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ON NUMERICAL SOLVING THE DIRICHLET GENERALIZED HARMONIC PROBLEM FOR REGULAR n-SIDED PYRAMIDAL DOMAINS BY THE PROBABILISTIC METHOD

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Abstract. In this paper, we describe how the probabilistic method (PM) can be applied to numerical solving the Dirichlet generalized harmonic problem in regular n-sided pyramidal domains. The term "generalized" indicates that a boundary function has a finite number of first kind discontinuity curves. In the considered case, the edges of the pyramid represent the curves. Application of the PM consists of the following main stages: a) computer modelling of the Wiener process; b) construction of an algorithm for finding the point of intersection of the trajectory of the simulated Wiener process and the surface of the pyramid; c) checking a scheme and a corresponding calculating program needed for numerical implementation and reliability of the obtained results; d) finding a probabilistic solution of generalized problems at any fixed points of the considered pyramid. The algorithm does not require approximation of a boundary function, which is main of its important properties. For illustration of the effectiveness and simplicity of the suggested method several numerical examples are considered and numerical results are presented.

1. Introduction

In the present paper, the PM for numerical solution of the Dirichlet harmonic problem in regular n-sided pyramidal domains with discontinuities in the boundary data is considered. It is known (see e.g., [1, 2, 6, 12, 15]) that in practical stationary problems (for example, for determination of the temperature of the thermal field or the potential of the electric field, and so on) there are cases in which the Dirichlet generalized harmonic problem requires consideration.

In general, it is known (see e.g., [6,7,18]) that the methods used to obtain an approximate solution to ordinary boundary problems are less suitable (or not suitable at all) for solving generalized boundary problems. In particular, the convergence of the approximate process is very slow in the neighborhood of boundary singularities and, consequently, the accuracy of the approximate solution of the generalized problem is very low.

The choice and construction of computational schemes (algorithms) depend mainly on a problem class, its dimension, geometry and location of singularities on the boundary, e.g., the Dirichlet generalized plane harmonic problems with concrete location of discontinuity points in the cases of simply connected domains are considered in [1, 2, 6, 8, 12, 15], and general cases for finite and infinite domains are studied in [9–11, 13, 14, 16].

In the case of 3D Dirichlet generalized harmonic problems, the difficulties become more significant. In particular, there does not exist a standard scheme suitable for a wide class of domains. In literature, the simplified, or the so-called "solvable" generalized problems (those whose "exact" solutions can be constructed by series with terms represented by special functions) are considered and some methods, such as separation of variables, particular solutions and heuristic method are applied for their solving, but the accuracy of the solutions is rather low. In the above-mentioned problems, the boundary conditions are mainly the constants, and in a general case, the analytic form of the "exact" solution is so complicated in the sense of numerical implementation, that it is of theoretical significance only (see,

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e.g., [1, 2, 6, 12, 15]). Therefore the construction of high accuracy and effectively realizable computational schemes for approximate solution of 3D Dirichlet generalized harmonic problems (applicable to a wide class of domains) have both theoretical and practical importance.

It should be noted that in literature (see e.g., [1,2,6,12,15]), while solving 3D Dirichlet generalized harmonic problems, the existence of discontinuity curves is often ignored. This fact and application of classical methods to solving generalized problems are the reasons of inaccuracies. Therefore, for numerical solution of 3D Dirichlet generalized harmonic problems we apply such methods which do not require approximation of a boundary function and in which the existence of discontinuity curves is not ignored. The suggested algorithm is one such method.

The paper is organized as follows. The mathematical formulation of 3D Dirichlet generalized harmonic problem is given in Section 2. In Section 3 the PM and simulation of the Wiener process are briefly described. The algorithm for determination of a point of intersection of the trajectory of the simulated Wiener process and the surface of the pyramid is given in Section 4. Results of numerical examples are presented in Section 5. Finally, in Section 6, some conclusions and ideas for further investigations are provided.

2. Mathematical Formulation of the Problem

Let D be a regular n-sided pyramid $P_n(H,R) \equiv P_n$ in the space R^3, where H is its height and R is a radius of a circumscribed circle on the base of the pyramid. Without loss of generality, we assume that the axis of P_n lies on Ox_3 of the Cartesian coordinate right-hand system $Ox_1x_2x_3$ and the base of P_n lies in the plane Ox_1x_2, and the axis Ox_1 passes through the vertex of P_n.

We consider the Dirichlet generalized harmonic problem for the Laplace equation.

Problem A. The function $g(y)$ given on the boundary S of the pyramid P_n is continuous everywhere, except the edges $l_1, l_2, \dots, l_{2n}$, of P_n, which represent the first kind discontinuity curves of the function $g(y)$. It is required to find a function $u(x) \equiv u(x_1,x_2,x_3) \in C^2(D) \bigcap C(\overline{D} \setminus \bigcup_{k=1}^{2n} l_k)$ satisfying the conditions

$$\Delta u(x) = 0, \quad x \in D, \tag{2.1}$$

$$u(y) = g(y), \quad y \in S, \quad y \overline{\in} l_k \subset S \ \ (k = \overline{1,2n}), \tag{2.2}$$

$$|u(y)| < c, \qquad y \in \overline{D}, \tag{2.3}$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and c is a real constant.

It is shown (see [5,18]) that Problem (2.1), (2.2), (2.3) has a unique solution depending continuously on the data, and for a generalized solution $u(x)$ the generalized extremum principal

$$\min_{x\in S} u(x) < \underset{x\in D}{u(x)} < \max_{x\in S} u(x) \tag{2.4}$$

is valid, where it is assumed that $x \overline{\in} l_k$ $(k = \overline{1,2n})$ for $x \in S$. Note (see [18]) that the additional requirement (2.3) of boundedness concerns actually only the neighborhoods of discontinuity curves of the function $g(y)$ which plays an important role in the extremum principle (2.4).

On the basis of (2.3), the values of $u(y)$ are, in general, not defined on the curves l_k. For example, if Problem A concerns the determination of the thermal (or the electric) field, then $u(y) = 0$ when $y \in l_k$, respectively. In this case, in physical sense, the curves l_k are non-conductors (or dielectrics).

It is evident that actually in the above-mentioned case the boundary function $g(y)$ has the following form

$$g(y) = \begin{cases} g_1(y), \quad y \in S_1, \\ g_2(y), \quad y \in S_2, \\ \dots\dots\dots\dots\dots \\ g_n(y), \quad y \in S_n, \\ g_{n+1}(y), \quad y \in S_{n+1}, \\ 0, \quad y \in l_k \ (k = \overline{1, 2n}), \end{cases} \tag{2.5}$$

where S_i $(i = \overline{1,n})$ and S_{n+1} are the lateral faces and the base of P_n without discontinuity curves(edges), respectively; the functions $g_i(y)$, $y \in S_i$ $(i = \overline{1, n+1})$ are continuous on the parts S_i of S. It is evident that $S = (\bigcup\limits_{i=1}^{n+1} S_i) \bigcup (\bigcup\limits_{k=1}^{2n} l_k)$.

Remark 1. a) If there is an emptiness inside the surface S, then we have the generalized problem with respect to closed pyramidal shells; b) In Problem A, it is not necessary for all edges of the pyramid to be dielectric, moreover, we can consider the cases where faces, apothems, base diagonals, etc. are taken as dielectrics.

3. The Probabilistic Method and Simulation of the Wiener Process

In this section, the essence of the suggested algorithm for numerical solving the problem of type A is given, and its detailed description can be find in [19]. The main theorem in realization of the PM is the following one (see e.g., [5])

Theorem 1. *If a finite domain $D \subset R^3$ is bounded by a piecewise smooth surface S and $g(y)$ is a continuous* (*or discontinuous*) *bounded function on S, then the solution of the Dirichlet ordinary* (*or generalized*) *boundary problem for the Laplace equation at the fixed point $x \in D$ has the form*

$$u(x) = E_x g(x(\tau)). \tag{3.1}$$

In (3.1), $E_x g(x(\tau))$ is the mathematical expectation of values of the boundary function $g(y)$ at random points of intersection of the trajectory of the Wiener process and the boundary S; τ is the random moment of the first exit of the Wiener process $x(t) = (x_1(t), x_2(t), x_3(t))$ from the domain D. It is assumed that the starting point of the Wiener process is always $x(t_0) = (x_1(t_0), x_2(t_0), x_3(t_0)) \in D$, where the value of the desired function is being determined. If number N of the random intersection points $y^j = (y_1^j, y_2^j, y_3^j) \in S$ $(j = \overline{1,N})$ is sufficiently large, then according to the law of large numbers, from (3.1), we have

$$u(x) \approx u_N(x) = \frac{1}{N} \sum_{j=1}^{N} g(y^j) \tag{3.2}$$

or $u(x) = \lim u_N(x)$ for $N \to \infty$, in probability. Thus if we have the Wiener process, the approximate value of the probabilistic solution to Problem A at a point $x \in D$ is calculated by formula (3.2).

In order to simulate the Wiener process, we use the following recursion relations(see, e.g., [18,19]):

$$\begin{gathered} x_1(t_k) = x_1(t_{k-1}) + \gamma_1(t_k)/nq, \\ x_2(t_k) = x_2(t_{k-1}) + \gamma_2(t_k)/nq, \\ x_3(t_k) = x_3(t_{k-1}) + \gamma_3(t_k)/nq, \\ (k = 1, 2, \dots), \quad x(t_0) = x, \end{gathered} \tag{3.3}$$

according to which the coordinates of the point $x(t_k) = (x_1(t_k), x_2(t_k), x_3(t_k))$ are being determined. In (3.3), $\gamma_1(t_k), \gamma_2(t_k), \gamma_3(t_k)$ are three normally distributed independent random numbers for the k-th step, with zero means and variances; nq is a number of quantifications such that $1/nq = \sqrt{t_k - t_{k-1}}$, and if $nq \to \infty$, then the discrete process approaches the continuous Wiener process. In the implementation, the random process is simulated at each step of the walk and continues until it crosses the boundary.

In the case under consideration, computations and generation of random numbers are performed in MATLAB.

4. Algorithm for Finding the Point of Intersection of the Trajectory of the Simulated Wiener Process and the Surface S of the Pyramid P_n

In order to determine the points $y^j = (y_1^j, y_2^j, y_3^j)$ $(j = \overline{1,N})$ of interaction of a trajectory of the Wiener process and the surface S (see Section 3), we operate in the following way. First of all, during the implementation of the Wiener process, for each current point $x(t_k)$ defined from (3.3), we have to check whether it belongs to $\overline{P_n}$ $(\overline{P_n} = P_n \bigcup S)$ or not.

For determination of the above-formulated question, we know only three parameters H, R, n of P_n. In addition, with the help of H, R, n, we must find: 1) the angle α of inclination of the lateral face with respect to the plane of the base of P_n; 2) equations of lines passing through the neighboring vertices of the base; 3) equations of lateral edges; 4) equations of lateral faces.

Without loss of generality, we assume that the vertices $A_1, A_2, \dots, A_n$ of the base of $P_n = MA_1 A_2 \dots A_n$ are located on the circle $C(O;R)$ in a counter-clockwise direction, so, $A_1 \in Ox_1$. In this case, for the angle α and the points A_m, A_{m+1} $(m = \overline{1,n},\ A_{n+1} \equiv A_1)$,we have

$$\begin{gathered}\alpha = \arctan\Big(\frac{H}{R\cos\frac{\pi}{n}}\Big),\\ A_m = \Big(R\cos\frac{2\pi(m-1)}{n}, R\sin\frac{2\pi(m-1)}{n}\Big),\\ A_{m+1} = \Big(R\cos\frac{2\pi m}{n}, R\sin\frac{2\pi m}{n}\Big).\end{gathered} \tag{4.1}$$

If we seek for equations of the lines passing through the points A_m and A_{m+1} $(m = \overline{1,n})$ in the form

$$x_2 = d_m x_1 + c_m, \tag{4.2}$$

then to define the constants d_m and c_m on the basis of (4.1), we have the following algebraic system:

$$\begin{cases} R\cos\frac{2\pi(m-1)}{n} d_m + c_m = R\sin\frac{2\pi(m-1)}{n}\\ R\cos\frac{2\pi m}{n} d_m + c_m = R\sin\frac{2\pi m}{n}.\end{cases} \tag{4.3}$$

It is easy to see that from (4.3), we get

$$d_m = -\cot\frac{\pi(2m-1)}{n}, \quad c_m = R\cos\frac{\pi}{n}/\sin\frac{\pi(2m-1)}{n}. \tag{4.4}$$

Since lateral edges of P_n are generatrices of a finite right circular cone whose height is H, and the base radius is R, therefore the sought equations are the same and they have the following form:

$$(x_1)^2 + (x_2)^2 - \Big(\frac{R}{H}\Big)^2 (H - x_3)^2 = 0, \ \ 0 \le x_3 \le H, \tag{4.5}$$

where $(x_1, x_2, x_3) \in S_c$ (S_c is a lateral surface of the cone).

It is easy to see that the equation of the plane passing through the points

$$\begin{gathered}M(0,0,H),\ A_m = \left(R\cos\frac{2\pi(m-1)}{n}, R\sin\frac{2\pi(m-1)}{n}, 0\right),\\ A_{m+1} = \Big(R\cos\frac{2\pi m}{n}, R\sin\frac{2\pi m}{n}, 0\Big),\end{gathered}$$

where $m = \overline{1,n}$ has the form (in our notations)

$$\begin{gathered}\Big(2H\cos\frac{\pi(2m-1)}{n}\sin\frac{\pi}{n}\Big)x_1 + \Big(2H\sin\frac{\pi(2m-1)}{n}\sin\frac{\pi}{n}\Big)x_2\\ +\Big(R\sin\frac{2\pi}{n}\Big)x_3 - RH\sin\frac{2\pi}{n} = 0.\end{gathered} \tag{4.6}$$

In particular, this is the equation of the m-th lateral face of P_n for $(x_1, x_2, x_3) \in \overline{S}_m$.

Now we have the needed information about the pyramid P_n in order to establish whether each current point $x(t_k)$ defined from (3.3) belongs to $\overline{P_n}$ or not. To this end, we operate in the following way. For each step of the simulated Wiener process we calculate the angles $\beta_m(m=\overline{1,n})$, which are the angles of inclination of the planes passing through $x(t_k), A_m, A_{m+1}$ points, with respect to the plane of the base of P_n. It is easy to see that

$$\beta_m = \arctan(x_3(t_k)/\Delta_m),$$

where Δ_m is a distance between the point $(x_1(t_k), x_2(t_k))$ and the line $A_m A_{m+1}$. It is known that

$$\Delta_m = \frac{|d_m x_1(t_k) - x_2(t_k) + c_m|}{\sqrt{((d_m)^2+1)}},$$
$$(m=\overline{1,n},\ k=1,2,\dots),$$

where d_m and c_m are given by (4.4). After calculating angles β_m, we compare them with the angle α. In particular: 1*) if $\beta_m < \alpha$ and $0 < x_3(t_k) < H$ for $(m=\overline{1,n})$, then the process continues until it crosses the boundary S; 2*) if for $m=p$, $\beta_p = \alpha$ and $0 < x_3(t_k) < H$, then $x(t_k) \in \overline{S_p}$ or $y^j = (y_1^j, y_2^j, y_3^j) = x(t_k)$; 3*) if $\beta_p > \alpha$ and $0 < x_3(t_k) < H$, this means that the trajectory of the modulated Wiener process intersect the p-th lateral face of P_n or $x(t_{k-1}) \in P_n$ for the moment $t = t_{k-1}$ and $x(t_k)\ \overline{\in}\ \overline{P_n}$ for the moment $t = t_k$. In this case, for approximate determination of the point y^j, a parametric equation of a line L passing through the points $x(t_{k-1})$ and $x(t_k)$ is firstly obtained, which has the form

$$\begin{cases} x_1 = x_1^{k-1} + (x_1^k - x_1^{k-1})\theta, \\ x_2 = x_2^{k-1} + (x_2^k - x_2^{k-1})\theta, \\ x_3 = x_3^{k-1} + (x_3^k - x_3^{k-1})\theta, \end{cases} \tag{4.7}$$

where (x_1, x_2, x_3) is the current point of L and θ is a parameter $(-\infty < \theta < \infty)$, and $x_i^{k-1} \equiv x_i(t_{k-1})$, $x_i^k \equiv x_i(t_k)$ $(i=1,2,3)$.

If we substitute the expressions of x_1, x_2, x_3 defined from (4.7) in (4.6), then we obtain an equation with respect to θ, in the form

$$A^*\theta = B^*. \tag{4.8}$$

In (4.8)

$$A^* = 2H\sin\Big(\frac{\pi}{n}\Big)\Big((x_1^k - x_1^{k-1})\cos\frac{\pi(2p-1)}{n} + (x_2^k - x_2^{k-1})\sin\frac{\pi(2p-1)}{n}\Big)$$
$$+R\sin\Big(\frac{2\pi}{n}\Big)(x_3^k - x_3^{k-1});$$
$$B^* = -2H\sin\Big(\frac{\pi}{n}\Big)\Big(x_1^{k-1}\cos\frac{\pi(2p-1)}{n} + x_2^{k-1}\sin\frac{\pi(2p-1)}{n}\Big)$$
$$+R\sin\Big(\frac{2\pi}{n}\Big)(H - x_3^{k-1}),\quad (p=\overline{1,n}).$$

In the considered case, due to the existence of the intersection point, $A^* \neq 0$ and $y^j = (x_1(\theta), x_2(\theta), x_3(\theta))$, where $\theta = B^*/A^*$.

Finally, consider the following cases: 4*) if $x_3(t_k) = 0$, then $y^j = (x_1(t_k), x_2(t_k), 0)$; 5*) if $x_3(t_k) < 0$, in this case, we find the point $y(y_1, y_2, 0)$ of intersection of the plane $x_3 = 0$ and a line L passing through the points $x(t_{k-1})$ and $x(t_k)$. Then if $(y_1, y_2, 0) \in \overline{S_{n+1}}$, we have $y^j = (y_1, y_2, 0)$.

Remark 2. In addition, during numerical implementation, it is checked whether the intersection point y^j is situated on the lateral edge or not in cases 2*) and 3*), and whether it lies or not on the base edge in cases 4*) and 5*), with the help of their equations (4.5) and (4.2), respectively.

5. Numerical Examples

It should be noted that test solutions for generalized problems of type A do not exist in a three-dimensional case, therefore for verification of a scheme needed for numerical solution of Problem A, the validity of the obtained results can be demonstrated in the following way.

If in boundary conditions (2.5) of Problem A we take $g_i(y) = 1/|y-x^0|$, where $y \in S_i$ $(i = \overline{1, n+1})$, $x^0 = (x_1^0, x_2^0, x_3^0) \,\overline{\in}\, \overline{D}$, and $|y-x^0|$ denotes the distance between the points y and x^0, then it is evident that the curves $l_k(k = \overline{1, 2n})$ represent removable discontinuity curves for the boundary function $g(y)$. Actually, in the above-mentioned case,instead of generalized problem A we obtain the following Dirichlet ordinary harmonic problem.

Problem B. Find a function $u(x) \equiv u(x_1, x_2, x_3) \in C^2(D) \bigcap C(\overline{D})$ satisfying the conditions

$$\Delta u(x) = 0, \quad x \in D,$$
$$u(y) = 1/|y - x^0|, \quad y \in S, \quad x^0 \;\overline{\in}\; \overline{D}.$$

We solve this question (by the PM) with application of calculating program to Problem A. It is well-known that Problem B is correct, i.e., its solution exists, is unique and depends continuously on the data. Evidently, an exact solution of Problem B is

$$u(x^0, x) = \frac{1}{|x - x^0|}, \quad x \in \overline{D}, \; x^0 \overline{\in}\, \overline{D}. \tag{5.1}$$

It should be noted that numerical solution of the Dirichlet ordinary harmonic problems by the PM is interesting and important (see e.g., [3, 4, 20]). In this paper, Problem B plays an auxiliary role. In particular, for Problem B, we first of all verify the validity both of the scheme needed for numerical solution of Problem A and of the corresponding calculating program (comparison of the obtained results with the exact solution) and then solve Problem A under the boundary conditions (2.5).

In the present paper, the PM is applied to three examples. In tables, N is a number of implementations of the Wiener process for the given points $x^i = (x_1^i, x_2^i, x_3^i) \in D$, and nq is the number of the quantifications. For simplicity, in the given examples, the values of nq and N are the same. In tables for problems of type B, we present the absolute errors Δ^i at the points $x^i \in D$ of $u_N(x)$, in the PM approximation, for $nq = 200$ and various values of N, and under notations of type $(E \pm k)$ we mean $10^{\pm k}$. In particular, $\Delta^i = |u_N(x^i) - u(x^0, x^i)|$, where $u_N(x^i)$ is the approximate solution of Problem B at the point x^i, which is defined by formula (3.2), and the exact solution $u(x^0, x^i)$ of the test problem is given by (5.1). In tables, for problems of type A, the probabilistic solution $u_N(x)$ is presented at the points x^i, defined by (3.2).

Remark 3. Problems A and B for ellipsoidal, spherical, cylindrical, conic and prismatic domains are considered in [17, 19].

Example 5.1. *In the first example, the domain D is the interior of the regular 4-sided pyramid $P_4(H, R)$, where H is its height, R is a radius of circumscribed circle on the base of P_4.*

It should be noted that all pyramids in the given examples have the same position in the system $Ox_1x_2x_3$ as described in Section 2.

Problems B and A are solved when $H = 2, R = 1, x^0 = (0, 0, -4)$, and in Problem A, the boundary function $g(y) \equiv g(y_1, y_2, y_3)$ has the form

$$g(y) = \begin{cases} 1, & y \in S_1, \\ 0, & y \in S_2, \\ 1, & y \in S_3, \\ 0, & y \in S_4, \\ 2, & y \in S_5, \\ 0, & y \in l_k \;\; (k = \overline{1, 8}). \end{cases} \tag{5.2}$$

In (5.2), $S_i(i = \overline{1, 4})$ and S_5 are the lateral faces and the base of P_4 without discontinuity curves, respectively; $l_k(k = \overline{1, 8})$ are the edges of P_4. It is evident that in the case under consideration, l_k, S_2, S_4 are non-conductors(or dielectrics) in a physical sense.

In all the considered by us examples, for determination of the points $y^i = (y_1^i, y_2^i, y_3^i)$ $(i = \overline{1, N})$ of intersection of the trajectory of the Wiener process and the surface S we use that scheme which is

TABLE 5.1B. Results for Problem B (in Example 5.1).

x^i	$(0,0,0.5)$	$(0,0,1)$	$(0,0,1.8)$	$(b,b,0.5)$	$(-b,-b,0.5)$
N	Δ^1	Δ^2	Δ^3	Δ^4	Δ^5
$5E+3$	$0.36E-3$	$0.22E-3$	$0.18E-4$	$0.40E-4$	$0.15E-3$
$1E+4$	$0.87E-4$	$0.38E-3$	$0.45E-4$	$0.93E-4$	$0.21E-4$
$5E+4$	$0.86E-4$	$0.87E-4$	$0.16E-4$	$0.24E-4$	$0.22E-4$
$1E+5$	$0.26E-4$	$0.51E-4$	$0.28E-4$	$0.39E-4$	$0.38E-4$
$5E+5$	$0.16E-4$	$0.39E-4$	$0.27E-4$	$0.26E-5$	$0.17E-4$
$1E+6$	$0.15E-4$	$0.50E-4$	$0.23E-4$	$0.22E-4$	$0.10E-4$

TABLE 5.1A. Results for Problem A (in Example 5.1).

x^i	$(0,0,0.5)$	$(0,0,1)$	$(0,0,1.8)$	$(b,b,0.5)$	$(-b,-b,0.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	0.86220	0.54020	0.50120	0.91920	0.92620
$1E+4$	0.85200	0.53760	0.50280	0.91460	0.92040
$5E+4$	0.85134	0.52926	0.49676	0.91752	0.91440
$1E+5$	0.84312	0.53110	0.49899	0.91724	0.91911
$5E+5$	0.84381	0.53133	0.49853	0.91561	0.91512
$1E+6$	0.84407	0.53129	0.49875	0.91646	0.91642

TABLE 5.2A. Results for Problem A (in Example 5.2).

x^i	$(0,0,0.5)$	$(0,0,1)$	$(0,0,1.8)$	$(0.5,0,0.5)$	$(-0.5,0,0.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.57880	1.07280	1.01880	1.26360	1.29660
$1E+4$	1.55200	1.07150	1.00520	1.30160	1.30190
$5E+4$	1.58008	1.07196	1.01616	1.29730	1.29350
$1E+5$	1.58114	1.07701	1.01746	1.29783	1.29142
$5E+5$	1.57953	1.07819	1.01594	1.29316	1.28986
$1E+6$	1.57866	1.07821	1.01615	1.28559	1.28673

described in Section 4. As noted above, for verification, we first of all solve the auxiliary Problem B with calculating program for Problem A.

In Table 5.1B, the absolute errors Δ^i of the approximate solution $u_N(x)$ of the test problem B at the points $x^i \in D$ $(i = \overline{1,5})$ are presented, for $b = 0.2$.

On the basis of the results presented in Table 5.1B, we can conclude that the calculating program for Problem A is correct.

In Section 3, it is noted that when $nq \to \infty$, the discrete process approaches the continuous Wiener process, respectively, the accuracy of probabilistic solution is increasing. We conducted a check experiment. Namely, we calculated the probabilistic solution of Problem B at the point (0,0,0.5) for $N = 1E+5$, $nq = 400$ and obtained $\Delta^1 = 0.80E-6$ (see, Table 5.1B). This result is in agreement with the above-noted. In general, if we need more accuracy, then calculations should be realized for sufficiently large values of nq and N. In this case, numerical implementation on a PC takes much time. We can avoid this difficulty by applying the method of parallel calculation. For this reason, a suitable computing technique is needed. Respectively, significantly less time will be needed for numerical implementation and, besides, the accuracy of the obtained results will be improved.

In Table 5.1A, the values of the approximate solution $u_N(x)$ to Problem A at the same points x^i $(i = \overline{1,5})$ are given, for $b = 0.2$. The boundary function (5.2) is symmetric with respect to the axis

Ox_3, respectively, for control in the role of x^i $(i = 4, 5)$, the points, symmetric with respect to the axis Ox_3 are taken. The results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture.

As it has been noted above, the calculating program for Problem A is correct, therefore in following examples only the values of the approximate solution $u_N(x)$ of Problem A are presented.

Example 5.2. *In this example the domain D is the interior of the regular 6-sided pyramid $P_6(H, R)$, where H and R have the same contents as in Example 5.1. On the above- mentioned basis, we solved Problem A directly when $H = 2$, $R = 1$, and the boundary function $g(y)$ has the form*

$$g(y) = \begin{cases} 2, & y \in S_1, \\ 1, & y \in S_2, \\ 2, & y \in S_3, \\ 0, & y \in S_4, \\ 1, & y \in S_5, \\ 0, & y \in S_6, \\ 3, & y \in S_7, \\ 0, & y \in l_k \;\; (k = \overline{1, 12}). \end{cases} \tag{5.3}$$

In (5.3), $S_i (i = \overline{1, 6})$ and S_7 are the lateral faces and the base of P_6 without discontinuity curves, respectively; $l_k (k = \overline{1, 12})$ are the edges of P_6. Besides, in this case, the edges l_k and S_4, S_6 are non-conductors.

The values of the approximate solution $u_N(x)$ of Problem A at the points $x^i \in D$ $(i = \overline{1, 5})$ are given in Table 5.2A. Since the boundary function (5.3) is symmetric with respect to the plane Ox_2x_3, therefore for a control in the role of x^i $(i = 4, 5)$, the points, symmetric with respect to the plane Ox_2x_3 are taken. The obtained results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture.

Example 5.3. *In the role of the domain D, the interior of a truncated regular 5-sided pyramid $TP_5(H, R, r)$ is taken, where H and R have the same contents as in the above-considered examples, and r is a radius of a circumscribed circle on the upper base of TP_5.*

We solved Problem A for TP_5, when $H = 1, R = 1, r = 0.5$ and the boundary function $g(y)$ has the form

$$g(y) = \begin{cases} 1, & y \in S_1, \\ 0, & y \in S_2, \\ 1.5, & y \in S_3, \\ 0, & y \in S_4, \\ 1, & y \in S_5, \\ 2, & y \in S_6, \\ 0.5, & y \in S_7, \\ 0, & y \in l_k \;\; (k = \overline{2, 15}), \end{cases} \tag{5.4}$$

In (5.4), S_i $(i = \overline{1, 5})$ and S_6, S_7, are the lateral faces and lower and upper bases of TP_5 without discontinuity curves, respectively; l_k $(k = \overline{2, 15})$ are the edges of TP_5; the edges l_k and S_2, S_4 are non-conductors, and the lateral edge l_1 corresponding to vertex A_1 (see Section 4) is a conductor.

The values of the approximate solution $u_N(x)$ of Problem A at the points $x^i \in D$ $(i = \overline{1, 3})$ are given in Table 5.3A. Since the boundary function (5.4) is symmetric with respect to the plane Ox_1x_3, therefore for a control in the role of x^i $(i = 4, 5)$, the points which are symmetric with respect to the plane Ox_1x_3 are taken. The obtained results have sufficient accuracy for many practical problems and are in good agreement with the real physical picture(see Table 5.3A).

In this work, we specially solved the problems of type A when the boundary functions $g_i(y)$ $(i = \overline{1, n+1})$ are the constants. This was caused by our interest in finding out how much the obtained

Table 5.3A. Results for Problem A (in Example 5.3).

x^i	$(0,0,0.2)$	$(0,0,0.5)$	$(0,0,0.8)$	$(0,-0.2,0.5)$	$(0,0.2,0.5)$
N	$u_N(x^1)$	$u_N(x^2)$	$u_N(x^3)$	$u_N(x^4)$	$u_N(x^5)$
$5E+3$	1.54320	1.00070	0.69310	0.91720	0.93600
$1E+4$	1.54205	1.01510	0.69425	0.92120	0.94400
$5E+4$	1.53879	1.01561	0.69385	0.93640	0.92748
$1E+5$	1.53636	1.01632	0.69292	0.92929	0.92925
$5E+5$	1.53647	1.01633	0.69197	0.93094	0.92778
$1E+6$	1.53567	1.01592	0.69276	0.92922	0.92852

results are in agreement with a real physical picture. It is evident that solution of Problem A under condition (2.5) is as easy as that of Problem B. In general, we can solve Problem A for all such locations of discontinuity curves, which make it possible to establish a part of surface S where the intersection point is located.

The analysis of the results of numerical experiments shows that the results obtained by the suggested algorithm are reliable and effective for numerical solution of problems of type A and B. In particular, the algorithm is sufficiently simple for numerical implementation.

6. Concluding Remarks

1. In this work, we have demonstrated that the probabilistic method (PM) is ideally suited for numerical solving of the Dirichlet generalized harmonic problem for regular n-sided pyramidal domains.

2. The PM does not require an approximation of a boundary function, which is one of its important properties.

3. It is easy to program, its computational cost is low, it is characterized by an accuracy which is sufficient for many practical problems.

4. In the future, we plan to investigate the following:

∗ Application of the proposed method to numerical solution of Dirichlet generalized harmonic problem for some axisymmetric domains with cylindrical hole.

∗ Application of the PM to the same type problem in domains which are bounded by several closed surfaces.

∗ Application of the PM to the same problem in infinite 3D domains.

References

1. B. M. Budak, A. A. Samarski, A. N. Tikhonov, *A Collection of Problems in Mathematical Physics.* (Russian) Third edition. Nauka, Moscow, 1980.
2. H. S. Carslaw, J. C. Jaeger, *Conduction of Heat in Solids.* Oxford University Press, London, 1959.
3. A. Chaduneli, Z. Tabagari, M. Zakradze, On solving the internal three-dimensional Dirichlet problem for a harmonic function by the method of probabilistic solution. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **2** (2008), no. 1, 25–28.
4. A. Sh. Chaduneli, M. V. Zakradze, Z. A. Tabagari, A method of probabilistic solution to the ordinary and generalized plane Dirichlet problem for the Laplace equation. Science and Computing. In: *Proc. Sixth TSTC Scientific Advisory Committee Seminar*, vol. 2, pp. 361–366, Moscow, 2003.
5. E. B. Duenkin, A. A. Yushkevich, *Theorems and Problems on Markov's Processes.* (Russian) Nauka, Moscow, 1967.
6. G. A. Grinberg, *Selected Problems of the Mathematical Theory of Electric and Magnetic Phenomena.* (Russian) Izd. Akad. Nauk SSSR, 1948.
7. L. V. Kantorovich̆, V. I. Krylov, *Approximate Methods of Higher Analysis.* (Russian) Fifth corrected edition Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow-Leningrad 1962.
8. A. Karageorghis, Modified methods of fundamental solutions for harmonic and biharmonic problems with boundary singularities. *Numer. Methods Partial Differential Equations* **8** (1992), no. 1, 1–19.
9. N. Koblishvili, M. Kublashvili, Z. Sanikidze, M. Zakradze, On solving the Dirichlet generalized problem for a harmonic function in the case of an infinite plane with a crack-type cut. *Proc. A. Razmadze Math. Inst.* **168** (2015), 53–62.
10. N. Koblishvili, Z. Tabagari, M. Zakradze, On reduction of the Dirichlet generalized boundary value problem to an ordinary problem for harmonic function. *Proc. A. Razmadze Math. Inst.* **132** (2003), 93–106.

11. N. Koblishvili, M. Zakradze, On solving the Dirichlet generalized problem for a harmonic function in the case of infinite plane with holes. *Proc. A. Razmadze Math. Inst.* **164** (2014), 71–82.
12. N. S. Koshlyakov, E. B. Gliner, M. M. Smirnov, *Equations in Partial Derivatives of Mathematical Physics.* (Russian) Vysshaya Shkola, Moscow, 1970.
13. M. Kublashvili, Z. Sanikidze and M. Zakradze, A method of conformal mapping for solving the generalized Dirichlet problem of Laplace's equation. *Proc. A. Razmadze Math. Inst.* **160** (2012), 71–89.
14. M. A. Lavrent'jev, B. V. Shabat, *Methods of the Theory of Functions of a Complex Variable.* (Russian) Nauka, Moscow, 1973.
15. W. R. Smythe, *Static and Dynamic Electricity.* second edition. New York, Toronto, London, 1950.
16. M. Zakradze, N. Koblishvili, A. Karageorghis and Y. Smyrlis, On solving the Dirichlet generalized problem for harmonic function by the method of fundamental solutions. *Semin. I. Vekua Inst. Appl. Math. Rep.* **34** (2008), 24–32, 124.
17. M. Zakradze, M. Kublashvili, N. Koblishvili, A. Chakhvadze, The method of probabilistic solution for determination of electric and thermal stationary fields in conic and prismatic domains. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 2, 235–246.
18. M. Zakradze, M. Kublashvili, Z. Sanikidze, N. Koblishvili, Investigation and numerical solution of some 3D internal Dirichlet generalized harmonic problems in finite domains. *Trans. A. Razmadze Math. Inst.* **171** (2017), no. 1, 103–110.
19. M. Zakradze, B. Mamporia M. Kublashvili, N. Koblishvili, The method of probabilistic solution for 3D Dirichlet ordinary and generalized harmonic problems in finite domains bounded with one surface. *Trans. A. Razmadze Math. Inst.* **172** (2018), no. 3, part A, 453–465.
20. M. Zakradze, Z. Sanikidze, Z. Tabagari, On solving the external three-dimensional Dirichlet problem for a harmonic function by the probabilistic method. *Bull. Georgian Natl. Acad. Sci. (N.S.)* **4** (2010), no. 3, 19–23.

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