of the singular integral Equation* (48) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Problem $(II)^+_{\mathbf{0},\mathbf{f}}$. Let us assume that ω is not an eigenfrequency of the BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. We seek a regular solution to the BVP $(II)^+_{\mathbf{0},\mathbf{f}}$ in the form of the single-layer potential

$$\mathbf{U}(\mathbf{x}) = \mathbf{W}^{(1)}(\mathbf{x},\mathbf{g}) \quad \text{for} \quad \mathbf{x}\in\Omega^+, \tag{52}$$

where $\mathbf{g}$ is the required nine-component vector function.

On the basis of Theorem 11, the vector function $\mathbf{U}$ is a solution of (38) for $\mathbf{x}\in\Omega^+$. Using the boundary condition (40) and keeping in mind the identity (33), from (52) we obtain a singular integral equation

$$\mathcal{L}^{(2)}\,\mathbf{g}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) \quad \text{for} \quad \mathbf{z}\in S \tag{53}$$

allowing us to determine the unknown vector $\mathbf{g}$. We prove that equation (53) is always solvable for an arbitrary vector $\mathbf{f}$.

Let us consider the homogeneous equation

$$-\frac{1}{2}\,\mathbf{g}_0(\mathbf{z}) + \mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{W}^{(1)}(\mathbf{z},\mathbf{g}_0) = \mathbf{0} \quad \text{for} \quad \mathbf{z}\in S, \tag{54}$$

where $\mathbf{g}_0$ is the required nine-component vector function. It will be sufficient to show that (54) has only the trivial solution. Using Theorem 11 and equation (54), the vector function $\mathbf{V}(\mathbf{x}) = \mathbf{W}^{(1)}(\mathbf{x},\mathbf{g}_0)$ is a regular solution of the internal homogeneous BVP $(II)^+_{\mathbf{0},\mathbf{0}}$. By Theorem 16 and the assumption that ω is not an eigenfrequency of the problem $(II)^+_{\mathbf{0},\mathbf{0}}$, this problem has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x}\in\Omega^+. \tag{55}$$

Then on the basis of Theorem 11 and the relation (55), we can write

$$\{\mathbf{V}(\mathbf{z})\}^- = \{\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for} \quad \mathbf{z}\in S,$$

i.e., by virtue of Theorem 11, the vector $\mathbf{V}(\mathbf{x})$ is a regular solution of problem $(I)^-_{\mathbf{0},\mathbf{0}}$. Now, using Theorem 17, the problem $(I)^-_{\mathbf{0},\mathbf{0}}$ has only the trivial solution, that is,

$$\mathbf{V}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for} \quad \mathbf{x}\in\Omega^-. \tag{56}$$

Combining equations (55) and (56) with identity (33), we can further conclude that

$$\mathbf{g}_0(\mathbf{z}) = \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^- - \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{V}(\mathbf{z})\}^+ = \mathbf{0} \quad \text{for} \quad \mathbf{z}\in S.$$

Thus the homogeneous Equation (54) has only the trivial solution and therefore, on the basis of Fredholm's theorem, the integral Equation (53) is always solvable for an arbitrary vector $\mathbf{f}$.

We have thereby proved the following consequences.

Theorem 19. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \le 1$, and ω is not an eigenfrequency of the BVP $(II)^+_{\mathbf{0},\mathbf{0}}$, then a regular solution of the internal BVP $(II)^+_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by single-layer potential* (52)*, where $\mathbf{g}$ is a solution of the singular integral Equation* (53) *which is always solvable for an arbitrary vector $\mathbf{f}$.*

Problem $(I)^-_{\mathbf{0},\mathbf{f}}$. In a similar manner as in Theorems 18 and 19, the following result is proved.

Theorem 20. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{1,\nu'}(S)$, $0 < \nu' < \nu \le 1$, then a regular solution $\mathbf{U}$ of the external BVP $(I)^-_{\mathbf{0},\mathbf{f}}$ exists, is unique and represented by a sum of double– and single-layer potentials*

$$\mathbf{U}(\mathbf{x}) = \mathbf{W}^{(2)}(\mathbf{x},\mathbf{g}) + (1-i)\mathbf{W}^{(1)}(\mathbf{x},\mathbf{g}) \quad \textit{for} \quad \mathbf{x} \in \Omega^-,$$

where $\mathbf{g}$ is a solution of the singular integral equation

$$\mathcal{L}^{(3)}\,\mathbf{g}(\mathbf{z}) + (1-i)\mathbf{W}^{(1)}(\mathbf{z},\mathbf{g}) = \mathbf{f}(\mathbf{z}) \quad \textit{for} \quad \mathbf{z} \in S$$

which is always solvable for an arbitrary vector $\mathbf{f}$.

Problem $(II)^-_{\mathbf{0},\mathbf{f}}$. Finally, we seek a regular solution to this problem in the form

$$\mathbf{U}(\mathbf{x}) = \mathbf{W}^{(1)}(\mathbf{x},\mathbf{h}) + \mathbf{U}^*(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega^-, \tag{57}$$

where $\mathbf{h}$ is the required nine-component vector function and the nine-component vector function $\mathbf{U}^*$ is a regular solution of the equation

$$\mathbf{A}(\mathbf{D_x})\mathbf{U}^*(\mathbf{x}) = \mathbf{0} \quad \text{for} \quad \mathbf{x} \in \Omega^-. \tag{58}$$

Combining equations (22), (33) and (57) for determining the unknown vector $\mathbf{h}$, we can obtain the following singular integral equation:

$$\mathcal{L}^{(4)}\,\mathbf{h}(\mathbf{z}) = \mathbf{f}^*(\mathbf{z}) \quad \text{for} \quad \mathbf{z} \in S, \tag{59}$$

where

$$\mathbf{f}^*(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}^*(\mathbf{z})\}^-. \tag{60}$$

Now we prove that equation (59) is always solvable for an arbitrary vector $\mathbf{f}$. We assume that the homogeneous equation

$$\mathcal{L}^{(4)}\,\mathbf{h}(\mathbf{z}) = \mathbf{0} \tag{61}$$

has m linearly independent orthonormal solutions $\{\mathbf{h}^{(l)}(\mathbf{z})\}_{l=1}^m$. By Fredholm's theorem, the solvability condition of equation (59) can be written as

$$\int_S \{\mathbf{P}(\mathbf{D_z},\mathbf{n})\mathbf{U}^*(\mathbf{z})\}^- \cdot \boldsymbol{\psi}^{(l)}(\mathbf{z}) d_{\mathbf{z}}S = N_l, \tag{62}$$

where

$$N_l = \int_S \mathbf{f}(\mathbf{z}) \cdot \boldsymbol{\psi}^{(l)}(\mathbf{z}) d_{\mathbf{z}}S$$

and $\{\boldsymbol{\psi}^{(l)}(\mathbf{z})\}_{l=1}^m$ is a complete system of solutions of the homogeneous associated equation of (61), i.e.,

$$\mathcal{L}^{(1)}\,\boldsymbol{\psi}^{(l)} = \mathbf{0}, \quad l = 1,2,\dots,m.$$

We can easily verify that condition (62) takes the form

$$\int_S \mathbf{h}^{(l)}(\mathbf{z}) \cdot \{\mathbf{U}^*(\mathbf{z})\}^- d_{\mathbf{z}}S = -N_l, \quad l = 1,2,\dots,m. \tag{63}$$

Let the vector $\mathbf{U}^*$ be a solution of (58) satisfying the boundary condition

$$\{\mathbf{U}^*(\mathbf{z})\}^- = \hat{\mathbf{f}}(\mathbf{z}), \tag{64}$$

where

$$\hat{\mathbf{f}}(\mathbf{z}) = \sum_{l=1}^m N_l \mathbf{h}^{(l)}(\mathbf{z}). \tag{65}$$

On the basis of Theorem 20, the BVP (58), (64) is always solvable. Then, in view of the orthonormalization of $\{\mathbf{h}^{(l)}(\mathbf{z})\}_{l=1}^m$, condition (63) is fulfilled automatically and the solvability of (59) is proved. Therefore the existence of regular solution of problem $(II)^-_{\mathbf{0},\mathbf{f}}$ is proved, too. Thus, the following theorem has been proved.

Theorem 21. *If $S \in C^{2,\nu}$, $\mathbf{f} \in C^{0,\nu'}(S)$, $0 < \nu' < \nu \leq 1$, then a regular solution $\mathbf{U}$ of the external BVP $(II)^{-}_{\mathbf{0},\mathbf{f}}$ exists, is unique and is represented by the sum* (57), *where $\mathbf{h}$ is a solution of the singular integral Equation* (59) *which is always solvable, $\mathbf{U}^*$ is the solution of BVP* (58), (64) *which is always solvable; and the vector functions $\mathbf{f}^*$ and $\hat{\mathbf{f}}$ are determined by* (60) *and* (65), *respectively.*

8. Concluding Remarks

1. In the present paper, the coupled linear model of elastic triple-porosity materials is considered in which the coupled phenomenon of the concepts of Darcy's extended law and the volume fractions of three levels of pore networks is proposed, and the following results are obtained:

(i) The fundamental solution of the system of steady vibration equations is constructed explicitly by means of elementary functions and its basic properties are established.

(ii) Green's identities are obtained and the uniqueness theorems of the basic internal and external BVPs of steady vibrations are proved.

(iii) The surface (single– and double-layer) and the volume potentials are introduced and their basic properties are established.

(iv) Finally, the existence theorems for regular (classical) solutions of the above- mentioned BVPs are proved by using the potential method and the theory of singular integral equations.

2. On the basis of results of this paper it is possible:

(i) to introduce a coupled linear model of thermoelastic triple-porosity materials in which the coupled phenomenon of the concepts of Darcy's law and the volume fraction is proposed;

(ii) to investigate the non-classical BVPs of this model by using the potential method and the theory of singular integral equations.

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