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RELATIVE UNIFORM CONVERGENCE OF DIFFERENCE SEQUENCE OF POSITIVE LINEAR FUNCTIONS

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Abstract. In this article, we introduce the notion of relative uniform convergence, relative uniform Cauchy of difference sequence of functions and study the relation between these two notions. We define the sequence spaces $\ell_\infty(\Delta, ru), c(\Delta, ru), c_0(\Delta, ru)$ and study their topological properties.

1. Introduction

Throughout the paper, $\ell_\infty,\ c,\ c_0$ denote bounded, convergent and null convergent sequence spaces of real or complex numbers. These are normed linear spaces, normed by

$$\|(x_k)\| = \sup_{k\in N} |x_k|.$$

Kizmaz (see [7]) defined the difference sequence spaces $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$ as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in Z\},$$

for $Z = \ell_\infty,\ c,\ c_0$, where $\Delta x_k = x_k - x_{k+1}$, $k \in N$.

These sequence spaces are the Banach spaces under the norm

$$\|(x_k)\|_\Delta = |x_1| + \sup_{k\in N} |\Delta x_k|.$$

Tripathy and Esi (see [12]) introduced the generalized notion of the difference operator $\Delta_m x_k$, for a fixed $m \in N$. They defined the difference sequence spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$, $c_0(\Delta_m)$ as follows:

$$Z(\Delta_m) = \{x = (x_k) : (\Delta_m x_k) \in Z\},$$

for $Z = \ell_\infty, c, c_0$, where $\Delta_m x_k = x_k - x_{k+m}$, $k \in N$. Difference sequence spaces are studied from different aspects by many other authers (see [9–11, 13]).

The notion of relative uniform convergence of a sequence of functions was introduced by E. H. Moore. Later on, Chittenden (see [1–3]) formulated the detailed definition of it as follows and carried out a systematic investigation on the topic.

A sequence (f_n) of real, single-valued functions f_n of a real variable x, ranging over a compact subset D of real numbers, converges uniformly relative on D in case there exist the functions g and σ defined on D, and for every $\varepsilon > 0$, there exists an integer n_0 (dependent on ε) such that for every $n \geq n_0$, the inequality

$$|g(x) - f_n(x)| < \varepsilon|\sigma(x)|,$$

holds for every element x of D.

The function σ of the above definition is called a scale function. The sequence (f_n) is said to relative uniformly convergent to the scale function σ. Relative uniform convergence was studied from different aspects by many others (see [4, 5, 8]). For the details of basics on the sequence spaces and summability theory, one may refer to the monograph by Kamthan and Gupta (see [6]).

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2. Definitions and Preliminaries

In this section, we procure some basic definitions that will be used for establishing the results of the article.

Definition 2.1. A sequence space D is said to be solid or normal if $(x_k) \in D$ implies $(\alpha_k x_k) \in D$, for all (α_k) with $|\alpha_k| \leq 1$, for all $k \in N$.

Definition 2.2. A sequence space D is said to be monotone if it contains the canonical pre-images of all its step spaces.

Definition 2.3. A difference sequence of functions $(\Delta f_n(x))$ of real, single-valued functions ranging over a compact subset D of real numbers converges uniformly relative on D if there exist the functions $g(x)$ and $\sigma(x)$ defined on D and for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$, the inequality

$$|g(x) - \Delta f_n(x)| < \varepsilon|\sigma(x)|,$$

where $\Delta f_n(x) = f_n(x) - f_{n+1}(x)$, holds for every element x of D.

Example 2.1. Consider the sequence of functions $(g_n(x))$ defined by

$$g_n(x) = \frac{x}{1+nx^2}, \quad x \in R;$$

$$\begin{aligned}\Rightarrow \Delta g_n(x) =& g_n(x) - g_{n+1}(x)\\ =& \frac{x}{1+nx^2} - \frac{x}{1+(n+1)x^2}\\ =& \frac{x^3}{(1+nx^2)(1+(n+1)x^2)} \sim \frac{x}{n}.\end{aligned}$$

$\Rightarrow \Delta g_n(x) = \frac{x}{n}$, doesn't converge uniformly to zero function on $x \in R$, but converges uniformly w. r. t. scale function x in R.

Definition 2.4. A difference sequence of functions $(\Delta f_n(x))$ of single, real-valued functions ranging over a compact subset D of real numbers is said to be relative uniformly Cauchy if there exists a function $\sigma(x)$ defined on D and for every $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that

$$|\Delta f_n(x) - \Delta f_m(x)| < \varepsilon|\sigma(x)|,$$

for all $n, m \geq n_0$, holds for every element x of D.

Definition 2.5. A difference sequence of functions $(\Delta f_n(x))$ defined on a compact domain D is said to be relative uniformly bounded if there exists a function $\sigma(x)$ defined on D such that

$$|\Delta f_n(x)| < M|\sigma(x)|,$$

for all $x \in D$ and $n \in N$.

We introduce the following difference sequence spaces:

$$Z(\Delta, ru) = \Big\{(f_n(x)) : (\Delta f_n(x)) \in Z \text{ relative uniformly } w.r.t.\ \sigma(x)\Big\},$$

for $Z = \ell_\infty, c$ and c_0, where $\Delta f_n(x) = f_n(x) - f_{n+1}(x)$.

The above spaces are normed by

$$\|f(x)\|_{(\Delta,ru)} = |f_1(x)| + \sup_{x\in D, n\in N} |\Delta f_n(x)\sigma(x)|.$$

3. Main Results

In this section, we establish the results of this article.

Theorem 3.1. *The difference sequence of functions $(\Delta f_n(x))$ converges relative uniformly w.r.t. scale function $\sigma(x)$ on a compact domain D if and only if it is relative uniformly Cauchy.*

Proof. Let $(\Delta f_n(x))$ be relative uniformly convergent sequence w.r.t. the scale function $\sigma(x)$.

Then $|\Delta f_n(x) - f(x)| < \frac{\varepsilon}{2}|\sigma(x)|$, for all $n \geq n_0$.

$$\begin{aligned}\Rightarrow |\Delta f_n(x) - \Delta f_m(x)| =& |\Delta f_n(x) - f(x) + f(x) - \Delta f_m(x)| \\ \leq& |\Delta f_n(x) - f(x)| + |\Delta f_m(x) - f(x)| \\ <& \frac{\varepsilon}{2}|\sigma(x)| + \frac{\varepsilon}{2}|\sigma(x)| < \varepsilon|\sigma(x)|.\end{aligned}$$

We get

$$|\Delta f_n(x) - \Delta f_m(x)| < \varepsilon|\sigma(x)|,$$

for all $n, m \geq n_0$.

Hence $(\Delta f_n(x))$ is relative uniformly Cauchy w.r.t. the scale function $\sigma(x)$.

Conversely, let $(\Delta f_n(x))$ be relative uniform Cauchy w.r.t. the scale function $\sigma(x)$.

Then $|\Delta f_n(x) - \Delta f_m(x)| < \frac{\varepsilon}{2}|\sigma(x)|$, for all $n, m \geq n_0$.

By Cauchy's general principle of convergence, $(\Delta f_n(x))$ converges pointwise w.r.t. $\sigma(x)$ for each $x \in D$, there exists $n_0 = n_0(\varepsilon)$ such that

$$|\Delta f_n(x) - f(x)| < \frac{\varepsilon}{2}|\sigma(x)|,$$

for $n \geq n_0$.

Then for all $n \geq n_0$,

$$\begin{aligned}|\Delta f_n(x) - f(x)| =& |\Delta f_n(x) - \Delta f_m(x) + \Delta f_m(x) - f(x)| \\ \leq& |\Delta f_n(x) - \Delta f_m(x)| + |\Delta f_m(x) - f(x)| \\ \leq& \frac{\varepsilon}{2}|\sigma(x)| + \frac{\varepsilon}{2}|\sigma(x)| < \varepsilon|\sigma(x)|.\end{aligned}$$

We get

$$|\Delta f_n(x) - f(x)| < \varepsilon|\sigma(x)|,$$

for all $n \geq n_0$ and for any $x \in D$.

$(\Delta f_n(x))$ is relative uniformly convergent w.r.t. scale function $\sigma(x)$. □

Theorem 3.2. *The class of sequences $\ell_\infty(\Delta, ru)$, $c(\Delta, ru)$, $c_0(\Delta, ru)$ are normed linear spaces normed by*

$$\|f(x)\|_{(\Delta,ru)} = |f_1(x)| + \sup_{x\in D, n\in N} |\Delta f_n(x)\sigma(x)|.$$

Proof. Let $(f_n(x)), (g_n(x)) \in \ell_\infty(\Delta, ru)$ and α, β be the scalars.

Then

$$\begin{aligned}(f_n(x)) \in \ell_\infty(\Delta, ru) \Rightarrow \sup_{x\in D, n\in N} |\Delta f_n(x)| \leq M_1|\sigma_1(x)| \quad \text{and} \\ (g_n(x)) \in \ell_\infty(\Delta, ru) \Rightarrow \sup_{x\in D, n\in N} |\Delta g_n(x)| \leq M_2|\sigma_2(x)|.\end{aligned}$$

We have

$$\sup_{x\in D, n\in N} |\Delta(\alpha f_n(x) + \beta g_n(x))| \leq \max(M_1, M_2) \max|\alpha_1(x), \alpha_2(x)|.$$

Hence $\ell_\infty(\Delta, ru)$ is a linear space.

To check for conditions of norm:

(i) $f(x) = 0 \Rightarrow \|f(x)\|_{(\Delta,ru)} = 0$.

Conversely, let $\|f(x)\|_{(\Delta,ru)} = 0$.

Then

$$\|f(x)\|_{(\Delta,ru)} = \sup_{x\in D}|f_1(x)| + \sup_{x\in D,n\in N}|\Delta f_n(x)\sigma(x)| = 0;$$
$$\Rightarrow f_1(x) = 0 \quad \text{and} \quad |\Delta f_n(x)\sigma(x)| = 0.$$

Let $n = 1$, then

$$(f_1(x) - f_2(x))\sigma(x) = 0;$$
$$\Rightarrow f_2(x)\sigma(x) = 0;$$
$$\Rightarrow f_2(x) = 0.$$

Proceeding this way, we get $f_n(x) = 0$, for all $n \in N$.

(ii)

$$\begin{aligned}\|f+g\|_{(\Delta,ru)} &= \sup_{x\in D}|f_1(x)+g_1(x)| + \sup_{x\in D,n\in N}|\Delta f_n(x)\sigma_1(x) + \Delta g_n(x)\sigma_2(x)| \\ &= \sup_{x\in D}|f_1(x)| + \sup_{x\in D,n\in N}|\Delta f_n(x)\sigma_1(x)| + \sup_{x\in D}|g_1(x)| + \sup_{x\in D,n\in N}|\Delta g_n(x)\sigma_2(x)|;\end{aligned}$$

$\Rightarrow \|f+g\|_{(\Delta,ru)} = \|f\|_{(\Delta,ru)} + \|g\|_{(\Delta,ru)}$.

(iii)

$$\begin{aligned}\|\lambda f(x)\|_{(\Delta,ru)} &= \sup_{x\in D}|\lambda f_1(x)| + \sup_{x\in D,n\in N}|\lambda\Delta f_n(x)\sigma(x)| \\ &= |\lambda|\sup_{x\in D}|f_1(x)| + |\lambda|\sup_{x\in D,n\in N}|\Delta f_n(x)\sigma(x)|;\\ \Rightarrow \|\lambda f(x)\|_{(\Delta,ru)} &= |\lambda|\|f(x)\|_{(\Delta,ru)}.\end{aligned}$$

Similarly, we can show for $c(\Delta, ru)$, $c_0(\Delta, ru)$.

Hence the three spaces are the normed linear spaces. □

Theorem 3.3. *The sequence spaces of functions $c(\Delta, ru)$, $c_0(\Delta, ru)$ and $\ell_\infty(\Delta, ru)$ are Banach spaces under the norm*

$$\|f(x)\|_{(\Delta,ru)} = |f_1(x)| + \sup_{x\in D,n\in N}|\Delta f_n(x)\sigma(x)|.$$

Proof. Let $(f^i(x))$ be a Cauchy sequence in $c(\Delta, ru)$, where $(f^i(x)) = ({f_n}^i(x)) = ({f_1}^i(x), {f_2}^i(x), \ldots) \in c(\Delta, ru)$, for each $n \in N$.

Then

$$\|f^i(x) - f^j(x)\|_{(\Delta,ru)} = \sup_{x\in D}|{f_1}^i(x) - {f_1}^j(x)| + \sup_{x\in D,n\in N}|(\Delta f^i(x) - \Delta f^j(x))\sigma(x)| \to 0 \text{ as } i,j\to\infty.$$

Hence for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$\begin{aligned}\|f^i(x) - f^j(x)\|_{(\Delta,ru)} &= \sup_{x\in D}|{f_1}^i(x) - {f_1}^j(x)| \\ &\quad + \sup_{x\in D,n\in N}|(\Delta f^i(x) - \Delta f^j(x))\sigma(x)| < \varepsilon, \text{ for all } i,\ j \geq n_0; \qquad (1)\end{aligned}$$

$$\Rightarrow ({f_n}^i(x)) \text{ is a Cauchy sequence in } D \text{ for } n \in N;$$
$$\Rightarrow ({f_n}^i(x)) \text{ is convergent in } D \text{ w.r.t. } \sigma(x) \text{ for } n \in N.$$

Let $\lim_{i\to\infty} {f_n}^i(x) = f_n(x)$, for each $n \in N$.

From (1), we have

$$|(\Delta {f_n}^i(x) - \Delta {f_n}^j(x))\sigma(x)| < \varepsilon, \text{ for all } i,j \geq n_0 \text{ and } n \in N.$$

Hence $(\Delta {f_n}^i(x))$ is a Cauchy sequence in D, for all $n \in N$.
$\Rightarrow (\Delta {f_n}^i(x))$ is convergent in D w.r.t. $\sigma(x)$, for each $n \in N$.
$\Rightarrow \lim_{i\to\infty} \Delta {f_n}^i(x) = g_n(x)$, for each $n \in N$.

Then

$$\lim_{m\to\infty}\sup_{x\in D}|{f_1}^j(x)-{f_1}^m(x)| = \sup_{x\in D}|{f_1}^j(x)-f_1(x)| < \varepsilon, \text{ for all } i\geq n_0 \text{ and}$$

$$\lim_{m\to\infty}\sup_{x\in D, n\in N}|(\Delta f^j(x)-\Delta f^m(x))\sigma(x)| = \sup_{x\in D, n\in N}|(\Delta f^j(x)-f_n(x))\sigma(x)| < \varepsilon, \text{ for all } j\geq n_0.$$

Thus

$$\|f^i(x)-f^m(x)\| = \|f^i(x)-f^j(x)+f^j(x)-f^m(x)\| < \varepsilon, \text{ for all } i,m\geq n_0.$$

$$\Rightarrow \|f^i(x)-f^m(x)\| \leq \|f^i(x)-f^j(x)\|+\|f^j(x)-f^m(x)\| < \varepsilon.$$

$$\Rightarrow \|f^i(x)-f^m(x)\| < \varepsilon.$$ □

Hence $c(\Delta, ru)$ is complete. Similarly we can show for $c_0(\Delta, ru)$ and $\ell_\infty(\Delta, ru)$.

Proposition 3.1. $c_0(\Delta, ru) \subset c(\Delta, ru) \subset \ell_\infty(\Delta, ru)$.

Proposition 3.2. *The spaces $c_0(\Delta, ru)$, $c(\Delta, ru)$ and $\ell_\infty(\Delta, ru)$ are not solid spaces.*

Proof. Proof of the proposed proposition follows from the following example. □

Example 3.1. Let us consider a sequence of functions $(f_n(x))$ defined by

$$f_n(x) = \frac{x}{n}, \quad x\in R.$$

Then

$$\begin{aligned}\Delta f_n(x) =& f_n(x)-f_{n+1}(x)\\ =&\frac{x}{n}-\frac{x}{n+1}\\ =&\frac{x}{n^2+n}\sim\frac{x}{n^2};\\ \Rightarrow \lim_{n\to\infty}\Delta f_n(x) =&0.\end{aligned}$$

Let us say that $(\Delta f_n(x))$ is uniformly convergent.
Then

$$|\Delta f_n(x)-f(x)|<\varepsilon, \text{ for all } n\geq n_0 \text{ and for any } x \text{ in } R;$$

$$\Rightarrow \left|\frac{x}{n^2}\right|<\varepsilon;$$

$$\Rightarrow n>\sqrt{\frac{x}{\varepsilon}}.$$

Here, n_0 is dependent on x. Therefore our assumption is wronged.
$(\Delta f_n(x))$ is not uniformly convergent, but converges relative uniformly to the scale function $\sigma(x) = x$ in R.
Hence $(f_n(x)) \in c(\Delta, ru)$ and $(f_n(x)) \in c_0(\Delta, ru)$.
$(\Delta f_n(x))$ is also uniformly bounded relative to the scale function $\sigma(x) = x$.
$\Rightarrow (f_n(x)) \in \ell_\infty(\Delta, ru)$.
Let us consider another sequence of functions $(g_n(x))$ defined by

$$g_n(x) = \frac{x^2+n}{n}, x\in R.$$

Then

$$\begin{aligned}\Delta g_n(x) =& g_n(x)-g_{n+1}(x)\\ =&\frac{x^2+n}{n}+\frac{x^2+n+1}{n+1}\\ =&\frac{x^2}{n^2+n}\sim\frac{x^2}{n^2}.\end{aligned}$$

$(\Delta g_n(x))$ is pointwise convergent to zero, but does not converge uniformly.

Let us say that $(\Delta g_n(x))$ is relative uniformly convergent w.r.t. the scale function $\sigma(x) = x, x \in R$. Then we have

$$\left|\frac{x^2}{n^2}\right| < \varepsilon|x|, \text{ for all } n \geq n_0 \text{ and for all } x \in R;$$

$$\Rightarrow n > \sqrt{\frac{x}{\varepsilon}}, \text{ for all } n \geq n_0 \text{ and for any } x \in R.$$

Here, n_0 is dependent on x. Therefore our assumption is wronged.

$(\Delta g_n(x))$ is not relative uniformly convergent w.r.t. the same scale function $\sigma(x) = x$.

Hence

$$(g_n(x)) \notin c(\Delta, ru) \text{ and } (g_n(x)) \notin c_0(\Delta, ru).$$

$(\Delta g_n(x))$ is also not uniformly bounded relative to the same scale function;

$$\Rightarrow (g_n(x)) \notin \ell_\infty(\Delta, ru).$$

Hence the three spaces are not solid.

Proposition 3.3. *The spaces $c_0(\Delta, ru), c(\Delta, ru)$ and $\ell_\infty(\Delta, ru)$ are the monotone spaces.*

Proof. Proof of the proposed proposition follows from the following example. □

Example 3.2. Let us consider the sequence of functions $(f_n(x))$ defined by

$$f_n(x) = \frac{x}{n}, x \in R$$

considered in Example 3.1. From the above example, we know that $(\Delta f_n(x))$ is relative uniformly convergent w.r.t. scale function $\sigma(x) = x$ and $(\Delta f_n(x))$ is also uniformly bounded relative to the scale function x.

Hence $(f_n(x)) \in c(\Delta, ru), (f_n(x)) \in c_0(\Delta, ru)$ and $(f_n(x)) \in \ell_\infty(\Delta, ru)$. Let $(g_n(x))$ be the pre-image of a sequence of functions $(f_n(x))$ defined by

$$g_n(x) = \begin{cases} \frac{x}{n}, & \text{if } n \text{ is even;} \\ 0, & \text{otherwise.} \end{cases}$$

$(\Delta g_n(x))$ is defined by

$$\Delta g_n(x) = \begin{cases} \frac{(-1)^n x}{n+1}, & \text{if } n \text{ is even;} \\ \frac{(-1)^n x}{n}, & \text{otherwise.} \end{cases}$$

We have seen that $(\Delta g_n(x))$ is also uniformly convergent and uniformly bounded relative to the same scale function $\sigma(x) = x$. Hence the three spaces $c_0(\Delta, ru)$, $c(\Delta, ru)$ and $\ell_\infty(\Delta, ru)$ are monotone.

Result 3.1. Let two difference sequence of functions $(\Delta f_n(x))$ and $(\Delta g_n(x))$ converge uniformly relative to a scale function $\sigma(x)$ on a compact domain D, then their sum also converges uniformly to their sum of the limit functions relative to the scale function $\sigma(x)$.

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