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THE PROBABILITY WHEN A FINITE COMMUTATIVE RING IS NIL-CLEAN

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Abstract. We define an indicator of the probability when a finite commutative ring is nil-clean, and calculate this probability for certain classes of finite commutative rings.

1. Introduction and Motivation

Throughout the present paper, all rings are assumed to be finite and commutative with identity 1 unless indicated otherwise. The most part of our notation and terminology are standard as the specific ones will be given explicitly in what follows.

In the existing literature there are numerous examples of articles (see, for instance, [3], [7] and [8]) which deal with a suitably defined probability calculating when a finite commutative ring possesses a given property.

On the other hand, referring to the original source [6], we recall that a ring is called *(uniquely) nil-clean* if every element is (uniquely) written as the sum of an idempotent and a nilpotent. It was independently proved in [6] and [5] that in the commutative case each nil-clean ring is uniquely nil-clean. Likewise, it has been shown in [10, Theorem 2.13] that both expressions are equivalent. Also, it follows that in nil-clean rings the ring element 2 is always a nilpotent. However, there is an abundance of finite rings which are surely *not* nil-clean, e.g., the rings $\mathbb{Z}/(k) \cong \mathbb{Z}_k$ whenever k is an odd integer.

That is why, as our motivating tool, we combine the presented above two things to arrive at a new notion concerning the probability of when a finite commutative ring is necessarily nil-clean. This will be done in the next sections.

2. Definitions and Preliminaries

Our purpose here is to introduce an appropriate and useful indicator which will somewhat show when a finite commutative ring may be nil-clean.

So, we come to our main key, motivating the writing of this article.

Definition 2.1. The symbol $pn(R)$, which indicates the probability that a ring R is nil-clean, means the following:

$$pn(R) = \frac{\left| r \in R : \quad r = e + q \ \text{ for some } e \in Id(R) \text{ and } q \in Nil(R) \right|}{|R|}.$$

One can easily see the validity of the following facts by a direct inspection of the stated above Definition 2.1.

Proposition 2.2. *For a ring R, the following statements are true:*

1. *R is nil-clean $\iff$ $pn(R) = 1$.*
2. $\frac{2}{|R|} \leq pn(R) = \frac{|Id(R)|.|Nil(R)|}{|R|}$.
3. *For a reduced ring R, $pn(R) = \frac{|Id(R)|}{|R|}$. Specifically, R is a reduced nil-clean ring $\iff$ R is a Boolean ring.*
4. *For an indecomposable ring R, e.g., a local ring*
$$pn(R) = \frac{2|Nil(R)|}{|R|}.$$

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(5) *If R is an indecomposable reduced ring, e.g., a domain, then* $pn(R) = \frac{2}{|R|}$.
(6) *R is an indecomposable reduced nil-clean ring* $\Longleftrightarrow$ *R is isomorphic to the ring* $\mathbb{Z}_2$.
(7) *If* $R = \prod_{i=1}^{n} R_i$, *then* $pn(R) = \prod_{i=1}^{n} pn(R_i)$.

Lemma 2.3. *Let R be a ring. Then:*

(1) *R is local if and only if $Zd(R)$, the set of zero divisors of R, is an ideal (and so, in particular, $Zd(R) = Nil(R)$ is the unique maximal ideal of R).*
(2) ([9, Theorem 2(ii)]) *If R is a local ring with the unique maximal ideal M, then $|R| = p^{nr}$, $|M| = p^{(n-1)r}$ for some prime p and positive integers n, r.*

Corollary 2.4. *For a local ring (R, M),*

$$pn(R) = \frac{2}{p^r},$$

in which $p^r = |R|/|M|$ for some prime p and a positive integer r.

Proof. The proof is clear by the utilization of Proposition 2.2(4) and Lemma 2.3. □

Corollary 2.5. *For a local ring (R, M), the following conditions are equivalent:*

(1) *R is nil-clean;*
(2) *$|R| = 2^n$ and $|M| = 2^{n-1}$ for some positive integer n.*

Proof. (1) $\Rightarrow$ (2). Assume that R is nil-clean. By a new application of Corollary 2.4, one has $|R|/|M| = p^r = 2$. Consequently, we conclude from Lemma 2.3(2) that $|R| = 2^n$ and $|M| = 2^{n-1}$ for some positive integer n.

(2) $\Rightarrow$ (1). Suppose that $|R| = 2^n$ and $|M| = 2^{n-1}$ for some positive integer n. Then $pn(R) = \frac{2}{p^r}$, where $p = 2$ and $r = 1$ by using Corollary 2.4. Thus R is a nil-clean ring. □

Recall that for a ring R and an R-module M, the *Nagata's idealization* of M, denoted by $R(+)M$, is a ring which is formed from the direct sum $R \oplus M$, but with multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$ for any elements (r, m) and (s, n) of $R(+)M$. These operations make $R(+)M$ a ring with the identity element $(1, 0)$, containing R. Also, all prime (maximal) ideals of $R(+)M$ have the form $P(+)M$ for some prime (maximal) ideals P of R.

Corollary 2.6. *Let p be a prime number and $k > i$ positive integers. Then $pn((\mathbb{Z}_{p^i})(+)(\mathbb{Z}_p)^{k-i}) = \frac{2}{p^i}$. Moreover, the ring $(\mathbb{Z}_{p^i})(+)(\mathbb{Z}_p)^{k-i}$ is nil-clean if and only if $p = 2$ and $i = 1$.*

Proof. Consider the ring $R = (\mathbb{Z}_{p^i})(+)(\mathbb{Z}_p)^{k-i}$. It is obvious that $|R| = p^k$ and $p\mathbb{Z}_{p^i}(+)(\mathbb{Z}_p)^{k-i}$ is the only maximal ideal of R and hence $|Nil(R)| = p^{k-i}$. It follows that $|R|/|Nil(R)| = p^i$ and so, with Corollary 2.4 at hand, one deduces that $pn(R) = \frac{2}{p^i}$, as required. □

Lemma 2.7. *Any finite ring R with exactly t distinct maximal ideals can be decomposed into a direct product of t local rings.*

Proof. See [2, Theorem 8.7]. □

The next assertion follows by direct and straightforward calculations, so we leave it to the interested reader for a verification.

Proposition 2.8. *Let R be a ring with exactly t distinct maximal ideals, so $R = R_1 \times R_2 \times \cdots \times R_t$. Then*

$$\frac{|R|}{|Nil(R)|} = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t},$$

where $p_i^{r_i} = \frac{|R_i|}{|Nil(R_i)|}$, for some (not necessarily distinct) primes $p_1, \ldots, p_t$ and positive integers $r_1, \ldots, r_t$.

We proceed by proving the following claim.

Theorem 2.9. *Let R be a ring and let $t \in \mathbb{N}$ be the number of maximal ideals of R. Then*

$$pn(R) = \frac{2^t|Nil(R)|}{|R|} = \frac{2^t}{p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t}},$$

where $R \cong \prod_{i=1}^{t} R_i$ and $p_i^{r_i} = \frac{|R_i|}{|Nil(R_i)|}$.

Proof. With the aid of Lemma 2.7, one finds that $R \cong \prod_{i=1}^{t} R_i$ such that R_i is a local ring for any $1 \leq i \leq t$. So, $pn(R) = \prod_{i=1}^{t} pn(R_i)$ according to Proposition 2.2(7). It follows now from Proposition 2.2(4) that $pn(R_i) = \frac{2|Nil(R_i)|}{|R_i|}$, whence $pn(R) = \frac{2^t \prod_{i=1}^{t}|Nil(R_i)|}{\prod_{i=1}^{t}|R_i|} = \frac{2^t|\prod_{i=1}^{t} Nil(R_i)|}{|R|} = \frac{2^t|Nil(|\prod_{i=1}^{t} R_i)|}{|R|} = \frac{2^t|Nil(R)|}{|R|}$. For the second part, by using the fact that $pn(R) = \prod_{i=1}^{t} pn(R_i)$ and Corollary 2.4, we get $pn(R) = \frac{2^t}{p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t}}$, where $p_i^{r_i} = \frac{|R_i|}{|Nil(R_i)|}$, as wanted. □

As two pivotal consequences, we obtain the following statements.

Corollary 2.10. *Let R be a ring with exactly t distinct maximal ideals, writing $R \cong R_1 \times R_2 \times \cdots \times R_t$. Then the following are equivalent:*

(1) *R is nil-clean.*
(2) *All of the local rings R_i are nil-clean.*
(3) $\frac{|R|}{|Nil(R)|} = 2^t$.
(4) *$|R| = 2^s$ and $|Nil(R)| = 2^{s-t}$ for some positive integer $s \geq t$.*

Proof. (1) $\Rightarrow$ (2), (3). Let $R = R_1 \times R_2 \times \cdots \times R_t$ be nil-clean. We conclude from Theorem 2.9 that $1 = pn(R) = \frac{2^m}{p_1^{r_1}p_2^{r_2}\cdots p_t^{r_t}}$ in which $p_i^{r_i} = \frac{|R_i|}{|Nil(R_i)|}$ for $1 \leq i \leq t$. Thus $p_1 = p_2 = \cdots = p_t = 2$ and $r_1 + r_2 + \cdots + r_t = t$. Since all r_i's are positive integers, we get $r_1 = r_2 = \cdots = r_t = 1$. So, $\frac{|R_i|}{|Nil(R_i)|} = 2$ and hence R_i's are nil-clean rings employing Corollary 2.5. Moreover, $\frac{|R|}{|Nil(R)|} = 2^t$ by virtue of Proposition 2.8.

(2) $\Rightarrow$ (1) is clear.

(3) $\Rightarrow$ (1) is trivially true by Theorem 2.9.

(1) $\Rightarrow$ (4). Suppose that R is nil-clean. Then by point (2), all of the local rings R_i are nil-clean. It follows from Corollary 2.5 that $|R_i| = 2^{n_i}$ and $|Nil(R_i)| = 2^{n_i-1}$, and so, $|R| = \prod_{i=1}^{t}|R_i| = 2^s$ and $|Nil(R)| = \prod_{i=1}^{t}|Nil(R_i)| = 2^{s-t}$, where $s = n_1 + n_2 + \cdots + n_t$.

(4) $\Rightarrow$ (3) is straightforward. □

Corollary 2.11. *For any ring R with exactly t maximal ideals, the following statements are equivalent:*

(1) *R is a reduced nil-clean ring.*
(2) $|R| = 2^t$.
(3) *R is a Boolean ring.*
(4) *R is a semi-primitive ring.*

Proof. Points (1) and (3) are equivalent owing to Theorem 2.2(3). Also, (1) implies (2) obviously. Now, let $|R| = 2^t$. Thus Theorem 2.9 and the fact that $pn(R) \leq 1$ show that $pn(R) = |Nil(R)| = 1$, i.e., R is a reduced nil-clean ring, as expected. □

3. The Probability $pn(\mathbb{Z}_n)$

Note that among the rings of the form $\mathbb{Z}_n \cong \mathbb{Z}/(n)$, where $n \in \mathbb{N}$, the local ones are all of the form $\mathbb{Z}_{p^k}$, where p is a prime and k, n are positive integers. Also, the ring of the residues modulo p^k, say $\mathbb{Z}_{p^k}$, has the unique maximal ideal $p\mathbb{Z}_{p^k}$. We start by characterizing the probability $pn(\mathbb{Z}_{p^k})$.

Lemma 3.1. *For a prime p and a positive integer k the following equality*

$$|Nil(\mathbb{Z}_{p^k})| = p^{k-1}, \text{ and } pn(\mathbb{Z}_{p^k}) = \frac{2}{p}$$

holds. Therefore the ring $\mathbb{Z}_{p^k}$ is nil-clean if and only if $p = 2$.

Proof. Since $Nil(\mathbb{Z}_{p^k}) = p\mathbb{Z}_{p^k}$ and p is the generator of the additive group $p\mathbb{Z}_{p^k}$, one writes that $|Nil(\mathbb{Z}_{p^k})| = |p\mathbb{Z}_{p^k}| = | < p > | = |p| = \frac{p^k}{\gcd(p^k,p)} = p^{k-1}$, where $|p|$ is the order of the element p in the group $\mathbb{Z}_{p^k}$. Therefore $\frac{|\mathbb{Z}_{p^k}|}{|Nil(\mathbb{Z}_{p^k})|} = p$. Thus we conclude from Corollary 2.4 that $pn(\mathbb{Z}_{p^k}) = \frac{2}{p}$, as promised. □

The following two results are the central ones in this section.

Theorem 3.2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ be the unique factorization of the positive integer n, where p_i a prime for each $1 \leq i \leq t$ with $p_i \neq p_j$ and for $i \neq j$ with $\alpha_j \in \mathbb{N}$. Then:*

(1) $pn(\mathbb{Z}_n) = \frac{2^t}{p_1 p_2 \cdots p_t}$.
Therefore the ring $\mathbb{Z}_n$ is nil-clean if and only if $n = 2^k$ for some positive integer k.
(2) $|Nil(\mathbb{Z}_n)| = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_t^{\alpha_t - 1}$.

Proof. (1) Let $R = \mathbb{Z}_n$. Then $R \simeq \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}}$. By applying Proposition 2.2(7) in this case, we get $pn(R) = \prod_{i=1}^{n} pn(\mathbb{Z}_{p_i^{\alpha_i}})$. So, by utilizing Lemma 3.1, we conclude that $pn(\mathbb{Z}_n) = \frac{2^t}{p_1 p_2 \cdots p_t}$.

(2) Since $\mathbb{Z}_n \simeq \mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_t^{\alpha_t}}$, one writes that

$$|Nil(\mathbb{Z}_n)| = \Big|\prod_{i=1}^{n} Nil(\mathbb{Z}_{p_i^{\alpha_i}})\Big| = \prod_{i=1}^{n} |Nil(\mathbb{Z}_{p_i^{\alpha_i}})|,$$

and so, $|Nil(\mathbb{Z}_n)| = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_t^{\alpha_t - 1}$ by the usage of Lemma 3.1, as required. □

Theorem 3.3. *For two positive integers k and n, the following statements are equivalent:*

(i) *There exist integers r, s such that $rk + sn = 1$, i.e., $(k, n) = 1$.*
(ii) *The ring $k\mathbb{Z}_{kn}$ has the identity element $\overline{rk}$ for some $r \in \mathbb{Z}$.*
(iii) *$k\mathbb{Z}_{kn} \simeq \mathbb{Z}_n$.*

Proof. (i) ⇒ (ii). Let k and n be two positive integers such that $(k, n) = 1$. Then $rk + sn = 1$ for $r, s \in \mathbb{Z}$. Now, it is easy to check that $k\mathbb{Z}_{kn}$ has exactly n elements as $\{\overline{0}, \overline{k}, \overline{2k}, \ldots, \overline{(n-1)k}\}$. Now, let $\overline{ik} \in k\mathbb{Z}_{kn}$ for some $0 \leq i \leq n-1$. We have $(\overline{ik})(\overline{rk}) = \overline{irk^2} = \overline{i(rk)k} = \overline{i(1-ns)k} = \overline{i(k-nsk)} = \overline{ik}$, as required.

(ii) ⇒ (i). If $\overline{rk}$ is the identity element of $k\mathbb{Z}_{kn}$, then $\overline{k} = \overline{k}.\overline{rk} = \overline{rk^2}$. This implies that $n \mid 1 - rk$, and so $(k, n) = 1$.

(ii) ⇒ (iii). Let the ring $k\mathbb{Z}_{kn}$ have an identity $\overline{rk}$ for some $r \in \mathbb{Z}$. We show that every subgroup of the ring $k\mathbb{Z}_{kn}$ is an ideal, and thus $k\mathbb{Z}_{kn} \simeq \mathbb{Z}_n$. Let H be a subgroup of $k\mathbb{Z}_{kn}$. Since $k\mathbb{Z}_{kn} \simeq \frac{k\mathbb{Z}}{kn\mathbb{Z}}$, there is an integer h such that $H \simeq \frac{hk\mathbb{Z}}{kn\mathbb{Z}}$, where $kn\mathbb{Z} \subseteq hk\mathbb{Z}$. Thus we have $H = hk\mathbb{Z}_{kn}$, where $h \mid n$.

(iii) ⇒ (i). Let $k\mathbb{Z}_{kn} \simeq \mathbb{Z}_n$ and $(k, n) \neq 1$. Then the ring $k\mathbb{Z}_{kn}$ does not have an identity element by (ii), a contradiction. □

4. The Probability $pn(R)$ for Commutative Rings Without Identity

Our basic instrument here is the following well-known concept.

Definition 4.1. Let R be a finite commutative ring (which does not necessarily have an identity). An element x of R is called *quasi-idempotent*, provided x^n is an idempotent for some $n \in \mathbb{N}$. We denote the set of all *quasi-idempotent* elements of R by $QI(R)$.

It is easy to see that every element in a finite commutative ring is either nilpotent or quasi-idempotent. So, we have the following result.

Lemma 4.2. *For a finite commutative ring R (which does not necessarily have an identity),*

$$|R| = |Nil(R)| + |QI(R)| - 1.$$

The following statement is a re-writing of [1, Lemma 4.4], which use we will need in the sequel.

Theorem 4.3. *Let R be a finite commutative ring. Then either*

(i) *R has an identity,*
or
(ii) *R is nilpotent,*
or
(iii) *there exists a positive integer l such that $R \simeq R_0e_0 \times R_1e_1 \times \cdots \times R_le_l \times S$, where $R_0 = R$, $e_i \in Id(R_i)$, $R_i = R_{i-1} - R_{i-1}e_{i-1}$, each $|R_i|$ is a divisor of $|R_{i-1}|$ for all $1 \le i \le l$ and $S = Nil(S) \neq 0$.*

As an immediate consequence, we extract the following

Corollary 4.4. *Let R be a finite commutative ring, which is neither unitary, nor nilpotent. Then $pn(R) = \prod_{i=0}^{l} pn(R_ie_i)$, where $e_i \in Id(R_i)$ and $R_i = R_{i-1} - R_{i-1}e_{i-1}$ for all $1 \le i \le l$ and $R_0 = R$.*

5. The Probability $pn(R)$ for Finite Cyclic Rings

Recall that a ring R is said to be cyclic if R^+ the additive group of R is a cyclic group. In [4, Corollary 2], it has been shown that R is a finite cyclic ring of order n if, and only if, there exists a positive divisor k of n such that R is isomorphic to $k\mathbb{Z}_{kn}$. Thus there are exactly $\tau(n)$ non-isomorphic finite cyclic rings of order n, where $\tau(n)$ is the number of the divisors of n. In addition, according to Theorem 3.3, the cyclic ring $k\mathbb{Z}_{kn}$ has an identity if, and only if, $k = 1$. So, from now on, we shall assume that $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ is the prime factorization of the positive integer n and $k = p_1^{\beta_1}p_2^{\beta_2}\cdots p_t^{\beta_t}$, where $0 \le \beta_i \le \alpha_i$ and $1 \le i \le t$. To investigate the probability that a finite cyclic ring is nil-clean, we now consider two different possibilities for k.

Case 1: If $\beta_i \neq 0$ for all $1 \le i \le t$, then we have the following results.

Theorem 5.1. *Let $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ be the prime factorization of the positive integer n. Then $k\mathbb{Z}_{kn} = Nil(k\mathbb{Z}_{kn})$ if, and only if, $k = p_1^{\beta_1}p_2^{\beta_2}\cdots p_t^{\beta_t}$, where $0 < \beta_i \le \alpha$ for all $1 \le i \le t$.*

Proof. Let $k = p_1^{\beta_1}p_2^{\beta_2}\cdots p_t^{\beta_t}$ with $0 < \beta_i \le \alpha$ and $\gamma_i = \alpha_i + \beta_i$. Then $p_1^{\gamma_1}p_2^{\gamma_2}\cdots p_t^{\gamma_t}$ is the prime factorization of the positive integer kn, and so, $\mathbb{Z}_{kn} \simeq \mathbb{Z}_{p_1^{\gamma_1}} \times \mathbb{Z}_{p_2^{\gamma_2}} \times \cdots \times \mathbb{Z}_{p_t^{\gamma_t}}$. Since $\beta_i \neq 0$, we find that $k\mathbb{Z}_{p_i^{\gamma_i}} \subseteq p_i\mathbb{Z}_{p_i^{\gamma_i}}$ for all $1 \le i \le t$. Therefore we have $k\mathbb{Z}_{kn} \simeq k\mathbb{Z}_{p_1^{\gamma_1}} \times k\mathbb{Z}_{p_2^{\gamma_2}} \times \cdots \times k\mathbb{Z}_{p_t^{\gamma_t}} \subseteq p_1\mathbb{Z}_{p_1^{\gamma_1}} \times p_2\mathbb{Z}_{p_2^{\gamma_2}} \times \cdots \times p_k\mathbb{Z}_{p_t^{\gamma_t}} \simeq Nil(\mathbb{Z}_{kn})$. On the other hand, $Nil(k\mathbb{Z}_{kn}) = Nil(\mathbb{Z}_{kn}) \cap k\mathbb{Z}_{kn}$, and hence $k\mathbb{Z}_{kn} = Nil(k\mathbb{Z}_{kn})$. Since the above steps are reversible, we have completed our proof. □

As a direct consequence, we yield the following:

Corollary 5.2. *If $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ and $k = p_1^{\beta_1}p_2^{\beta_2}\cdots p_t^{\beta_t}$, where $0 < \beta_i \le \alpha_i$ for all $1 \le i \le t$, then $pn(k\mathbb{Z}_{kn}) = 1$.*

Case 2: Suppose that $\beta_i = 0$ for some $1 \leq i \leq t$.

So, an inference from [4, Theorem 13] to determine the prime ideals of finite cyclic rings is as follows:

Lemma 5.3. *Let P be an ideal of the ring $k\mathbb{Z}_{kn}$. Then P is prime if, and only if, there exists a prime number p with $(k,p) = 1$ such that $P =< \overline{pk} >$ and $|P| = \frac{|R|}{p}$.*

Referring to Lemma 5.3, one can see that there is exactly one prime ideal of the ring $k\mathbb{Z}_{kn}$, for each prime number p, satisfying the condition $(k,p) = 1$.

If $\beta_i = 0$ for every $1 \leq i \leq t$, then $k = 1$ and so, $pn(k\mathbb{Z}_{kn}) = pn(\mathbb{Z}_n)$ as mentioned earlier. Hence in the following statement we shall assume that there is at least one i such that $\beta_i \neq 0$.

Theorem 5.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ and $k = p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}$ in which $\beta_i = 0$ for some $1 \leq i \leq t$, and also there is at least one $0 \leq i \leq t$ such that $\beta_i \neq 0$. If $H = \{p_j : \beta_j = 0 \text{ for some } 1 \leq j \leq t\}$, then $pn(k\mathbb{Z}_{kn}) = \frac{2^s}{\prod_{p_j \in H} p_j}$, where $s = |H|$.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$ and $k = p_1^{\beta_1} p_2^{\beta_2} \cdots p_t^{\beta_t}$ such that $\beta_i \neq 0$ for at least one $0 \leq i \leq t$. Let $\gamma_i = \alpha_i + \beta_i$. Then $k\mathbb{Z}_{kn} \cong k\mathbb{Z}_{p_1^{\gamma_1}} \times k\mathbb{Z}_{p_2^{\gamma_2}} \times \cdots \times k\mathbb{Z}_{p_t^{\gamma_t}}$. For all $p_j \in H$, we have $(k, p_j) = 1$ and so, it is easy to verify that $k\mathbb{Z}_{p_j^{\gamma_j}} \cong \mathbb{Z}_{p_j^{\gamma_j}}$, where $p_j \in H$. So, we conclude from Lemma 3.1 that $pn(k\mathbb{Z}_{p_j^{\gamma_j}}) = \frac{2}{p_j}$, for all $p_j \in H$. On the other hand, we can easily obtain that $k\mathbb{Z}_{p_i^{\gamma_i}} = Nil(k\mathbb{Z}_{p_j^{\gamma_j}})$, and hence $pn(k\mathbb{Z}_{p_j^{\gamma_j}}) = 1$ for all $p_i \notin H$. Thus

$$pn(k\mathbb{Z}_{kn}) = \prod_{p_j \in H} \frac{2}{p_j} = \frac{2^s}{\prod_{p_j \in H} p_j}. \qquad \square$$

By application of Theorem 3.2(1), Corollary 5.2 and Theorem 5.4, we could build several examples of finite cyclic rings that are necessarily nil-clean.

We finish our work with the following intriguing question.

Problem 5.5. Define and calculate the probability of when a finite commutative ring is weakly nil-clean as stated in [5].

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