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DYNAMICAL THERMOSTABILITY OF SHELLS OF REVOLUTION WITH AN ELASTIC FILLER AND UNDER THE ACTION OF MERIDIONAL FORCES, NORMAL PRESSURE AND TEMPERATURE

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Abstract. Dynamical thermostability of closed shells of revolution, close by their form to cylindrical ones, with an elastic filler and under the action of meridional stresses, external pressure and temperature, is studied. The shells of middle length whose midsurface generatrix is a parabolic function, are considered. The shells of positive and negative Gaussian curvature are investigated. Formulas to find lower frequencies and boundaries of regions of dynamical instability depending on the Gaussian curvature, initial stress, temperature and amplitude of shell deviation from cylindrical form, are obtained.

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The present work considers dynamic thermostability of closed shells of revolution, close by their form to cylindrical ones, with an elastic filler, and under the action of meridional stresses distributed uniformly over the end-walls of the shell, external pressure and temperature. We consider a light filler for which the effect of tangential stresses on the contact surface and inertia forces may be neglected. The shell is assumed to be thin and elastic. Temperature in the shell body is uniformly distributed. An elastic filler is simulated by Winkler's base, its extension caused by heating is not taken into account. We investigate the shells of middle length for which the shape of a midsurface generatrix is a parabolic function, and also the shells of positive and negative Gaussian curvature. The boundary conditions on the end-walls correspond to a free support admitting certain radial shift in the initial state.

In solving the problems under consideration the main attention is paid to the finding of the most dangerous area of dynamical instability and to lower frequencies, which are practically most important. Formulas in dimensionless form and universal curves of dependence of lower frequency, shape of wave formation and boundaries of regions of dynamical instability on the Gaussian curvature, prestress, temperature and shell deviation amplitude from a cylinder, are derived. It is shown that in the presence of an elastic filler and prestresses, the temperature may change considerably lower frequencies and boundaries of regions with dynamical instability.

1. We consider the shell whose midsurface is formed by the rotation of square parabola around the z-axis of the Cartesian coordinate system xyz with origin at the midpoint of the rotation axis segment. It is assumed that the cross-section radius R of the midsurface of the shell is defined by the equality

$$R = r + \delta_0[1 - \xi^2(r/l)^2], \tag{1.1}$$

where r is the end section radius, δ_0 is maximal deviation (for $\delta_0 > 0$, the shell is convex, and for $\delta_0 < 0$, is concave); $L = 2l$ is the shell length, $\xi = z/r$. We consider the midlength shells [6] and it is assumed that

$$(\delta/r)^2, (\delta_0/l)^2 \ll 1. \tag{1.2}$$

The equations of the theory of shallow shells were taken as the basic equations of oscillations [5]. For the midlength shells under consideration, the oscillation modes corresponding to lower frequencies have weak variability in longitudinal direction in comparison with circumferential, therefore the relation

$$\delta^2 f/\partial\xi^2 \ll \partial^2 f/\partial\varphi^2 \quad (f = w, \psi) \tag{1.3}$$

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is valid, where w and ψ are the functions of radial displacement and stress, respectively. As a result, the system of equations for the shells under consideration reduces to the following equation:

$$\varepsilon\frac{\partial^8 w}{\partial\varphi^8}+\frac{\partial^4 w}{\partial\xi^4}+4\delta\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2}+4\delta^2\frac{\partial^4 w}{\partial\varphi^4}-t_1^0\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4}-t_2^0\frac{\partial^6 w}{\partial\varphi^6}-2s^0\frac{\partial^8 w}{\partial\xi\partial\varphi^5}$$
$$+\gamma\frac{\partial^4 w}{\partial\varphi^4}+\frac{\rho r^2}{E}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big)=0, \tag{1.4}$$
$$\varepsilon=h^2/12r^2(1-\nu^2),\ \ \delta=\delta_0 r/l^2,\ \ \gamma=\beta r^2/Eh,\ \ t_1^0=\frac{T_i^0}{Eh},\ \ s^0=S_0/Eh\ \ (i=1,2),$$

where E and ν are the modulus of elasticity and the Poisson coefficient, respectively; T_1^0, T_2^0, S^0 are maridinal, circumferential and shear force in the initial state; ρ is density of the shell material; γ is the "bed" coefficient of the elastic filler (charactrizing elastic rigidity of the filler); φ is angular coordinate; t is time.

The initial state is assumed to be momentless. Relying on the corresponding solution and taking into account the filler reaction and also relation (1.2), we get the following approximate expressions:

$$\begin{aligned}T_1^0=&P_1\Big[1+\frac{\delta_0}{r}\Big(\xi^2(r/l)^2-1\Big)\Big]-q\delta_0\Big[\xi^2(r/l)^2-1\Big],\\ T_2^0=&-2P_1\delta_0 r/l^2-qr+\beta_0 r w_0,\quad S=0,\end{aligned} \tag{1.5}$$

where ω_0 and β_0 are, respectively, the deflection and "bed" coefficient of the filler in the initial state. Taking into account that

$$\Big|\xi^2(r/l)^2-1\Big|\frac{\partial^2 w}{\partial\xi^2}\ll 2(r/l)^2\frac{\partial^2 w}{\partial\varphi^2},$$
$$\frac{\delta_0}{r}\Big|\xi^2(r/l)^2-1\Big|\frac{\partial^2 w}{\partial\xi^2}\ll\frac{\partial^2 w}{\partial\varphi^2},$$

the expressions (1.5), after substitution into equation (1.4), can be simplified. Thus they take the form

$$\frac{T_1^0}{Eh}=\frac{P_1}{Eh},\quad \frac{T_2^0}{Eh}=-2\frac{P_1}{Eh}\delta-\frac{qr}{Eh}+w_0\frac{\beta_0 r}{Eh},\quad T_i=\sigma_i^0 h\ \ (i=1,2). \tag{1.6}$$

Bearing in mind that in the initial state the shell deformation in the circumferential direction ε_φ^0 is defined by the equality

$$\varepsilon_\varphi^0=\frac{\sigma_2^0-\nu\sigma_1^2}{E}+\alpha T,\quad \varepsilon_\varphi^0=-\frac{w_0}{r},$$

where α is coefficient of linear extension and T is temperature, we get

$$w_0=(-\sigma_2^0+\nu\sigma_1^0)\frac{r}{E}-\alpha Tr. \tag{1.7}$$

Substituting (1.7) into the second equality (1.6), we obtain

$$\frac{T_2}{Eh}=\frac{\sigma_2^0}{Eh}-2\frac{P_1}{Eh}\delta+\frac{\beta_0 r^2}{Eh}(-\sigma_2^0+\nu\sigma_1^0)\frac{1}{E}-\frac{\alpha T\beta_0 r^2}{Eh}$$

whence

$$\frac{\sigma_2^0}{E}\Big(1+\frac{\beta r^2}{Eh}\Big)=-\frac{qr}{Eh}-2\frac{P_1}{Eh}\delta+\nu\frac{\sigma_1^0}{E}\frac{\beta_0 r^2}{Eh}-\alpha T\frac{\beta_0 r^2}{Eh}.$$

Introduce the notation

$$\frac{qr}{Eh}=\overline{q},\quad \frac{P_1}{Eh}=-p,\quad \frac{\beta_0 r}{Eh}=\gamma_0,\quad 1+\gamma_0=g.$$

Then expressions (1.6) take the form

$$-\frac{\sigma_1^0}{E}=p,\quad -\frac{\sigma_2^0}{E}=(\overline{q}-2p\delta+\nu p\gamma_0+\alpha T\gamma_0)g^{-1}. \tag{1.8}$$

As a result, equation (1.4) can be written as

$$\varepsilon\frac{\partial^8 w}{\partial\varphi^8}+\frac{\partial^4 w}{\partial\xi^4}+4\delta\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2}+4(\delta^2+\gamma/4)\frac{\partial^4 w}{\partial\varphi^4}+(\overline{q}-2p\delta+\nu p\gamma_0+\alpha T\gamma_0)g^{-1}\frac{\partial^6 w}{\partial\varphi^6}$$
$$+p\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4}+\frac{\beta r^2}{E}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big)=0. \tag{1.9}$$

First, consider harmonic oscillations. The above boundary conditions of free support and equation (1.9) are satisfied by the solution

$$w=A_{mn}\cos\lambda_m\xi\sin n\varphi\cos\omega_{mn}t,\quad \lambda_m=m\pi r/2l$$
$$(m=2i+1,\quad i=0,1,2,\dots). \tag{1.10}$$

Substituting (1.10) into equation (1.9), to find eigenfrequencies we obtain the following equality (in the sequel, for ω_{mn}, the indices m and n will be omitted):

$$\omega^2=\frac{E}{\rho r^2}\Big[\varepsilon n^4+\lambda_m^4 n^{-4}+4\delta\lambda_m^2 n^{-2}+4(\delta^2+\gamma/4)-p(\lambda_m^2-2\widetilde{\delta}n^2)-(\overline{q}+\gamma_0\alpha T)g^{-1}n^2\Big]. \tag{1.11}$$

Introduce the notation

$$\overline{\delta}^2=\delta^2+\gamma/4,\quad \widetilde{\delta}=(\delta-0,5\nu\gamma_0)g^{-1},\quad \widetilde{q}=(\overline{q}+\gamma_0\alpha T)g^{-1}.$$

Then (1.11) takes the form

$$\omega^2=\frac{E}{\rho r^2}\Big[\varepsilon n^4+\lambda_m^4 n^{-4}+4\delta\lambda_m^2 n^{-2}+4\overline{\delta}^2-p(\lambda_m^2-2\widetilde{\delta}n^2)-\widetilde{q}n^2\Big]. \tag{1.12}$$

Thus it can be seen that for $p=0$, $\delta>0$, the value $m=1$ corresponds to the least frequency. It can also be shown that this condition holds for $\delta<0$, taking into account inequalities (1.2) and also the fact that $\omega^2>0$. Thus we first consider the modes of oscillastions under which one half-wave ($m=1$) is formed along the length of the shell and n waves in circumferential direction. For compression, $p>0$, and for tension, $p<0$; q is a normal pressure and regarded as positive, if external.

Represent expression (1.12) for $m=1$ in dimensionless form and towards this end, we introduce the following dimensionless quantities:

$$N=n^2/n_0^2,\;\; P=p/p_{0*},\;\; \widetilde{Q}=\widetilde{q}/\overline{q}_{0*},\;\; p_{0*}=\frac{(1-\nu^2)^{-1/2}}{\sqrt{3}}\frac{h}{r},\;\; \overline{q}_{0*}=0,855(1-\nu^2)^{-3/4}\Big(\frac{h}{r}\Big)^{3/2}\frac{r}{L},$$
$$\delta_*=\delta\varepsilon_*^{-1/2}\quad \varepsilon_*=(1-\nu^2)^{-1/2}\Big(\frac{r}{L}\Big)^2\frac{h}{2},\quad n_0^2=\lambda_1\varepsilon^{-1/4},\quad \lambda_1=\pi r/L,\quad \overline{\delta}_*^2=\delta_*^2+\gamma/4, \tag{1.13}$$
$$\gamma_*=\gamma\varepsilon_*^{-1},\quad \widetilde{\delta}_*=(\delta-0,5\nu\gamma_0)\varepsilon_*^{1/2}g^{-1},\quad \omega_*^2=2\lambda_1^2\varepsilon^{1/3}\frac{E}{\rho r^2},\quad \frac{\widetilde{q}}{\overline{q}_{0*}}=\Big(\frac{\overline{q}}{\overline{q}_{0*}}+\frac{\gamma_0\alpha T}{\overline{q}_{0*}}\Big)g^{-1},$$

where ω_*, p_{0*}, $\overline{q}_{0*}$ are, respectively, the least frequency, critical loading under compression and critical pressure for the cylindrical shell of middle length [2,6]. As a result, equality (1.12) can be written in the following dimensionless form:

$$\omega^2(N)/\omega_*^2=0,5\Big[N^2+N^{-2}+2,37\delta_*N^{-1}+1,404\overline{\delta}_*^2-2P(1-1,185\widetilde{\delta}_*N)-1,755\widetilde{Q}N\Big]. \tag{1.14}$$

The least frequency (for $\omega^2(N)>0$) is determined from the condition $[\omega^2(N)]=0$. Thus we obtain either

$$0,8775\widetilde{Q}-1,185\widetilde{\delta}_*P=N-1,185\delta_*N^{-2}-N^{-3},$$

or

$$N^4-(0,8775\widetilde{Q}-1,185\delta_*P)N^3-1,185\delta_*N-1=0, \tag{1.15}$$

whence for $P=\widetilde{Q}=0$, we have the well-known equation

$$N^4-1,185\delta_*N-1=0,$$

whose roots were obtained explicitly in [3]. Moreover, from (1.15), for $\delta_*=0$, $\widetilde{Q}=0$ ($\delta=\gamma_0=0$, $q=0$) we obtain the equation $N^4-1=0$ with the root $N=1$. Consequently, for the cylindrical midlength shell the least frequency is realized for $N=1$, regardless of P, which is in a full agreement with [4].

Let us consider equation (1.15) and write it in the form

$$N^4 + bN^3 + dN + e = 0, \quad b = 1,185\delta_* P - 0,8775\widetilde{Q}, \quad d = -1,185\delta_*, \quad e = -1. \tag{1.16}$$

The roots of this equation coincide with the roots of the two quadratic equations

$$N^2 + (b + B_{1,2})\frac{N}{2} + \Big(y_1 + \frac{by_1 - d}{B_{1,2}}\Big) = 0, \quad B_{1,2} = \pm\sqrt{b(y_1 + b^2/8)}.$$

Introduce the notation

$$\gamma_1 = y_1 + b^2/8, \quad \gamma_2 = y_1 - b^2/4.$$

Then the roots of the these equations take the form

$$N_{1,2} = -\frac{\sqrt{8\gamma_1} + b}{4} \pm \sqrt{-\frac{by_1 - d}{\sqrt{8\gamma_1}} + \frac{b\sqrt{8\gamma_1} - 4\gamma_2}{8}}, \tag{1.17}$$

$$N_{3,4} = \frac{\sqrt{8\gamma_1} - b}{4} \pm \sqrt{\frac{by_1 - d}{\sqrt{8\gamma_1}} - \frac{b\sqrt{8\gamma_1} + 4\gamma_2}{8}}, \tag{1.18}$$

where y_1 is any real root of the cubic equation

$$y^3 + 3py + 2q = 0,$$

$$3p = 1 - \frac{1,185^2\widetilde{\delta}^2 PM}{4}, \quad 2q = -\frac{1,185^2\widetilde{\delta}^2(1 - P^2M^2)}{8}, \quad M = 1 - 0,7405\widetilde{Q}/\widetilde{\delta}_* P$$

for

$$\frac{1,185^2\widetilde{\delta}_*^2|PM|}{4} \ll 1 \quad (\delta_* \le 0.5, \ \ |PM| \le 0,5) \tag{1.19}$$

$$p = \frac{1}{3}, \quad q = -1,185^2\widetilde{\delta}_*^2(1 - P^2M^2)/16$$

since the dinscriminant of this equation is $D > 0$, we have one real root

$$y_1 = (-q + \sqrt{D})^{1/3} + (-q - \sqrt{D})^{1/3}, \quad \sqrt{D} = \sqrt{1 + 0,208\widetilde{\delta}_*^4(1 + P^2M^2)^2}/3^{3/2}. \tag{1.20}$$

Taking

$$0,208\widetilde{\delta}_*^4(1 - P^2M^2)^2 \ll 1 \tag{1.21}$$

and expanding into series the expressions appearing in (1.20) neglecting therein the values of the second order of smallness, we obtain $y_1 = 0,1755\widetilde{\delta}^2(1 - P^2M^2)$. Under the restrictions (1.19), inequality (1.21) is fulfilled all the more.

Substituting the values y_1, b, d, γ_1, γ_2 into expressions (1.17) and (1.18) and taking also into account inequality (1.19), we find that for $d > 0$ ($\delta_* < 0$) only the root N_1 is positive, whereas for $d < 0$ ($\delta_* > 0$), positive is the root N_3. Thus we have

$$N = [1 + 0,1755\widetilde{\delta}_*^2 PM_1(1 - P^2M_1^2) - 0,0877\widetilde{\delta}_*^2(1 + 2PM_1 - 2P^2M_1^2)]^{1/2} \\ +0,2962(1 - PM_1) \ \ (\delta_* > 0). \tag{1.22}$$

$$N = [1 + 0,1755\widetilde{\delta}_*^2 PM_2(1 - P^2M_2^2) - 0,0877\widetilde{\delta}_*^2(1 + 2PM_2 - 2P^2M_2^2)]^{1/2} \\ -0,2962(1 - PM_2) \ \ (\delta_* < 0). \tag{1.23}$$

$$M_1 = 1 - 0,7405\widetilde{Q}/\widetilde{\delta}_* P, \quad M_2 = 1 + 0,7405\widetilde{Q}/\widetilde{\delta}_* P.$$

For $\widetilde{\delta}_* > 0$, $P/\widetilde{Q} > 0$, the value $M_1 = 0$, if $\widetilde{\delta}_* = 0,7405P/\widetilde{Q}$; for $\widetilde{\delta}_* < 0$, $P/\widetilde{Q} < 0$, the value $M_2 = 0$, if $|\widetilde{\delta}_*| = -0,7405P/\widetilde{Q}$. In addition, formulas (1.22) and (1.23) take the form

$$N = \sqrt{1 - 0,0877\widetilde{\delta}_*^2 + 0,2962\widetilde{\delta}_*} \quad (\delta_* > 0),$$

$$N = \sqrt{1 - 0,0877\widetilde{\delta}_*^2 - 0,2962|\widetilde{\delta}_*|} \quad (\delta_* < 0).$$

It should be noted that this case of the defined values $\widetilde{\delta}_*$ corresponds to the cases under which normal cicumferential stresses caused by meridional loading, external pressure and temperature, mutually neutralize each other.

For $\gamma_* = 0$, we have $\widetilde{\delta}_* = \delta_*$ and for N, we obtain the formulas given in [3]. In what follows, we take $\gamma = \gamma_0$.

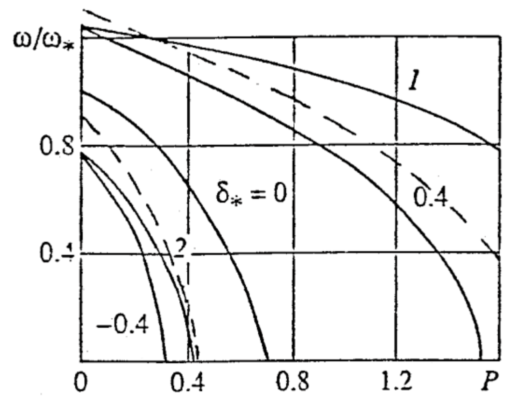

FIGURE 1

Substituting the values of N on the basis of formulas (1.22), (1.23) for fixed $(\delta_*, P, \widetilde{Q}, \gamma_*)$ into formula (1.14), we obtain the least value of dimensionless frequency ω/ω_*. Figure 1 shows the values ω/ω_* depending on P for the relation $\widetilde{Q} = 0,54P$ (for $\delta_* = 0,4; 0; -0,4$) and $(\gamma_* = 0; 1,272)$. The corresponding dependencies for $\gamma_* = 0$ are given by solid curves and for $\gamma_* = 1,272$ by dashed curves. Moreover, we can see, for comparison, the curves of the least frequency dependence on P, when $\widetilde{Q} = 0$, $\gamma = 0$ for $\delta_* = 0,4; -0,4$, which are denoted, respectively, by 1 and 2.

For $\omega = 0$, from equality (1.14), we get

$$1,755\widetilde{Q} = N + N^{-3} + 2,37\delta_* N^{-2} - 2P(N^{-1} - 1,185\widetilde{\delta}_*). \tag{1.24}$$

The least value $\widetilde{Q} > 0$ depending on N is realized for $\widetilde{Q}^i_N = 0$, whence it follows that

$$N^4 + cN^2 + dN + e = 0, \quad c = 2P - 1,404\overline{\delta}_*^2, \quad d = -4,74\delta_*, \quad e = -3. \tag{1.25}$$

The roots of equation (1.25) coincide with those of the two quadratic equations

$$N^2 + \frac{A_{1,2}}{2}N + \Big(y_1 - \frac{d}{A_{1,2}}\Big) = 0, \quad A_{1,2} = \pm\sqrt{8\alpha}$$

$$N_{1,2} = -\sqrt{\frac{\alpha}{2}} \pm \sqrt{\frac{d}{\sqrt{8\alpha}} - \frac{\alpha_1}{2}}, \quad N_{3,4} = \sqrt{\frac{\alpha}{2}} \pm \sqrt{\frac{-d}{\sqrt{8\alpha}} - \frac{\alpha_1}{2}} \tag{1.26}$$

$$\alpha = y_1 - \frac{c}{2}, \quad \alpha_1 = y_1 + \frac{c}{2}, \tag{1.27}$$

where y_1 is any real root of the cubic equation

$$y^3 - \frac{c}{2}y^2 - ey + \Big(\frac{ce}{2} - \frac{d^2}{8}\Big) = 0, \tag{1.28}$$

or

$$z^3 + 3pz + 2q = 0 \quad (z = y - c/6), \tag{1.29}$$

$$p = 1 - (2P - 1,404\overline{\delta}_*^2)^2/36, \quad q = -\frac{1}{2}(2P + 1,404\overline{\delta}_*^2)\Big[1 - \frac{(2P - 1,404\overline{\delta}_*^2)^3}{108(2P + 1,404\overline{\delta}_*^2)}\Big]. \tag{1.30}$$

If we take

$$(2P - 1,404\overline{\delta}_*^2)^2 \ll 1, \tag{1.31}$$

then expressions (1.30) have the form

$$p = 1, \quad q = \frac{1}{2}(2P + 1,404\overline{\delta}_k^2)\Big[1 - \frac{(2P-1,404\overline{\delta}_*^2)^3}{108(2P+1,404\overline{\delta}_*^2)}\Big].$$

Since the discriminant of equation (1.29) is $D = q^2 + p^3 > 0$, we have one real root

$$z = (-q + \sqrt{q^2+p^3}) + (-q - \sqrt{q^2+p^3}). \tag{1.32}$$

Taking

$$(2P + 1,404\overline{\delta}_*2)^2/36 \ll 1, \tag{1.33}$$

expanding the expresseins appearing in (1.32) into series and omitting the values of the second order of smallness, we obtain $z = [2P + 1,404(\overline{\delta}_*2 + \frac{3}{4}\gamma_*)]/3$. Then relying on (1.25), (1.27), we get

$$\alpha = z - c/3 = 2\cdot 1,404\overline{\delta}_*2/3, \quad \alpha_1 = z + \frac{2}{3}c = 2P + 1,404\Big(\overline{\delta}_*2 + \frac{3}{4}\gamma_*\Big)/3. \tag{1.34}$$

Since $N^2 = n^2/n_0^2$, therefore of our interest are only the positive roots of equation (1.25). Taking into account inequality (1.31) and also the fact that y_1 is the root of equation (1.28), we find that for $\delta_* > 0$ ($d < 0$), positive is only the root of N_3, and for $\delta_* < 0$ ($d > 0$), positive is only the root of N_1.

Substituting the values d, α, α_1 according to equalities (1.25) and (1.34) into expressions (1.26), we obtain

$$\begin{aligned} N &= \sqrt{\sqrt{3} + 0,234\Big(\overline{\delta}_*^2 + \frac{3}{4}\gamma_*\Big) - P} + 0,684\delta_* \quad (\delta_* > 0), \\ N &= \sqrt{\sqrt{3} + 0,234\Big(\overline{\delta}_*^2 + \frac{3}{4}\gamma_*\Big) - P} - 0,684|\delta_*| \quad (\delta_* < 0). \end{aligned} \tag{1.35}$$

As a result, we get

$$n_{1,2} = \Big(\sqrt{\sqrt{3} + 0,2703\varepsilon^{-1}\Big(\Big[\Big(\frac{\delta_0}{l}\Big)^2 + \frac{3\gamma}{4}\Big(\frac{l}{r}\Big)^2\Big] - P} \pm 0,735\varepsilon^{-1/4}\frac{|\delta_0|}{l}\Big)\lambda_1\varepsilon^{1/4}, \tag{1.36}$$

where index (1) corresponds to $\delta_0 > 0$, and index (2) to $-\delta_0 < 0$. In particular, for $\delta_0 = \gamma_0 = p = 0$, we obtain the known formula for a critical number of waves of the cylindrical shell of middle length $n_*^2 = \sqrt[4]{3}\lambda_1\varepsilon^{-1/4}$ [6].

From formula (1.36), it is not difficult to notice that under the action of compressive forces the number of critical waves around the circumference decreases, whereas under tensile forces it increases.

Formula (1.35), as it has been mentioned above, holds when condition (1.33) is fulfilled. In case this condition is not fulfilled, it is necessary to proceed from full expressions (1.26). Defining thus the values of N_* (for fixed δ_*, γ_*, P) and substituting them into (1.24), we obtain the corresponding critical value of $\widetilde{Q}_*$.

Figure 2 shows in dimensionless form critical values of N_* depending on P for $\delta_* = -0,4; 0; 0,4$ and $\gamma_* = 0; 1,272$. The corresponding graphs for $\gamma_* = 0$ are represented by solid curves, and for $\gamma_* = 1,272$ by dashed curves. The values $\widetilde{Q}_*(\delta_*, \gamma_*, P)$ denoted, respectively, by solid and dashed curves are given in Figure 3. Note that the curve $\widetilde{Q}_*(\delta_*, \gamma_*, P)$ for $P > 0$ in Figure 3 coincides practically with that of the work [4].

Further, consider the value $m > 1$. Using notation (1.13), formula (1.12) can be represented as follows:

$$\begin{aligned} \omega^2/\omega_*^2 = 0,5m^2\Big[&Q^2 + Q^{-2} + 2,37\delta_*\theta^{-1}m^{-1} + 1,404(\delta_*^2 + \gamma_*/4)m^{-2} \\ &-2P(1 - 1,185\widetilde{\delta}_*\theta m^{-1}) - 1,755\widetilde{\theta}\theta m^{-1}\Big]. \end{aligned} \tag{1.37}$$

$$\theta = N/m. \tag{1.38}$$

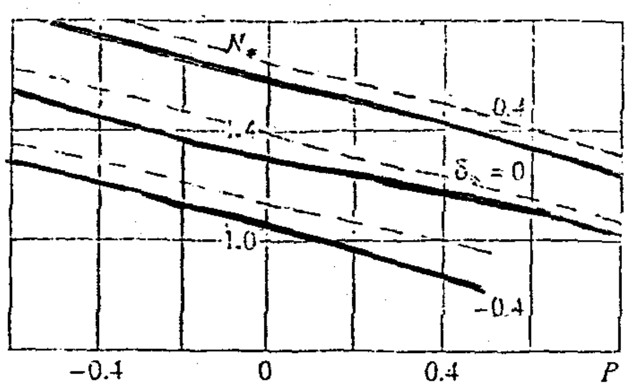

FIGURE 2

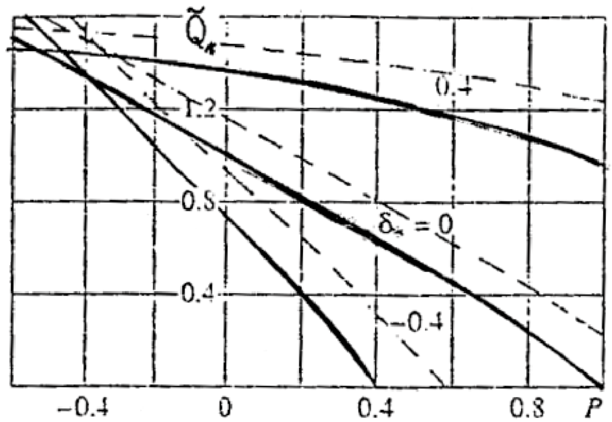

FIGURE 3

Consider now the expression for finding critical loading $\widetilde{Q} > 0$. The right-hand side in relation (1.37) vanishes for

$$1,755\widetilde{Q} = m(\theta^2 + \theta^{-2} - 2PQ^{-1}) + 2,37\delta_*\theta^{-2} + 1,404\overline{\delta}_*^2\theta^{-1}m^{-1} + 2,37P\widetilde{\delta}_*. \tag{1.39}$$

Next, taking into account inequality (1.2), we restrict ourselves to $|\delta_*| \lesssim 1$.

The quantity Q, realizing the least value of $\widetilde{Q}$ (for fixed m) is defined by a positive root of the equation

$$\theta^4 + (2P - 1,404\overline{\delta}_\nu^2)\theta^2 - 4,74\delta_\nu\theta - 3, \quad \delta_\nu = \delta_*/m, \quad \overline{\delta}_\nu^2 = \Big(\delta_*^2 + \frac{\gamma_*}{4}\Big)|m^2.$$

Similarly to the above-said, taking into account inequality (1.33) (replacing δ_* by δ_ν), we obtain

$$\theta = \sqrt{\sqrt{3} + 0,234\Big(\overline{\delta}_*^2 + \frac{3}{4}\gamma_*\Big)/m^2 - P} + 0,684\delta_*/m \quad (\delta_* > 0),$$

$$\theta = \sqrt{\sqrt{3} + 0,234\Big(\overline{\delta}_*^2 + \frac{3}{4}\gamma_*\Big)/m^2 - P} - 0,684|\delta_*|/m \quad (\delta_* < 0).$$

In case δ_*, γ_*, P do not satisfy inequality (1.33), we have to proceed from the full expressions for the roots of Q_1, Q_3. They are of the same form as N_1, N_3 defined by equalities (1.26), where δ_* should be replaces by δ_ν, $\overline{\delta}_*^2$ and $\overline{\delta}_\nu^2$.

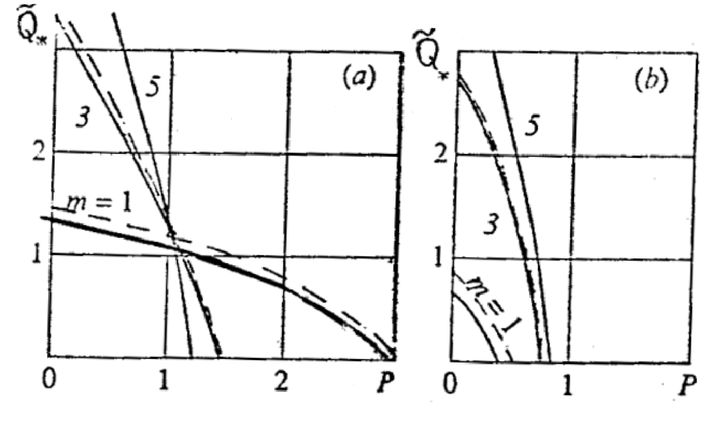

FIGURE 4

Figure 4 gives critical values of $\widetilde{Q}_*(m, P)$ under $m = 1, 3, 5$ ($\gamma_* = 0; 1,272$) for $\delta_* = 0,4$ (Figure 4a) and under $\delta_* = -0,4$ (Figure 4b). Corresponding dependencies for $\gamma_* = 0$ are given by solid curves, and for $\gamma_* = 1,272$ by dashed curves. It can be seen that for $\delta_* > 0$ and $P < 0$, the least value of $\widetilde{Q}_*$, irrespective of γ_*, is realized for $m = 1$, whereas for P approaching from above to unity, the critical value of $\widetilde{Q}_*$ is realized for large m. For $\delta_* < 0$, the least value of $\widetilde{Q}_*$ is realized for $m = 1$ when $0 \le P \le P_*$ (P_* is a critical value of P for $\widetilde{Q} = 0$).

Consider now expression (1.37). The least value ω^2 with respect to Q (for fixed m) is defined from the condition

$$(\omega^2)'_\theta = 0,5m^2(2\theta - 2\theta^{-3} - 2,37\delta_\nu\theta^{-2} + 2,37P\delta_\nu - 1,755\widetilde{\theta}_\nu) = 0$$
$$\widetilde{Q}_\nu = \widetilde{Q}/m, \quad \delta_\nu = \delta_*/m,$$

which implies that

$$\theta^4 + b\theta^3 + d\theta + e = 0, \quad b = 1,185\widetilde{\delta}_\nu P - 0,8775\widetilde{Q}_\nu, \quad d = -1,185\delta_\nu, \quad e = 1.$$

This equation is of the same form as (1.16), where δ_* should be replaced by δ_ν; $\widetilde{\delta}_*$ by $\widetilde{\delta}_\nu$, and $\widetilde{Q}$ by $\widetilde{Q}_\nu$. Therefore, analogously to the above-said, we find that

$$\begin{aligned} \theta &= [1 + 1,755\widetilde{\delta}_*^2 PM_1(1 - P^2M_1^2) - 0,08775\widetilde{\delta}_\nu^2(1 + 2PM_1 - 2P^2M_1^2)]^{1/2} \\ &\quad +0,2962\delta_\nu(1 - PM_1), \ (\delta_* > 0), \\ \theta &= [1 + 1,755\widetilde{\delta}_*^2 PM_2(1 - P^2M_1^2) - 0,08775\widetilde{\delta}_\nu^2(1 + 2PM_2 - 2P^2M_2^2)]^{1/2} \\ &\quad +0,2962\delta_\nu(1 - PM_2), \ (\delta_* < 0), \end{aligned} \tag{1.40}$$

where

$$M_{1,2} = 1 \mp (0,7405\widetilde{\theta}/|\widetilde{\delta}_*|P);$$

indices (1) and (2) correspond to $\delta_* > 0$ and $\delta_* < 0$, respectively.

Figure 5 presents the least values of frequency $\omega(m, P, \widetilde{Q})$ for $\widetilde{Q} = 0,54P$ under $m = 1, 3, 5$; $\gamma_* = 0$, $\gamma_* = 1,272$ for $\delta_* = 0,4$ (Figure 5a) and $\delta_* = -0,4$ (Figure 5b). Corresponding dependencies for $\gamma_* = 0$ are given by solid curves and for $\gamma_* = 1,272$ by dashed curves. It is not difficult to see that for $\delta_* > 0$ and P, varying in the interval $0 \le P < 1$, the least is the frequency for $m = 1$, whereas for P, approaching from above to unity, the least frequency is realized for large values m. For $\delta_* < 0$ and P varying in the interval $0 \le P \le P_*$ (P_* is the critical value of P for $\widetilde{Q} = 0$), the least frequency is realized for $m = 1$.

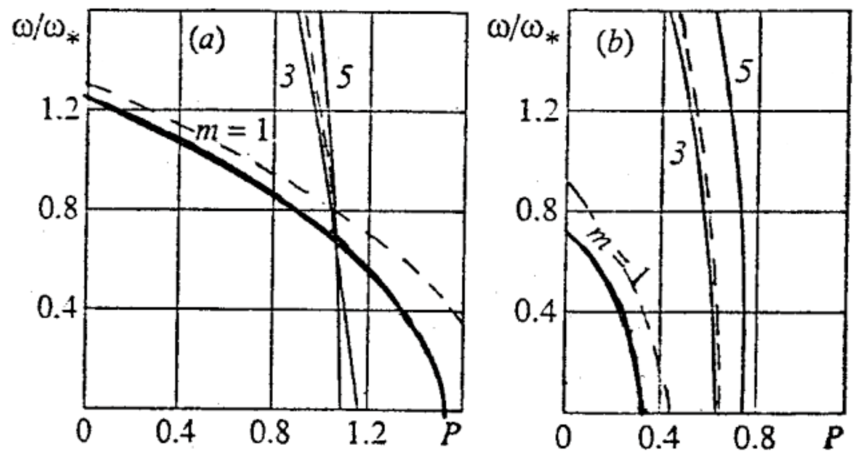

FIGURE 5

It follows from the above formulas (1.38) and (1.40), that for $m > 1$, the values θ are close to unity ($\theta \approx 1$), i.e., when $n^2 \approx \lambda_m \varepsilon^{-1/4}$. Therefore the given results are valid only for sufficiently thin shells, when $\varepsilon^{-1/4} \gg \lambda_m$, then the relation $n^2 \gg \lambda_m^2$ holds and the given theory is valid. Moreover, by formula (1.37), we find that for comparatively large m, when $Q \approx 1$, $\omega^2/\omega_*^2 \approx 0,5m^2(\theta^2+\theta^{-2}-2P) = m^2(1-P)$ i.e., the influence of δ_* and $\widetilde{\theta}$ may be neglected.

Thus it is shown that if the stresses arising in the shells under the action of external pressure, temperature and filler constraint significantly change the lower frequencies, then the influence of these factors on the higher frequencies is practically insignificant. At the same time, the influence of meridional loading is significant both on the lower and on the higher frequencies.

2

Consider now the case for

$$q_1 = q_0 + q_t \cos\Omega t, \quad P_1 = P_0 + P_t \cos\Omega t, \quad T_1 = T_0 + T_t \cos\Omega t.$$

We seek for a solution of equation (1.9) in the form

$$w = f_{mn}(t) \cos\lambda_m \xi \sin n\varphi.$$

Substituting the given solution into (1.9) and requiring that the latter be satisfied for any ξ and φ, we get

$$\frac{d^2 f_{mn}}{dt^2} + \frac{E}{\rho r^2}\Big\{\varepsilon n^4 + \lambda_m^{-4} n^{-4} + 4\delta\lambda_m^2 n^{-2} + 4\delta^2 + \frac{\gamma}{4} - P_1(t)(\lambda_m^2 - 2\delta n^2)$$
$$-[q_1(t) + \alpha\gamma T_1(t)]\Big\} g^{-1} n^2 f_{mn} = 0. \tag{2.1}$$

Frequencies of natural oscillations of the shell (for $q_1 = q_0$, $P_1 = P_0$, $T_1 = T_0$) are defined from equation (2.1) by putting $f_{mn}t = C\sin\omega_{mn}t$ and expressed by formula (1.11). Since equation (2.1) is identical for all forms of oscillations, the indices m and n may be neglected.

Analogously to the above-said, let us introduce dimensionless values (1.13), (1.38) and write equation (2.1) as follows:

$$\frac{d^2 f}{dt^2} + 0,5m^2\omega_*^2\{\theta^2 + \theta^{-2} + 2,37\delta_\nu\theta^{-1} + 1,404(\delta_\nu^2 + \gamma_\nu/4) - 2(P_0 + P_t\cos\Omega t)(1 - 1,185_\nu\theta)$$
$$-1,755[(Q_0 + \alpha\gamma\overline{T}_0) + (Q_t + \alpha\gamma\overline{T}_t)\cos\Omega t]g^{-1}Q_m^{-1}\}f = 0, \tag{2.2}$$

where

$$Q_i = \overline{q}_i/\overline{q}_{0*}, \quad P_i = p_i/p_{0*}, \quad \overline{T}_i = T_i/\overline{q}_{0*} \ (i = 0, t), \quad \delta_\nu = \delta_*/m, \quad \gamma_\nu = \gamma/m^2.$$

Further, we introduce the notation $\widetilde{Q}_i = (Q_i + \widetilde{T}_i)g^{-1}$, $\widetilde{T}_j = \alpha\gamma\overline{T}_i$ and reduce equation (2.2) to the standard form of the Mathieu equation

$$d^2f/dt^2 + \omega^2(\theta)[1 - 2\mu(\theta)\cos\Omega t]f = 0, \tag{2.3}$$

$$\omega^2(\theta) = \omega_0^2(\theta)[1 - M_0(\theta)], \quad \omega_0^2(\theta) = 0,5\omega_*^2 m^2 D(\theta), \quad \omega_*^2 = 2\lambda_1^2\varepsilon^{1/2}E/(\rho r^2), \tag{2.4}$$

$$D(\theta) = \theta^2 + \theta^{-2} + 2,37\delta_\nu\theta^{-1} + 1,404\overline{\delta}_\nu^2, \quad \overline{\delta}_\nu^2 = \delta_\nu^2 + \gamma_\nu/4,$$

$$\mu(\theta) = \frac{M_t(\theta)}{2[1 - M_0(\theta)]}, \quad M_0(\theta) = \frac{P_0}{P(\theta)} + \frac{\widetilde{Q}_0}{\widetilde{Q}(\theta)}, \quad M_t(\theta) = \frac{P_t}{P(\theta)} + \frac{\widetilde{Q}_t}{\widetilde{Q}(\theta)}, \tag{2.5}$$

$$P(\theta) = D(\theta)/2(1 - 1,185\delta_\nu\theta), \quad \widetilde{Q}(\theta) = D(\theta)/1,755m^{-1}\theta. \tag{2.6}$$

If $\frac{P(t)}{\widetilde{Q}(t)} = \chi$, then $\frac{P_i}{\widetilde{Q}_i} = \chi$ ($\widetilde{Q}_i = Q_i + \alpha\gamma\overline{T}$; $i = 0, t$) and, in addition, we obtain

$$M_0 = \frac{\widetilde{Q}_0}{\widetilde{Q}_c}, \quad M_t = \frac{\widetilde{Q}_i}{\widetilde{Q}_c}, \quad \widetilde{Q}_c = D(\theta)/2\chi(1 - 1.185\delta_\nu\theta) + 1,755m^{-1}\theta.$$

The value μ is usually called an energizing coefficient. The solution of equation (2.3) has been investigated in a number of works where it was mentioned that under certain relations between μ, Ω, ω and $t \to \infty$ the solution of equation (2.2) is infinitely increasing in the regions of instability. Generalizing the results of [1] to the shell under consideration, below we present the following formulas. To elucidate the influence of temperature on the location of regions of dynamical instability, let us consider first the case for $P_t \to 0$ ($\mu \to 0$).

Thus we find that these regions are located in the vicinity of frequencies

$$\Omega_* = 2\omega(\theta)/k.$$

Depending on a number k, we distinguish the first, second, third and so on regions of dynamical instability. The region of instability ($k = 1$) lying close to $\Omega_* = 2\omega(\theta)$, when $\omega(\theta)$ takes the least value, is the most dangerous and hence of greatest practical importance. This region is called a principal region of dynamical instability. If P_t is other than zero, then for the boundaries of the principal region of instability we obtain the following formula:

$$\Omega_* = 2\omega(\theta)\sqrt{1 \pm \mu(\theta)},$$

Taking into account the resistance forces, proportional to the first time derivative with respect to displacement (with damping factor ε), the formula for finding the boundaries of the principal region of instability takes the form

$$\Omega_* = 2\omega(\theta)\sqrt{1 \pm \sqrt{\mu^2(\theta) - (\Delta/\pi)^2}}, \quad \Delta = 2\pi\varepsilon/\omega(\theta), \tag{2.7}$$

where the terms involving higher degrees Δ/π are omitted, taking into account that the damping decrement Δ is usually very small as compared with unity. The values of $\omega(\theta)$, $P(\theta)$, $\mu(\theta)$ are defined by formulas(2.4), (2.5), (2.6), where m and θ correspond to the least value of $\omega(\theta)$. For $m = 1$, on the basis of formula (1.38), $\theta = N$, the corresponding values $\omega(N)$ depending on P_0 and δ_* are presented in Figure 1.

It follows from (2.7) that the minimal value of the energizing coefficient (critical), for which undamped oscillations are still possible, is defined by the equality

$$\underset{*1}{\mu} = \Delta/\pi.$$

For the boundary of the second region of instability ($k = 2$), the following formula

$$\Omega_* = \omega(\theta)\sqrt{1 + \mu^2(\theta) \pm \sqrt{\mu^4(\theta) - (\Delta/\pi)^2}[1 - \mu^2(\theta)]} \tag{2.8}$$

holds.

In this case, a critical value of the energizing coefficient is defined approximately by the equality $\underset{*2}{\mu} = (\Delta/\pi)^{1/2}$. Analogously, generalizing the results of [1], we can likewise give formulas for the boundaries of the third region of instability which is practically rarely realized.

On the basis of our formulas and graphs, it is not difficult to determine intervals of change energizing frequencies (depending on δ_*, P_0, P_t, Q_0, Q_t, T_0, T_t) falling into the regions of dynamical instability. Thus, for example, for $\delta_* = 0,4$; $P_i/(Q_i + \widetilde{T}_i) = 1,85$, $(i = 0, t)$, $P_0 = 0,2$, $P_t = 0,05$ $(Q_0 = Q_t = 0$, $\widetilde{T}_0 = 0,108$, $\widetilde{T}_t = 0,027)$, $\Delta = 0,01$ we find that the least frequency is realized for $m = 1$, $\theta = M = 1,11$, $\omega(N) = 1,167\omega_*$, $T_c(N) = 0,842$, $\mu(N) = 0,0183$, $\underset{*1}{\mu} = 0,00318$, $\underset{*2}{\mu} = 0,0564$.

Then, by formula (2.7) we find that the values Ω, appearing in the interval $2,292\omega_* \leq \Omega \leq 2,355\omega_*$, lie in the principal region of dynamical instability. Since $\mu(N) < \underset{*2}{\mu}$, the second region of instability is not attained.

In case $\delta_* = -0,4$ and for the same values of external loading and temperature the least frequency is realized for $m = 1$, $\theta = N = 0,924$, $\omega(N) = 0,486\omega_*$, $T_c(N) = 0,1761$, $\mu(N) = 0,1978$. In addition, by formula (2.7), we find that the values Ω, appearing in the interval $0,7798\omega_* \leq \Omega \leq 1,1643\omega_*$, fall into the principal region of dynamical instability. In the given case, the second region of dynamical instability is attained, since $\mu(N) > \underset{*2}{\mu}$. In addition, by formula (2.8), we find that the values Ω appearing in the interval $0,486\omega_* \leq \Omega \leq 0,5046\omega_*$ fall into the second region of dynamical instability.

The above formulas for the above posed questions allow one to define in a sufficiently simple way to what extent temperature and acting loadings affect the regions of dynamical instability.

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