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BOUNDARY-TRANSMISSION PROBLEMS OF THE THERMO-PIEZO-ELECTRICITY THEORY WITHOUT ENERGY DISSIPATION

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Abstract. In the paper, we study Dirichlet, Neumann and mixed type interaction problems of pseudo-oscillations between thermo-elastic and thermo-piezo-elastic bodies. The model under consideration is based on the Green–Haghdi theory of thermo-piezo-electricity without energy dissipation. This theory allows the thermal waves to propagate only with a finite speed. Using the potential theory and the method of boundary pseudodifferential equations, we prove the existence and uniqueness of solutions and analyze their smoothness.

1. Introduction

In this paper, we investigate the boundary-transmission problems, i.e., the Dirichlet, Neumann and mixed type interaction problems of pseudo-oscillations between thermo-elastic and thermo-piezo-elastic bodies. The model under consideration is based on the Green–Haghdi theory of thermo-piezo-electricity without energy dissipation. This theory allows the thermal waves to propagate only with a finite speed.

Other models of thermo-piezo-electricity, in particular, Foigt and Mindlin's model is well known. Our model is refined, it takes into account microrotation and microstretch of a particle.

Almost complete historical and bibliographical notes in this direction can be found in [14], where the dynamical equations of the thermo-piezo-electricity without energy dissipation are derived on the basis of the GreenNaghdi theory established in [11, 12] and Eringen's results obtained in [7, 8]. In the present paper, we consider the pseudo-oscillation equations obtained by the Laplace transform from the dynamical equations derived by Iean in [14] for homogeneous isotropic solids possessing thermo-piezo-electricity properties without energy dissipation.

Using the potential theory and the method of boundary pseudodifferential equations, we prove the existence and uniqueness theorems of solutions in appropriate function spaces. We prove regularity results of the Dirichlet and Neumann boundary-transmission problems. Further, we analyze regularity of solutions of mixed boundary-transmission problem near the exceptional curve, where different type boundary conditions collide. This regularity of solutions depends on the material constants and does not depend on the geometry of the exceptional curve. If these constants meet certain conditions, then the smoothness of solutions is $C^{\frac{1}{2}}$ (see [2–6]).

2. Thermo-elastic Field Equations and Thermo-piezo-elastic Field Equations without Energy Dissipation

The model under consideration is based on the Green–Haghdi theory of thermo-piezo-electricity without energy dissipation.

Consider disjoint bounded domains Ω_1 and Ω_2 in the Euclidean space $\mathbb{R}^3$ with sufficiently smooth boundaries $\partial\Omega_1 = S_1$ and $\partial\Omega_2 = S_1 \cup S_2$ ($S_1 \cap S_2 = \varnothing$). Throughout the paper $n = (n_1, n_2, n_3)$ stands for the exterior unit normal vector to $\partial\Omega_1 = S_1$ and vector $\nu = (\nu_1, \nu_2, \nu_3)$ is exterior unit normal vector to $\partial\Omega_2 = S_1 \cup S_2$.

Suppose the domain Ω_1 is filled with a homogeneous thermo-elastic material, then the system of governing differential equations of pseudo-oscillations with respect to the sought vector function $U^{(1)} =$

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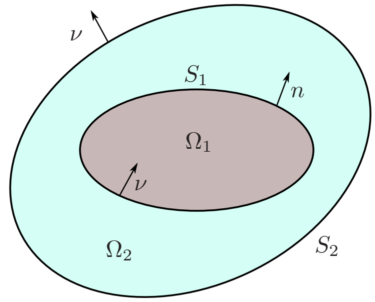

FIGURE 1. Composed body

$(u^{(1)},\vartheta^{(1)})^\top$, where $u^{(1)}=(u_1^{(1)},u_2^{(1)},u_3^{(1)})^\top$ is the displacement vector and $\vartheta^{(1)}$ is the temperature, has the following form (see [15]):

$$(\mu^{(1)}+\varkappa^{(1)})\Delta u^{(1)}+(\lambda^{(1)}+\mu^{(1)})\operatorname{grad}\operatorname{div}u^{(1)}-\tau^2\rho_1u^{(1)}-\tau\beta_0^{(1)}\operatorname{grad}\vartheta^{(1)}=(F_1^{(1)},F_2^{(1)},F_3^{(1)})^\top,\quad(2.1)$$

$$k^{(1)}\Delta\vartheta^{(1)}-\tau^2a^{(1)}\vartheta^{(1)}-\tau\beta_0^{(1)}\operatorname{div}u^{(1)}=F_4^{(1)},\quad(2.2)$$

where $(F_1^{(1)},F_2^{(1)},F_3^{(1)})^\top$ is a mass force density, $F_4^{(1)}$ is a heat source density, ρ_1 is the mass density, $\mu^{(1)}$, $\varkappa^{(1)}$, $\lambda^{(1)}$, $\beta_0^{(1)}$, $k^{(1)}$, and $a^{(1)}$ are the thermo-elastic constants satisfying the conditions

$$\varkappa^{(1)}>0,\ \ \varkappa^{(1)}+2\mu^{(1)}>0,\ \ \varkappa^{(1)}+2\mu^{(1)}+3\lambda^{(1)}>0,\ \ k^{(1)}>0,\ \ \rho_1>0,\ \ a^{(1)}>0,$$
$$\beta_0^{(1)}>0,\ \ \tau=\sigma+i\omega,\ \ \sigma>\sigma_0>0,\ \ \omega\in\mathbb{R}.$$

The stress operator for a homogeneous isotropic system of equations is defined as follows:

$$\begin{aligned}T^{(1)}&=T^{(1)}(\partial_x,n,\tau)=\big[T_{ij}^{(1)}(\partial_x,n,\tau)\big]_{4\times4}\\&:=\begin{bmatrix}[\lambda^{(1)}n_i\partial_j+\mu^{(1)}n_j\partial_i+\delta_{ij}(\mu^{(1)}+\varkappa^{(1)})n_k\partial_k]_{3\times3}, & [-\tau\beta_0^{(1)}n]_{3\times1}\\ [0]_{1\times3} & k^{(1)}n_l\partial_l\end{bmatrix}_{4\times4}.\end{aligned}$$

We can write the above system (2.1)–(2.2) of equations for pseudo-oscillations of the theory of homogeneous isotropic thermo-elasticity in the following matrix form:

$$A^{(1)}(\partial_x,\tau)U^{(1)}=F^{(1)},$$

where $U^{(1)}=(u^{(1)},\vartheta^{(1)})^\top$, $F^{(1)}=(F_1^{(1)},F_2^{(1)},F_3^{(1)},F_4^{(1)})^\top$, and $A^{(1)}(\partial_x,\tau)$ is the 4-dimensional matrix differential operator of the generalized thermo-elasticity:

$$\begin{aligned}A^{(1)}(\partial_x,\tau)&=\big[A_{ij}^{(1)}(\partial_x,\tau)\big]_{4\times4}\\&:=\begin{bmatrix}[\delta_{ij}(\lambda^{(1)}+\mu^{(1)})\Delta+(\lambda^{(1)}+\varkappa^{(1)})\partial_i\partial_j-\tau^2\rho_1\delta_{ij}]_{3\times3}, & -\tau\beta_0^{(1)}[\partial_i]_{3\times1}\\ -\tau\beta_0^{(1)}[\partial_j]_{1\times3} & -\tau a^{(1)}+k^{(1)}\Delta\end{bmatrix}_{4\times4},\end{aligned}$$

where δ_{ij} is the Kronecker delta.

The domain Ω_2 is filled with a thermo-electro-elastic material. The corresponding system of differential equations of pseudo-oscillations with respect to the sought vector function $U^{(2)}$ has the following form (see [14]):

$$\begin{aligned}&(\mu^{(2)}+\varkappa^{(2)})\partial_j\partial_ju_i^{(2)}+(\lambda^{(2)}+\mu^{(2)})\partial_i\partial_ju_j^{(2)}-\rho_2\tau^2u_i^{(2)}+\varkappa^{(2)}\varepsilon_{ijk}\partial_j\phi_k^{(2)}\\&\qquad+\lambda_0^{(2)}\partial_i\varphi^{(2)}-\tau\beta_0^{(2)}\partial_i\vartheta^{(2)}=-\rho_2f_i,\quad i=1,2,3,\end{aligned}\quad(2.3)$$

$$k^{(2)}\partial_j\partial_j\vartheta^{(2)}-\tau^2a^{(2)}\vartheta^{(2)}-\tau\beta_0^{(2)}\partial_ju_j^{(2)}-\tau c_0^{(2)}\varphi^{(2)}+\nu_1^{(2)}\partial_j\partial_j\varphi^{(2)}-\nu_3^{(2)}\partial_j\partial_j\psi^{(2)}=-\frac{1}{T_0}\rho_2Q,\quad(2.4)$$

$$\gamma^{(2)}\partial_j\partial_j\phi_i^{(2)}+(\alpha^{(2)}+\beta^{(2)})\partial_j\partial_i\phi_j^{(2)}-\tau^2I_0^{(2)}\phi_i^{(2)}+\varkappa^{(2)}\varepsilon_{ijk}\partial_ju_k^{(2)}$$

$$-2\varkappa^{(2)}\phi_i^{(2)} = -\rho_2 X_i, \quad i=1,2,3, \tag{2.5}$$

$$(a_0^{(2)}\partial_j\partial_j - \xi_0^{(2)})\varphi^{(2)} - j_0^{(2)}\tau^2\varphi^{(2)} - \lambda_2^{(2)}\partial_j\partial_j\psi^{(2)} + \nu_1^{(2)}\partial_j\partial_j\vartheta^{(2)}$$
$$+ c_0^{(2)}\tau\vartheta^{(2)} - \lambda_0^{(2)}\partial_j u_j^{(2)} = -\rho_2 F, \tag{2.6}$$

$$\lambda_0^{(2)}\partial_j\partial_j\varphi^{(2)} + \chi^{(2)}\partial_j\partial_j\psi^{(2)} + \nu_3^{(2)}\partial_j\partial_j\vartheta^{(2)} = -f, \tag{2.7}$$

where $U^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \vartheta^{(2)}, \phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top$, $u^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})^\top$ is the displacement vector, $\vartheta^{(2)}$ is the temperature, $\phi^{(2)} = (\phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)})^\top$ is the vector of microrotation, $\varphi^{(2)}$ is the microstretch, $\psi^{(2)}$ is the electric field potential, and (f_1, f_2, f_3) is the external body force per unit mass, Q is the external rate of supply of heat per unit mass, X_i is the external body couple per unit mass, F is the microstretch body force, f is the density of free charge, T_0 is the initial reference temperature, ε_{ijk} is the Levi-Civita symbol and ρ_2 is the mass density.

Due to the positiveness of internal energy, the coefficients of system (2.3)–(2.7) must satisfy the following conditions:

$$\varkappa^{(2)} > 0, \ \ \varkappa^{(2)} + 2\mu^{(2)} > 0, \ \ \varkappa^{(2)} + 2\mu^{(2)} + 3\lambda^{(2)} > 0,$$
$$\xi_0^{(2)}(\varkappa^{(2)} + 2\mu^{(2)} + 3\lambda^{(2)}) > 3(\lambda_0^{(2)})^2,$$
$$\gamma^{(2)} > |\beta^{(2)}|, \ \ a_0^{(2)}k^{(2)} - (\nu_1^{(2)})^2 > 0, \ \ \beta^{(2)} + \gamma^{(2)} + 3\alpha^{(2)} > 0,$$
$$\chi^{(2)} > 0, \ \ a^{(2)} > 0, \ \ k^{(2)} > 0, \ \ a_0^{(2)} > 0, \ \ a_0^{(2)}(\gamma^{(2)} - \beta^{(2)}) > 2(b_0^{(2)})^2,$$
$$(\gamma^{(2)} - \beta^{(2)})[a_0^{(2)}k^{(2)} - (\nu_1^{(2)})^2] + 4b_0^{(2)}\nu_1^{(2)}\nu_2^{(2)} - 2a_0^{(2)}(\nu_2^{(2)})^2 - 2k^{(2)}(b_0^{(2)})^2 > 0,$$
$$\rho_2 > 0, \ \ I_0^{(2)} > 0, \ \ j_0^{(2)} > 0, \ \ \beta_0^{(2)} > 0.$$

Denote by

$$A^{(2)}(\partial_x, \tau) = [A_{ij}^{(2)}(\partial_x, \tau)]_{9\times 9}$$

the matrix differential operator generated by the left-hand side expressions in (2.3)–(2.7),

$$A_{ij}^{(2)}(\partial_x,\tau) = \delta_{ij}(\mu^{(2)} + \varkappa^{(2)})\partial_l\partial_l + (\lambda^{(2)} + \mu^{(2)})\partial_i\partial_j - \tau^2\rho_2\delta_{ij},$$
$$A_{i4}^{(2)}(\partial_x,\tau)) = -\tau\beta_0^{(2)}\partial_i, \quad A_{i,j+4}^{(2)}(\partial_x,\tau) = -\varkappa^{(2)}\varepsilon_{ijl}\partial_l,$$
$$A_{i8}^{(2)}(\partial_x,\tau) = \lambda_0^{(2)}\partial_i, \quad A_{i9}^{(2)}(\partial_x,\tau) = 0,$$
$$A_{4j}^{(2)}(\partial_x,\tau) = -\tau\beta_0^{(2)}\partial_j, \quad A_{44}^{(2)}(\partial_x,\tau) = k^{(2)}\partial_l\partial_l - \tau^2 a^{(2)},$$
$$A_{4,j+4}^{(2)}(\partial_x,\tau) = 0, \quad A_{48}^{(2)}(\partial_x,\tau) = \nu_1^{(2)}\partial_l\partial_j - \tau c_0^{(2)}, \quad A_{49}^{(2)}(\partial_x,\tau) = -\nu_3^{(2)}\partial_l\partial_l,$$
$$A_{i+4,j}^{(2)}(\partial_x,\tau) = -\varkappa^{(2)}\varepsilon_{ijl}\partial_l, \ \ A_{i+4,4}(\partial_x,\tau) = 0,$$
$$A_{i+4,j+4}^{(2)}(\partial_x,\tau) = \delta_{ij}\gamma^{(2)}\partial_l\partial_l + (\alpha^{(2)} + \beta^{(2)})\partial_i\partial_j - (2\varkappa^{(2)} + \tau^2 I_0^{(2)})\delta_{ij},$$
$$A_{i+4,8}(\partial_x,\tau) = 0, \quad A_{i+4,9}^{(2)}(\partial_x,\tau) = 0,$$
$$A_{8j}^{(2)}(\partial_x,\tau) = -\lambda_0^{(2)}\partial_j, \quad A_{84}^{(2)}(\partial_x,\tau) = \nu_1^{(2)}\partial_l\partial_l + \tau c_0^{(2)},$$
$$A_{8,j+4}^{(2)}(\partial_x,\tau) = 0, \quad A_{88}^{(2)}(\partial_x,\tau) = a_0^{(2)}\partial_l\partial_l - (\xi_0^{(2)} + \tau^2 j_0^{(2)}),$$
$$A_{89}^{(2)}(\partial_x,\tau) = -\lambda_2^{(2)}\partial_l\partial_l, \quad A_{9j}^{(2)}(\partial_x,\tau) = 0, \quad A_{94}^{(2)}(\partial_x,\tau) = \nu_3^{(2)}\partial_l\partial_l,$$
$$A_{9,j+4}^{(2)}(\partial_x,\tau) = 0 \quad A_{98}^{(2)}(\partial_x,\tau) = \lambda_2^{(2)}\partial_l\partial_l, \quad A_{99}(\partial_x,\tau) = \chi^{(2)}\partial_l\partial_l, \ \ i,j = 1,2,3.$$

The stress differential operator of thermo-electro-elasticity is defined as follows:

$$T^{(2)} = T^{(2)}(\partial_x, \nu, \tau) := [T_{ij}^{(2)}(\partial_x, \nu, \tau)]_{9\times 9},$$

where

$$T_{ij}^{(2)}(\partial_x,\nu,\tau) = \lambda^{(2)}\nu_i\partial_j + \mu^{(2)}\nu_j\partial_i + \delta_{ij}(\mu^{(2)} + \varkappa^{(2)})\nu_k\partial_k, \quad T_{i4}^{(2)}(\partial_x,\nu,\tau) = -\tau\beta_0^{(2)}\nu_i,$$
$$T_{i,j+4}^{(2)}(\partial_x,\nu,\tau) = -\varkappa^{(2)}\varepsilon_{ijk}\nu_k, \quad T_{i8}^{(2)}(\partial_x,\nu,\tau) = \lambda_0^{(2)}\nu_i, \quad T_{i9}^{(2)}(\partial_x,\nu,\tau) = 0,$$

$$
\begin{aligned}
&T^{(2)}_{4,j}(\partial_x,\nu,\tau)=0, \quad T^{(2)}_{44}(\partial_x,\nu,\tau)=k^{(2)}\nu_l\partial_l,\\
&T^{(2)}_{4,j+4}(\partial_x,\nu,\tau)=-\nu_2^{(2)}\varepsilon_{ljk}\nu_l\partial_k, \quad T^{(2)}_{48}(\partial_x,\nu,\tau)=\nu_1^{(2)}\nu_k\partial_k,\\
&T^{(2)}_{49}(\partial_x,\nu,\tau)=-\nu_3^{(2)}\nu_k\partial_k, \quad T^{(2)}_{i+4,j}(\partial_x,\nu,\tau)=0,\\
&T^{(2)}_{i+4,4}(\partial_x,\nu,\tau)=\nu_2^{(2)}\varepsilon_{lik}\nu_l\partial_k, \quad T^{(2)}_{i+4,j+4}(\partial_x,\nu,\tau)=\alpha^{(2)}\nu_i\partial_j+\beta^{(2)}\nu_j\partial_i+\delta_{ij}\gamma^{(2)}\nu_k\partial_k,\\
&T^{(2)}_{i+4,8}(\partial_x,\nu,\tau)=b_0^{(2)}\varepsilon_{lik}\nu_l\partial_k, \quad T^{(2)}_{i+4,9}(\partial_x,\nu,\tau)=\lambda_1^{(2)}\varepsilon_{lik}\nu_l\partial_k,\\
&T^{(2)}_{8j}(\partial x,\nu,\tau)=0, \quad T^{(2)}_{84}(\partial_x,\nu,\tau)=\nu_1^{(2)}\nu_k\partial_k,\\
&T^{(2)}_{8,j+4}(\partial_x,\nu,\tau)=-b_0^{(2)}\varepsilon_{lik}\nu_l\partial_k, \; T^{(2)}_{88}(\partial_x,\nu,\tau)=a_0^{(2)}\nu_k\partial_k, \quad T^{(2)}_{89}(\partial_x,\nu,\tau)=-\lambda_2^{(2)}\nu_k\partial_k,\\
&T^{(2)}_{9j}(\partial x,\nu,\tau)=0, \quad T^{(2)}_{94}(\partial_x,\nu,\tau)=\nu_3^{(2)}\nu_k\partial_k,\\
&T^{(2)}_{9,j+4}(\partial_x,\nu,\tau)=-\lambda_1^{(2)}\varepsilon_{ljk}\nu_l\partial_k, \quad T^{(2)}_{98}(\partial_x,\nu,\tau)=\lambda_2^{(2)}\nu_k\partial_k,\\
&T^{(2)}_{99}(\partial_x,\nu,\tau)=\chi^{(2)}\nu_k\partial_k, \;\; i,j=1,2,3.
\end{aligned}
$$

The system of equations (2.3)–(2.7) can be written in a matrix form

$$A^{(2)}(\partial_x,\tau)U^{(2)}=\Phi,$$

where

$$
\begin{aligned}
U^{(2)}&=(u_1^{(2)},u_2^{(2)},u_3^{(2)},\vartheta^{(2)},\phi_1^{(2)},\phi_2^{(2)},\phi_3^{(2)},\varphi^{(2)},\psi^{(2)})^\top,\\
\Phi&=-\Big(\rho_2f_1,\rho_2f_2,\rho_2f_3,\frac{1}{T_0}\rho_2Q,\rho_2X_1,\rho_2X_2,\rho_2X_3,\rho_2F,f\Big)^\top
\end{aligned}
$$

and $A^{(2)}(\partial_x,\tau)$ is the 9-dimensional matrix differential operator corresponding to system (2.3)–(2.7).

3. Formulation of Boundary-transmission Pseudo-oscillation Problems

By H^s with $s\in\mathbb{R}$, we denote the Sobolev-Slobodetsky space. Let $\mathscr{M}_0$ be a smooth surface without boundary. For a proper sub-manifold $\mathscr{M}\subset\mathscr{M}_0$, we denote by $\widetilde{H}^s(\mathscr{M})$ the subspace of $H^s(\mathscr{M}_0)$,

$$\widetilde{H}^s(\mathscr{M})=\big\{g\,:\,g\in H^s(\mathscr{M}_0),\; supp\, g\subset\overline{\mathscr{M}}\big\},$$

while $H^s(\mathscr{M})$ stand for the space of restriction on $\mathscr{M}$ of functions from $H^s(\mathscr{M}_0)$.

3.1. **Formulation of the Dirichlet boundary-transmission problem $(TD)_\tau$ of pseudo-oscillations.** We are looking for a solution

$$
\begin{aligned}
&U^{(1)}=(u^{(1)},\vartheta^{(1)})^\top=(u^{(1)},u_4^{(1)})^\top\in[H^1(\Omega_1)]^4,\\
&U^{(2)}=(u^{(2)},\vartheta^{(2)},\phi^{(2)},\varphi^{(2)},\psi^{(2)})^\top=(u^{(2)},u_4^{(2)},u_5^{(2)},\dots,u_9^{(2)})^\top\in[H^1(\Omega_2)]^9
\end{aligned}
$$

of the pseudo-oscillation equations

$$
\begin{aligned}
A^{(1)}(\partial_x,\tau)U^{(1)}&=0 \quad \text{in } \Omega_1,\\
A^{(2)}(\partial_x,\tau)U^{(2)}&=0 \quad \text{in } \Omega_2,
\end{aligned}
$$

which satisfy on the surface S_1 the following transmission conditions:

$$
\begin{aligned}
&\{u_j^{(1)}\}^+-\{u_j^{(2)}\}^+=f_j^{(1)} && \text{on } S_1, \;\; j=\overline{1,4},\\
&\{T^{(1)}(\partial_x,\nu,\tau)U^{(1)}\}_j^+ + \{T^{(2)}(\partial_x,\nu,\tau)U^{(2)}\}_j^+=f_j^{(2)} && \text{on } S_1, \quad j=\overline{1,4}, \;\; \nu=-n,
\end{aligned}
$$

and the boundary conditions

$$\{u_j^{(2)}\}^+=Q_j^{(2)}, \;\; j=\overline{5,9},$$

while on the surface S_2, the Dirichlet boundary conditions

$$\{U^{(2)}\}^+=p^{(2)} \quad \text{on } S_2,$$

where

$$f_j^{(1)}\in H^{\frac12}(S_1), \quad f_j^{(2)}\in H^{-\frac12}(S_1), \;\; j=\overline{1,4},$$

$$Q_j^{(2)} \in H^{\frac{1}{2}}(S_1), \quad j=\overline{5,9}, \quad p^{(2)} \in [H^{\frac{1}{2}}(S_2)]^9.$$

3.2. **Formulation of the Neumann boundary-transmission problem $(TN)_\tau$ of pseudo-oscillations.** We are looking for a solution

$$U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top = (u^{(1)}, u_4^{(1)})^\top \in [H^1(\Omega_1)]^4,$$
$$U^{(2)} = (u^{(2)}, \vartheta^{(2)}, \phi^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top = (u^{(2)}, u_4^{(2)}, u_5^{(2)}, \dots, u_9^{(2)})^\top \in [H^1(\Omega_2)]^9$$

of the pseudo-oscillation equations

$$A^{(1)}(\partial_x, \tau) U^{(1)} = 0 \quad \text{in } \Omega_1,$$
$$A^{(2)}(\partial_x, \tau) U^{(2)} = 0 \quad \text{in } \Omega_2,$$

which satisfy on the surface S_1 the following transmission conditions:

$$\{u_j^{(1)}\}^+ - \{u_j^{(2)}\}^+ = f_j^{(1)} \quad \text{on } S_1, \quad j=\overline{1,4},$$
$$\{T^{(1)}(\partial_x,\nu,\tau)U^{(1)}\}_j^+ + \{T^{(2)}(\partial_x,\nu,\tau)U^{(2)}\}_j^+ = f_j^{(2)} \quad \text{on } S_1, \quad j=\overline{1,4}, \quad \nu=-n,$$

and the boundary conditions

$$\{u_j^{(2)}\}^+ = Q_j^{(2)}, \quad j=\overline{5,9},$$

while on the surface S_2, the Neumann boundary conditions

$$\{T^{(2)}(\partial_x,\nu,\tau)U^{(2)}\}^+ = q^{(2)} \quad \text{on } S_2,$$

where

$$f_j^{(1)} \in H^{\frac{1}{2}}(S_1), \quad f_j^{(2)} \in H^{-\frac{1}{2}}(S_1), \quad j=\overline{1,4},$$
$$Q_j^{(2)} \in H^{\frac{1}{2}}(S_1), \quad j=\overline{5,9}, \quad q^{(2)} \in [H^{-\frac{1}{2}}(S_2)]^9.$$

3.3. **Formulation of the mixed boundary-transmission problem $(TM)_\tau$ of pseudo-oscillations.** We are looking for a solution

$$U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top = (u^{(1)}, u_4^{(1)})^\top \in [H^1(\Omega_1)]^4,$$
$$U^{(2)} = (u^{(2)}, \vartheta^{(2)}, \phi^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top = (u^{(2)}, u_4^{(2)}, u_5^{(2)}, \dots, u_9^{(2)})^\top \in [H^1(\Omega_2)]^9$$

of the pseudo-oscillation equations

$$A^{(1)}(\partial_x, \tau) U^{(1)} = 0 \quad \text{in } \Omega_1,$$
$$A^{(2)}(\partial_x, \tau) U^{(2)} = 0 \quad \text{in } \Omega_2,$$

which satisfy on the surface S_1) the following transmission conditions:

$$\{u_j^{(1)}\}^+ - \{u_j^{(2)}\}^+ = f_j^{(1)} \quad \text{on } S_1, \quad j=\overline{1,4},$$
$$\{T^{(1)}(\partial_x,\nu,\tau)U^{(1)}\}_j^+ + \{T^{(2)}(\partial_x,\nu,\tau)U^{(2)}\}_j^+ = f_j^{(2)} \quad \text{on } S_1, \quad j=\overline{1,4}, \quad \nu=-n,$$

and the boundary conditions

$$\{u_j^{(2)}\}^+ = Q_j^{(2)} \quad \text{on } S_1, \quad j=\overline{5,9},$$

while on the surface S_2, the mixed boundary conditions

$$\{U^{(2)}\}^+ = p_2^{(D)} \quad \text{on } S_2^{(D)},$$
$$\{T^{(2)}(\partial_x,\nu,\tau)U^{(2)}\}^+ = q_2^{(N)} \quad \text{on } S_2^{(N)},$$

where

$$S_2 = \overline{S_2^{(D)}} \cup \overline{S_2^{(N)}}, \quad S_2^{(D)} \cap S_2^{(N)} = \varnothing,$$
$$\ell_m = \partial S_2^{(D)} = \partial S_2^{(N)} \in C^\infty, \quad f_j^{(1)} \in H^{\frac{1}{2}}(S_1), \quad f_j^{(2)} \in H^{-\frac{1}{2}}(S_1), \quad j=\overline{1,4},$$
$$Q_j^{(2)} \in H^{\frac{1}{2}}(S_1), \quad j=\overline{5,9}, \quad p_2^{(D)} \in [H^{\frac{1}{2}}(S_2^{(D)}]^9, \quad q_2^{(N)} \in [H^{-\frac{1}{2}}(S_2^{(N)}]^9.$$

4. Uniqueness Theorems for Solutions of Boundary-transmission Problems of Pseudo-oscillations

Theorem 4.1. *The boundary-transmission problems $(TD)_\tau$ and $(TN)_\tau$ cannot have two different solutions in the class of regular vector functions $U^{(1)} \in [C^2(\Omega_1)]^4 \cap [C^1(\overline{\Omega_1})]^4$, $U^{(2)} \in [C^2(\Omega_2)]^9 \cap [C^1(\overline{\Omega_2})]^9$ and also in the Sobolev space $[H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9$.*

Proof. It is sufficient to show that the homogeneous problems $(TD)_\tau$ and $(TN)_\tau$ have only the trivial solution. Indeed, suppose $(U^{(1)}, U^{(2)})$ is a regular solution to the homogeneous problem $(TD)_\tau$ or $(TN)_\tau$. Let us write Green's formulas for the vector functions $U^{(1)}$ and $U^{(2)}$ in the domains Ω_1 and Ω_2, respectively:

$$\int_{\Omega_1} A^{(1)}(\partial_x,\tau)U^{(1)}\cdot U^{(1)}dx + \int_{\Omega_1} E_\tau^{(1)}(U^{(1)},\overline{U}^{(1)})dx = \int_{S_1}\{T^{(1)}U^{(1)}\}^+\cdot\{U^{(1)}\}^+ds, \tag{4.1}$$

$$\int_{\Omega_2} A^{(2)}(\partial_x,\tau)U^{(2)}\cdot U^{(2)}dx + \int_{\Omega_2} E_\tau^{(2)}(U^{(2)},\overline{U}^{(2)})dx = \int_{S_1\cup S_2}\{T^{(2)}U^{(2)}\}^+\cdot\{U^{(2)}\}^+ds, \tag{4.2}$$

where

$$U^{(1)} = (u^{(1)},\vartheta^{(1)})^T, \quad U^{(2)} = (u^{(2)},\vartheta^{(2)},\phi^{(2)},\varphi^{(2)},\psi^{(2)})^T,$$
$$E_\tau^{(1)}(U^{(1)},\overline{U}^{(1)}) = \mathscr{E}(u^{(1)},\overline{u}^{(1)}) + \rho_1\tau^2|u^{(1)}|^2 - \tau\beta_0^{(1)}\vartheta^{(1)}\operatorname{div}\overline{u}^{(1)} + k^{(1)}|\operatorname{grad}\vartheta^{(1)}|^2$$
$$+\tau\beta_0^{(1)}\operatorname{div}u^{(1)}\overline{\vartheta}^{(1)} + \tau^2a^{(1)}|\vartheta^{(1)}|^2,$$
$$\mathscr{E}(u^{(1)},\overline{u}^{(1)}) = (\mu^{(1)}+\varkappa^{(1)})|\operatorname{grad}u^{(1)}|^2 + (\lambda^{(1)}+\mu^{(1)})|\operatorname{div}u^{(1)}|.$$

Here and in what follows, $a\cdot b$ denotes the scalar product of two, in general, complex-valued vectors

$$a\cdot b = \sum_{k=1}^N a_k\overline{b}_k, \quad a,b\in\mathbb{C}^N.$$

Obviously, $\mathscr{E}(u^{(1)},\overline{u}^{(1)}) > 0$,

$$E_\tau^{(2)}(U^{(2)},\overline{U}^{(2)}) = B(u^{(2)},\overline{u}^{(2)}) + 2i\lambda_1^{(2)}\varepsilon_{ijk}\operatorname{Im}(\partial_k\psi^{(2)}\partial_i\overline{\phi}_j^{(2)}) + 2i\lambda_2^{(2)}\operatorname{Im}(\partial_j\varphi^{(2)}\partial_j\overline{\psi}^{(2)})$$
$$+2i\nu_3^{(2)}\operatorname{Im}(\partial_j\vartheta^{(2)}\partial_j\overline{\psi}^{(2)}) + 2i\tau\beta_0^{(2)}\operatorname{Im}(\partial_ju_j^{(2)}\overline{\vartheta}^{(2)}) + 2i\tau c_0^{(2)}\operatorname{Im}(\varphi^{(2)}\overline{\vartheta}^{(2)})$$
$$+\tau^2(\rho_2|u^{(2)}|^2 + I_0^{(2)}|\phi^{(2)}|^2 + j_0^{(2)}|\varphi^{(2)}|^2 + a^{(2)}|\vartheta^{(2)}|^2);$$

here, $B(u^{(2)},\overline{u}^{(2)}) > 0$ $\forall u^{(2)}\neq 0$ (for the definition of this form see [5] formula (2.19)).

Adding Green's formulas (4.1) and (4.2) and taking into account the fact that $(U^{(1)},U^{(2)})$ is a solution to the homogeneous transmission problem $(TD)_\tau$ or $(TN)_\tau$, we get

$$\int_{\Omega_1} E_\tau^{(1)}(U^{(1)},\overline{U}^{(1)})dx + \int_{\Omega_2} E_\tau^{(2)}(U^{(2)},\overline{U}^{(2)})dx$$
$$= \int_{S_1}\sum_{j=1}^4\{T^{(1)}U^{(1)}\}_j^+\{\overline{U}^{(2)}\}_j^+ds + \int_{S_1}\sum_{j=1}^4\{T^{(2)}U^{(2)}\}_j^+\{\overline{U}^{(2)}\}_j^+ds$$
$$= \int_{S_1}\sum_{j=1}^4\big(\{T^{(1)}U^{(1)}\}_j^+ + \{T^{(2)}U^{(2)}\}_j^+\big)\{\overline{U}^{(2)}\}_j^+ds = 0.$$

Therefore we obtain

$$\int_{\Omega_1} E_\tau^{(1)}(U^{(1)},\overline{U}^{(1)})dx + \int_{\Omega_2} E_\tau^{(2)}(U^{(2)},\overline{U}^{(2)})dx = 0. \tag{4.3}$$

Similarly we get (see [5])

$$\int\limits_{\Omega_1} E_\tau^{(1)}(U^{(1)},\overline{U}^{(1)})dx+\int\limits_{\Omega_2} \widetilde{E}_\tau^{(2)}(U^{(2)},\overline{U}^{(2)})dx=0, \tag{4.4}$$

where

$$\widetilde{E}_\tau^{(2)}(U^{(2)},\overline{U}^{(2)}):=B^{(2)}(u^{(2)},\overline{u}^{(2)})+2i\tau\beta_0^{(2)}\operatorname{Im}(\partial_j u_j^{(2)}\overline{\vartheta}^{(2)})+2i\tau c_0^{(2)}\operatorname{Im}(\varphi^{(2)}\overline{\vartheta}^{(2)})$$
$$+\tau^2(\rho_2|u|^2+I_0^{(2)}|\phi|^{(2)}+j_0^{(2)}|\varphi^{(2)}|^2+a^{(2)}|\vartheta^{(2)}|^2). \tag{4.5}$$

Now, let us take first the real part of equality (4.3), and then the imaginary part, where

$$\tau=\sigma+i\omega,\quad \tau^2=(\sigma^2-\omega^2)+2i\sigma\omega,\quad \sigma>\sigma_0>0,\quad \omega\in\mathbb{R}.$$

Thus we obtain the following integral equalities:

$$\int\limits_{\Omega_1}\Big[\mathscr{E}(u^{(1)},u^{(1)})+(\sigma^2-\omega^2)\rho_1|u^{(1)}|^2-2\beta_0^{(1)}\omega\operatorname{Im}(\overline{\vartheta}^{\,(1)}\operatorname{div}u^{(1)})+k^{(1)}|\operatorname{grad}\vartheta^{(1)}|^2$$
$$+a^{(1)}(\sigma^2-\omega^2)|\vartheta^{(1)}|^2\Big]dx$$
$$+\int\limits_{\Omega_2}\Big[B(u^{(2)},u^{(2)})-2\omega\beta_0^{(2)}\operatorname{Im}(\overline{\vartheta}^{\,(2)}\operatorname{div}u^{(2)})-2\omega c_0^{(2)}\operatorname{Im}(\varphi^{(2)}\overline{\vartheta}^{\,(2)})+(\sigma^2-\omega^2)\big(\rho_2|u^{(2)}|^2$$
$$+I_0^{(2)}|\phi^{(2)}|^2+j_0^{(2)}|\varphi^{(2)}|^2+a^{(2)}|\vartheta^{(2)}|^2\big)\Big]dx=0, \tag{4.6}$$

$$\int\limits_{\Omega_1}\Big[2\sigma\omega\rho_1|u^{(1)}|^2+2a^{(1)}\sigma\omega|\vartheta^{(1)}|^2+2\beta_0^{(1)}\sigma\operatorname{Im}(\overline{\vartheta}^{\,(1)}\operatorname{div}u^{(1)})dx\Big]$$
$$+\int\limits_{\Omega_2}\Big[2\sigma\beta_0^{(2)}\operatorname{Im}(\overline{\vartheta}^{\,(2)}\operatorname{div}u^{(2)})+2\sigma c_0^{(2)}\operatorname{Im}(\varphi^{(2)}\overline{\vartheta}^{\,(2)})+2\sigma\omega(\rho_2|u^{(2)}|^2$$
$$+I_0^{(2)}|\phi^{(2)}|^2+j_0^{(2)}|\varphi^{(2)}|^2+a^{(2)}|\vartheta^{(2)}|^2)\Big]dx=0. \tag{4.7}$$

Multiplying (4.7) by $\frac{\omega}{\sigma}$ and adding equality (4.6), we get

$$\int\limits_{\Omega_1}\Big[\mathscr{E}(u^{(1)},u^{(1)})+(\sigma^2+\omega^2)\rho_1|u^{(1)}|^2+k^{(1)}|\operatorname{grad}\vartheta^{(1)}|^2+a^{(1)}(\sigma^2+\omega^2)|\vartheta^{(1)}|^2\Big]dx$$
$$+\int\limits_{\Omega_2}\Big[B(u^{(2)},u^{(2)})+(\sigma^2+\omega^2)(\rho_2|u^{(2)}|^2+I_0^{(2)}|\phi^{(2)}|^2+j_0^{(2)}|\varphi^{(2)}|^2+a^{(2)}|\vartheta^{(2)}|^2)\Big]dx=0. \tag{4.8}$$

Since

$$\mathscr{E}(u^{(1)},u^{(1)})>0,\ \ \forall u^{(1)}\neq 0,$$
$$B(u^{(2)},u^{(2)})>0,\ \ \forall u^{(2)}\neq 0,$$
$$\rho_1>0,\ \ k^{(1)}>0,\ \ a^{(1)}>0,\ \ \rho_2>0,\ \ I_0^{(2)}>0,\ \ j_0^{(2)}>0,\ \ a^{(2)}>0,$$

we obtain

$$|u^{(1)}|=|\vartheta^{(1)}|=0,$$
$$|u^{(2)}|=|\phi^{(2)}|=|\varphi^{(2)}|=|\vartheta^{(2)}|=0,$$
$$\int\limits_{\Omega_2}\chi^{(2)}|E|^2dx=0,$$

where $E=-\operatorname{grad}\psi^{(2)}$ (see formula (2.3) in [5]), hence

$$u^{(1)}=0,\quad \vartheta^{(1)}=0,\quad \text{in }\ \Omega_1,$$

$$u^{(2)} = 0, \quad \phi^{(2)} = 0, \quad \varphi^{(2)} = 0, \quad \vartheta^{(2)} = 0 \quad \text{and} \quad \psi^{(2)} = b \quad \text{in } \Omega_2,$$

here, b is an arbitrary constant.

For the function $\psi^{(2)}$, the Dirichlet homogeneous condition on the surface S_1 implies that $b = 0$. Therefore for the homogeneous transmission problem $(TD)_\tau$, we obtain

$$U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top = 0 \quad \text{in } \Omega_1,$$

$$U^{(2)} = (u^{(2)}, \vartheta^{(2)}, \phi^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top = 0 \quad \text{in } \Omega_2,$$

while for the homogeneous transmission problem $(TN)_\tau$, for the function $\psi^{(2)}$, the Dirichlet homogeneous condition on the surface S_1 implies that $b = 0$, and we get

$$U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top = 0 \quad \text{in } \Omega_1,$$

$$U^{(2)} = (u^{(2)}, \vartheta^{(2)}, \phi^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top = 0 \quad \text{in } \Omega_2.$$

Note that the uniqueness theorem of the transmission problems $(TD)_\tau$ and $(TN)_\tau$ in the Sobolev space $[H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9$ can be proved similarly. □

For the mixed boundary-transmission problem $(TM)_\tau$, the following theorem holds.

Theorem 4.2. *The mixed boundary-transmission problem $(TM)_\tau$ cannot have two different solutions in the Sobolev space $[H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9$.*

Proof. It is sufficient to show that the homogeneous problem $(TM)_\tau$ has only the trivial solution.

Indeed, suppose $(U^{(1)}, U^{(2)})$ is a solution to the homogeneous problem $(TM)_\tau$. Let us write Green's formulas for the vector functions $U^{(1)}$ and $U^{(2)}$ in the domains Ω_1 and Ω_2, respectively:

$$\int_{\Omega_1} A^{(1)}(\partial_x, \tau) U^{(1)} \cdot U^{(1)} dx + \int_{\Omega_1} E_\tau^{(1)}(U^{(1)}, \overline{U}^{\,(1)}) dx = \langle \{T^{(1)} U^{(1)}\}^+, \{U^{(1)}\}^+ \rangle_{S_1}, \tag{4.9}$$

$$\int_{\Omega_2} A^{(2)}(\partial_x, \tau) U^{(2)} \cdot U^{(2)} dx + \int_{\Omega_2} E_\tau^{(2)}(U^{(2)}, \overline{U}^{\,(2)}) dx = \langle \{T^{(2)} U^{(2)}\}^+, \{U^{(2)}\}^+ \rangle_{S_1 \cup S_2}, \tag{4.10}$$

where the symbols $\langle \cdot, \cdot \rangle_{S_1}$ and $\langle \cdot, \cdot \rangle_{S_1 \cup S_2}$ denote the duality between the function spaces $[H^{-\frac{1}{2}}(S_1))]^4$ and $[H^{\frac{1}{2}}(S_1))]^4$, and the function spaces $[H^{-\frac{1}{2}}(S_1 \cup S_2)]^9$ and $[H^{\frac{1}{2}}(S_1 \cup S_2)]^9$, respectively.

Adding Green's formulas (4.9) and (4.10) and taking into account that $(U^{(1)}, U^{(2)})$ is a solution to the homogeneous transmission problem $(TM)_\tau$, we get

$$\begin{aligned}
&\int_{\Omega_1} E_\tau^{(1)}(U^{(1)}, \overline{U}^{(1)}) dx + \int_{\Omega_2} E_\tau^{(2)}(U^{(2)}, \overline{U}^{(2)}) dx \\
&\qquad = \sum_{j=1}^{4} \langle \{T^{(1)} U^{(1)}\}_j^+, \{U^{(1)}\}_j^+ \rangle_{S_1} + \sum_{j=1}^{9} \langle \{T^{(2)} U^{(2)}\}_j^+, \{U^{(2)}\}_j^+ \rangle_{S_1} \\
&\qquad = \sum_{j=1}^{4} \langle \{T^{(1)} U^{(1)}\}_j^+ + \{T^{(2)} U^{(2)}\}_j^+, \{U^{(2)}\}_j^+ \rangle_{S_1} = 0.
\end{aligned}$$

Therefore we obtain

$$\int_{\Omega_1} E_\tau^{(1)}(U^{(1)}, \overline{U}^{(1)}) dx + \int_{\Omega_2} E_\tau^{(2)}(U^{(2)}, \overline{U}^{(2)}) dx = 0, \tag{4.11}$$

and

$$\int_{\Omega_1} E_\tau^{(1)}(U^{(1)}, \overline{U}^{(1)}) dx + \int_{\Omega_2} \widetilde{E}_\tau^{(2)}(U^{(2)}, \overline{U}^{(2)}) dx = 0.$$

Now, if we repeat the reasoning in Theorem 4.1, we get

$$u^{(1)} = 0, \quad \vartheta^{(1)} = 0 \quad \text{in } \Omega_1,$$

$$u^{(2)} = 0, \quad \phi^{(2)} = 0, \quad \varphi^{(2)} = 0, \quad \vartheta^{(2)} = 0 \quad \text{and} \quad \psi^{(2)} = b \quad \text{in } \Omega_2,$$

where b is an arbitrary constant.

For the function $\psi^{(2)}$, the Dirichlet homogeneous condition on the surface $S_D^{(2)}$ implies that $b=0$. Therefore for the homogeneous transmission problem $(TM)_\tau$, we obtain

$$U^{(1)}=(u^{(1)},\vartheta^{(1)})^\top=0 \quad \text{in } \Omega_1,$$
$$U^{(2)}=(u^{(2)},\vartheta^{(2)},\phi^{(2)},\varphi^{(2)},\psi^{(2)})^\top=0 \quad \text{in } \Omega_2. \qquad \square$$

5. Properties of Potentials and Boundary Operators

The single layer potentials are defined as follows (their properties see [15], [5]):

$$V_{S_1}^{(1)}(g)(x)=\int\limits_{S_1}\Gamma^{(1)}(x-y)g(y)d_yS,$$
$$V_{S_1}^{(2)}(f)(x)=\int\limits_{S_1}\Gamma^{(2)}(x-y)f(y)d_yS,$$
$$V_{S_2}^{(2)}(h)(x)=\int\limits_{S_2}\Gamma^{(2)}(x-y)h(y)d_yS,$$

where $\Gamma^{(1)}(x-y)$ and $\Gamma^{(2)}(x-y)$ are the fundamental solutions of the differential operators $A^{(1)}(\partial_x,\tau)$ and $A^{(2)}(\partial_x,\tau)$ respectively (see [15], [5]).

The following theorem holds (see [15], [5], [17]).

Theorem 5.1. *Let $g\in\big[H^{-\frac12}(S_1)\big]^4$, $f\in\big[H^{-\frac12}(S_1)\big]^9$, $h\in\big[H^{-\frac12}(S_2)\big]^9$, then the following jump relations hold:*

$$\{T^{(1)}(\partial_x,n,\tau)V_{S_1}^{(1)}(g)\}^\pm=\Big(\mp\frac12 I_4+\mathscr{K}_{S_1}^{(1)}\Big)(g) \quad \text{on } S_1,$$
$$\{T^{(2)}(\partial_x,\nu,\tau)V_{S_1}^{(2)}(f)\}^\pm=\Big(\mp\frac12 I_9+\mathscr{K}_{S_1}^{(2)}\Big)(f) \quad \text{on } S_1,$$
$$\{T^{(2)}(\partial_x,\nu,\tau)V_{S_2}^{(2)}(h)\}^\pm=\Big(\mp\frac12 I_9+\mathscr{K}_{S_2}^{(2)}\Big)(h) \quad \text{on } S_2,$$

where

$$\mathscr{K}_{S_1}^{(1)}(g)(z)=\int\limits_{S_1}T^{(1)}(\partial_z,n(z),\tau)\Gamma^{(1)}(z-y)g(y)d_yS, \quad z\in S_1,$$
$$\mathscr{K}_{S_1}^{(2)}(f)(z)=\int\limits_{S_1}T^{(2)}(\partial_z,\nu(z),\tau)\Gamma^{(2)}(z-y)f(y)d_yS, \quad z\in S_1,$$
$$\mathscr{K}_{S_2}^{(2)}(h)(z)=\int\limits_{S_2}T^{(2)}(\partial_z,\nu(z),\tau)\Gamma^{(2)}(z-y)h(y)d_yS, \quad z\in S_2,$$

and

$$\mathscr{H}_{S_1}^{(1)}(g)(z)=\{V_{S_1}^{(1)}(g)(z)\}^+=\{V_{S_1}^{(1)}(g)(z)\}^-, \quad z\in S_1,$$
$$\mathscr{H}_{S_1}^{(2)}(f)(z)=\{V_{S_1}^{(2)}(f)(z)\}^+=\{V_{S_1}^{(2)}(f)(z)\}^-, \quad z\in S_1,$$
$$\mathscr{H}_{S_2}^{(2)}(h)(z)=\{V_{S_2}^{(2)}(h)(z)\}^+=\{V_{S_2}^{(2)}(h)(z)\}^-, \quad z\in S_2.$$

Here we collect some theorems describing the mapping properties of potentials and corresponding boundary (pseudodifferential) operators. The proof of these theorems can be found in references [5], [17], [9].

Theorem 5.2. *Let $s\in\mathbb{R}$. Then the single layer potentials can be extended to the continuous operators*

$$V_{S_1}^{(1)}:\big[H^s(S_1)\big]^4\longrightarrow\big[H^{s+\frac32}(\Omega_1)\big]^4,$$

$$V_{S_1}^{(2)} : \big[H^s(S_1)\big]^9 \longrightarrow \big[H^{s+\frac{3}{2}}(\Omega_1)\big]^9,$$
$$V_{S_2}^{(2)} : \big[H^s(S_2)\big]^9 \longrightarrow \big[H^{s+\frac{3}{2}}(\Omega_2)\big]^9.$$

Theorem 5.3. *Let $s \in \mathbb{R}$. Then the pseudodifferential operators of order -1*

$$\mathscr{H}_{S_1}^{(1)} : \big[H^s(S_1)\big]^4 \longrightarrow \big[H^{s+1}(S_1)\big]^4,$$
$$\mathscr{H}_{S_1}^{(2)} : \big[H^s(S_1)\big]^9 \longrightarrow \big[H^{s+1}(S_1)\big]^9,$$
$$\mathscr{H}_{S_2}^{(2)} : \big[H^s(S_2)\big]^9 \longrightarrow \big[H^{s+1}(S_2)\big]^9$$

are invertible.

Theorem 5.4. *Let $s \in \mathbb{R}$. Then the singular integral operators*

$$\mathscr{K}_{S_1}^{(1)} : \big[H^s(S_1)\big]^4 \longrightarrow \big[H^s(S_1)\big]^4,$$
$$\mathscr{K}_{S_1}^{(2)} : \big[H^s(S_1)\big]^9 \longrightarrow \big[H^s(S_1)\big]^9,$$
$$\mathscr{K}_{S_2}^{(2)} : \big[H^s(S_2)\big]^9 \longrightarrow \big[H^s(S_2)\big]^9$$

are continuous.

6. Existence of Solutions to the Dirichlet Boundary-transmission Problem $(TD)_\tau$ of Pseudo-oscillations

We look for a solution of the boundary-transmission problem $(TD)_\tau$ in the form of the single layer potentials

$$U^{(1)} = V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1} g^{(1)} \quad \text{in } \Omega_1,$$
$$U^{(2)} = V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)} + V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)} \quad \text{in } \Omega_2,$$

where the unknown densities $g^{(1)}$, $g^{(2)}$ and $h^{(2)}$ belong to the following Sobolev spaces:

$$g^{(1)} = (g_1^{(1)}, \dots, g_4^{(1)})^\top \in [H^{\frac{1}{2}}(S_1)]^4, \quad g^{(2)} = (g_1^{(2)}, \dots, g_9^{(2)})^\top \in [H^{\frac{1}{2}}(S_1)]^9,$$
$$h^{(2)} = (h_1^{(2)}, \dots, h_9^{(2)})^\top \in [H^{\frac{1}{2}}(S_2)]^9.$$

Taking into account the boundary and boundary-transmission conditions of the contact problem $(TD)_\tau$, for the vector-functions $g^{(1)}$, $g^{(2)}$ and $h^{(2)}$ we obtain the following system of equations:

$$g_j^{(2)} + r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)}]_j = Q_j^{(2)} \quad \text{on } S_1, \ \ j = \overline{5,9}, \tag{6.1}$$
$$g_j^{(1)} - g_j^{(2)} - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)}]_j = f_j^{(1)} \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{6.2}$$
$$\Big[\Big(-\frac{1}{2} I_4 + K_{S_1}^{(1)}\Big)(\mathscr{H}_{S_1}^{(1)})^{-1} g^{(1)}\Big]_j + \Big[\Big(-\frac{1}{2} I_9 + \mathscr{K}_{S_1}^{(2)}\Big)(\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)}\Big]_j$$
$$+ r_{S_1}[T^{(2)} V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)}]_j = f_j^{(2)} \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{6.3}$$
$$r_{S_2}[V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)}] + h^{(2)} = p^{(2)} \quad \text{on } S_2, \tag{6.4}$$

where r_{s_j} $(j = 1, 2)$ is the restriction operator on the surface S_j $(j = 1, 2)$.

Let us change positions of equations (6.1) and (6.2) of system (6.1)–(6.4) and multiply equation (6.1) by -1, i.e., rewrite system (6.1)–(6.4) in the form of the following equivalent system of equations:

$$g_j^{(1)} - g_j^{(2)} - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)}]_j = f_j^{(1)} \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{6.5}$$
$$-g_j^{(2)} - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)}]_j = -Q_j^{(2)} \quad \text{on } S_1, \ \ j = \overline{5,9}, \tag{6.6}$$
$$\Big[\Big(-\frac{1}{2} I_4 + \mathscr{K}_{S_1}^{(1)}\Big)(\mathscr{H}_{S_1}^{(1)})^{-1} g^{(1)}\Big]_j + \Big[\Big(-\frac{1}{2} I_9 + \mathscr{K}_{S_1}^{(2)}\Big)(\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)}\Big]_j$$
$$+ r_{S_1}[T^{(2)} V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)}]_j = f_j^{(2)} \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{6.7}$$
$$r_{S_2}[V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)}] + h^{(2)} = p^{(2)} \quad \text{on } S_2. \tag{6.8}$$

The operator corresponding to system (6.5)–(6.8) has the following matrix form:

$$\mathscr{N} := \begin{bmatrix} I_{9\times 4} & -I_9 & -r_{S_1}[V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}]_{9\times 9} \\ \mathscr{A}^{(1)}_{S_1} & [\mathscr{A}^{(2)}_{S_1}]_{4\times 9} & r_{S_1}[T^{(2)}V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}]_{4\times 9} \\ [0]_{9\times 4} & r_{S_2}[V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}]_{9\times 9} & I_9 \end{bmatrix}_{22\times 22},$$

where

$$\mathscr{A}^{(1)}_{S_1} := \Big(-\frac{1}{2}I_4 + \mathscr{K}^{(1)}_{S_1}\Big)(\mathscr{H}^{(1)}_{S_1})^{-1}, \quad \mathscr{A}^{(2)}_{S_1} := \Big(-\frac{1}{2}I_9 + \mathscr{K}^{(2)}_{S_1}\Big)(\mathscr{H}^{(2)}_{S_1})^{-1}$$

are the Poincaré–Steklov type operators. These operators are strongly elliptic pseudodifferential operators of order 1, and

$$I_{9\times 4} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\top}.$$

The operator

$$\mathscr{N} : \mathscr{X} \to \mathscr{Y}$$

is bounded, where

$$\mathscr{X} := [H^{\frac{1}{2}}(S_1)]^{13} \times [H^{\frac{1}{2}}(S_2)]^9,$$
$$\mathscr{Y} := [H^{\frac{1}{2}}(S_1)]^9 \times [H^{-\frac{1}{2}}(S_1)]^4 \times [H^{\frac{1}{2}}(S_2)]^9.$$

The following theorem holds.

Theorem 6.1. *The operator*

$$\mathscr{N} : \mathscr{X} \to \mathscr{Y}$$

is invertible.

Proof. First, we show that the operator $\mathscr{N} : \mathscr{X} \to \mathscr{Y}$ is Fredholm with index zero. Indeed, obviously, the operators

$$r_{S_1}[V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}]_{9\times 9} : [H^{\frac{1}{2}}(S_2)]^9 \to [H^{\frac{1}{2}}(S_1)]^9,$$
$$r_{S_1}[T^{(2)}V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}]_{4\times 9} : [H^{\frac{1}{2}}(S_2)]^9 \to [H^{-\frac{1}{2}}(S_1)]^4,$$
$$r_{S_2}[V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}]_{9\times 9} : [H^{\frac{1}{2}}(S_1)]^9 \to [H^{\frac{1}{2}}(S_2)]^9$$

are compact, since $S_1 \cap S_2 = \varnothing$.

Now, let us consider the operator

$$\mathscr{N}_1 := \begin{bmatrix} I_{9\times 4} & -I_9 & [0]_{9\times 9} \\ \mathscr{A}^{(1)}_{S_1} & [\mathscr{A}^{(2)}_{S_1}]_{4\times 9} & [0]_{4\times 9} \\ [0]_{9\times 4} & [0]_{9\times 9} & I_9 \end{bmatrix}_{22\times 22},$$

where the operator

$$\mathscr{N} - \mathscr{N}_1 : \mathscr{X} \to \mathscr{Y}$$

is compact.

Write corresponding system of the operator $\mathscr{N}_1$ as follows:

$$\widetilde{g}^{\,(1)}_j - \widetilde{g}^{\,(2)}_j = \widetilde{f}^{\,(1)}_j \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{6.9}$$
$$-\widetilde{g}^{\,(2)}_j = \widetilde{F}_j \quad \text{on } S_1, \ \ j = \overline{5,9}, \tag{6.10}$$
$$[\mathscr{A}^{(1)}_{S_1}\widetilde{g}^{\,(1)}]_j + [\mathscr{A}^{(2)}_{S_1}\widetilde{g}^{\,(2)}]_j = \widetilde{f}^{\,(2)}_j \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{6.11}$$
$$\widetilde{h}^{\,(2)} = \widetilde{p}^{\,(2)} \quad \text{on } S_2. \tag{6.12}$$

System (6.9)–(6.12) is equivalent to the following system:

$$\widetilde{g}^{\,(1)}_j - \widetilde{g}^{\,(2)}_j = \widetilde{f}^{\,(1)}_j \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{6.13}$$

$$-\widetilde{g}_j^{(2)} = \widetilde{F}_j \qquad \text{on } S_1, \;\; j=\overline{5,9}, \tag{6.14}$$

$$(\mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)})\widetilde{g}_j^{(1)} = \Psi \quad \text{on } S_1, \tag{6.15}$$

$$\widetilde{h}^{(2)} = \widetilde{p}^{(2)} \qquad \text{on } S_2. \tag{6.16}$$

where

$$\overline{\mathscr{A}}_{S_1}^{(2)} := \big[\mathscr{A}_{S_{1,pq}}^{(2)}\big]_{4\times 4}, \quad p,q=\overline{1,4},$$

and $\Psi = (\psi_1, \dots, \psi_4)^\top$,

$$\psi_j = \widetilde{f}_j^{(2)} + [\mathscr{A}_{S_1}^{(2)}(\widetilde{f}_1^{(1)}, \dots, \widetilde{f}_4^{(1)}, \widetilde{F}_5, \dots, \widetilde{F}_9)^\top]_j, \quad j=\overline{1,4}. \tag{6.17}$$

The operator corresponding to system (6.13)–(6.16), has the following form

$$\mathscr{N}_2 := \begin{bmatrix} I_{9\times 4} & -I_9 & [0]_{9\times 9} \\ \mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)} & [0]_{4\times 9} & [0]_{4\times 9} \\ [0]_{9\times 4} & [0]_{9\times 9} & I_9 \end{bmatrix}_{22\times 22}.$$

Obviously, the operator

$$\mathscr{N}_2 : \mathscr{X} \to \mathscr{Y}$$

is bounded.

Consider the composition

$$\mathscr{N}_3 := \mathscr{N}_2 \circ \mathscr{Q},$$

where

$$\mathscr{Q} := \begin{bmatrix} [0]_{4\times 9} & I_4 & [0]_{4\times 9} \\ I_9 & [0]_{9\times 4} & [0]_{9\times 9} \\ [0]_{9\times 9} & [0]_{9\times 4} & I_9 \end{bmatrix}_{22\times 22}.$$

Obviously, the operator

$$\mathscr{Q} : \mathscr{X} \to \mathscr{X}$$

is invertible.

The operator $\mathscr{N}_3$ has the form

$$\mathscr{N}_3 := \begin{bmatrix} -I_9 & I_{9\times 4} & [0]_{9\times 9} \\ [0]_{4\times 9} & \mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)} & [0]_{4\times 9} \\ [0]_{9\times 9} & [0]_{9\times 4} & I_9 \end{bmatrix}_{22\times 22}.$$

To show that the operator

$$\mathscr{N} : \mathscr{X} \to \mathscr{Y}$$

is Fredholm with zero index, it suffices to show that the operator

$$\mathscr{N}_3 : \mathscr{X} \to \mathscr{Y}$$

is Fredholm with zero index.

Indeed, since the operator $\mathscr{N}_3$ is triangular diagonal, it suffices to show that the following Lemma holds.

Lemma 6.2. *The operator*

$$\mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)} : [H^{\frac{1}{2}}(S_1)]^4 \to [H^{-\frac{1}{2}}(S_1)]^4$$

is Fredholm with zero index.

Proof. Using Green's formula and Korn's inequality (see [10]), for an arbitrary vector-function $U^{(1)} \in [H^1(\Omega_1)]^4$ which is a solution of the homogeneous differential equation in Ω_1

$$A^{(1)}(\partial_x, \tau)U^{(1)} = 0 \quad \text{in } \Omega_1,$$

we get

$$\operatorname{Re}\langle \{T^{(1)}U^{(1)}\}^+, \{U^{(1)}\}^+\rangle_{S_1} \geq c_1 \|U^{(1)}\|^2_{[H^1(\Omega_1)]^4} - c_2\|U^{(1)}\|^2_{[L_2(\Omega_1)]^4}. \tag{6.18}$$

Here and below, the constants $c_i > 0$. Substituting $U^{(1)} = V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1}g^{(1)}$ in (6.18) and applying the trace theorem, we obtain

$$\begin{aligned}\mathrm{Re}\langle \mathscr{A}_{S_1}^{(1)} g^{(1)}, g^{(1)}\rangle_{S_1} &\geq c_1 \|V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1}g^{(1)}\|^2_{[H^1(\Omega_1)]^4} - c_2\|V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1}g^{(1)}\|^2_{[L_2(\Omega_1)]^4}\\ &\geq c_3\|g^{(1)}\|^2_{[H^{\frac{1}{2}}(S_1)]^4} - c_2\|V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1}g^{(1)}\|^2_{[L_2(\Omega_1)]^4}. \end{aligned} \tag{6.19}$$

Now, using the boundedness of the single layer potential

$$\|V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1}g^{(1)}\|_{[L_2(\Omega_1)]^4} \leq c_4\|g^{(1)}\|_{[H^{-\frac{1}{2}}(S_1)]^4},$$

from (6.19), we obtain the coercivity of the operator $\mathscr{A}_{S_1}^{(1)}$:

$$\mathrm{Re}\langle \mathscr{A}_{S_1}^{(1)} g^{(1)}, g^{(1)}\rangle_{S_1} \geq c_3\|g^{(1)}\|^2_{[H^{\frac{1}{2}}(S_1)]^4} - c_5\|g^{(1)}\|^2_{[H^{-\frac{1}{2}}(S_1)]^4} \quad \forall g^{(1)} \in [H^{\frac{1}{2}}(S_1)]^4. \tag{6.20}$$

Similarly, we can obtain the coercivity of the operator $\mathscr{A}_{S_1}^{(2)}$.

Indeed, using Green's formula and Korn's inequality, for an arbitrary vector-function $U^{(2)} \in [H^1(\Omega_2)]^9$, which is a solution of the homogeneous equation

$$A^{(2)}(\partial_x, \tau)U^{(2)} = 0 \quad \text{in } \Omega_2$$

and $\{U^{(2)}\}^+ = 0$ on S_2, we get

$$\mathrm{Re}\langle \{T^{(2)}U^{(2)}\}^+, \{U^{(2)}\}^+\rangle_{S_1} \geq c_6\|U^{(2)}\|^2_{[H^1(\Omega_2)]^9} - c_7\|U^{(2)}\|^2_{[L_2(\Omega_2)]^9}. \tag{6.21}$$

Substituting $U^{(2)} = V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}$ in (6.21), we obtain

$$\mathrm{Re}\langle \mathscr{A}_{S_1}^{(2)} g^{(2)}, g^{(2)}\rangle_{S_1} \geq c_6\|V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}\|^2_{[H^1(\Omega_2)]^9} - c_7\|V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}\|^2_{[L_2(\Omega_2)]^9}. \tag{6.22}$$

Using the boundedness of the single layer potential

$$\|V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}\| \leq c_8\|g^{(2)}\|_{[H^{-\frac{1}{2}}(S_1)]^9}$$

and the trace theorem, from (6.22), we get

$$\mathrm{Re}\langle \mathscr{A}_{S_1}^{(2)} g^{(2)}, g^{(2)}\rangle_{S_1} \geq c_9\|g^{(2)}\|^2_{[H^{\frac{1}{2}}(S_1)]^9} - c_{10}\|g^{(2)}\|^2_{[H^{-\frac{1}{2}}(S_1)]^9} \quad \forall g^{(2)} \in [H^{\frac{1}{2}}(S_1)]^9. \tag{6.23}$$

Now, substitute $g^{(2)} = (g^{(1)}, 0, 0, 0, 0, 0)^\top$ in (6.23), where $g^{(1)} = (g_1^{(1)}, g_2^{(1)}, g_3^{(1)}, g_4^{(1)})^\top$, then (6.23) can be rewritten as follows:

$$\mathrm{Re}\langle \overline{\mathscr{A}}_{S_1}^{(2)} g^{(1)}, g^{(1)}\rangle_{S_1} \geq c_9\|g^{(1)}\|^2_{[H^{\frac{1}{2}}(S_1)]^4} - c_{10}\|g^{(1)}\|^2_{[H^{-\frac{1}{2}}(S_1)]^4}. \tag{6.24}$$

Adding inequalities (6.20) and (6.24), we get

$$\mathrm{Re}\langle (\mathscr{A}_1^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)} g^{(1)}, g^{(1)}\rangle_{S_1} \geq c_{11}\|g^{(1)}\|^2_{[H^{\frac{1}{2}}(S_1)]^4} - c_{12}\|g^{(1)}\|^2_{[H^{-\frac{1}{2}}(S_1)]^4} \ \ \forall g^{(1)} \in [H^{\frac{1}{2}}(S_1)]^4,$$

i.e., the operator

$$\mathscr{A}_1^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)} : [H^{\frac{1}{2}}(S_1)]^4 \to [H^{-\frac{1}{2}}(S_1)]^4$$

is coercive, and therefore it is Fredholm with zero index (see [13, 16]). Thus we obtain the validity of Lemma 6.2. □

It follows from Lemma 6.2 that the operator

$$\mathscr{N}_3 : \mathscr{X} \to \mathscr{Y}$$

is Fredholm with zero index. Then the operators

$$\mathscr{N}_2,\ \mathscr{N}_1 : \mathscr{X} \to \mathscr{Y}$$

are also Fredholm with zero index, and hence the operator

$$\mathscr{N} : \mathscr{X} \to \mathscr{Y}$$

is Fredholm with zero index.

Now we show that the operator

$$\mathscr{N}:\mathscr{X}\to\mathscr{Y}$$

is invertible.

It can be easily shown that the invertibility of the operator $\mathscr{N}$ follows from the uniqueness of solutions of the boundary-transmission problem $(TD)_\tau$. Indeed, let $(g^{(1)},g^{(2)},h^{(2)})^\top$ be a solution of the homogeneous equation

$$\mathscr{N}(g^{(1)},g^{(2)},h^{(2)})^\top=0.$$

We construct the following potentials:

$$U^{(1)}=V^{(1)}_{S_1}(\mathscr{H}^{(1)}_{S_1})^{-1}g^{(1)}, \tag{6.25}$$

$$U^{(2)}=V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}g^{(2)}+V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}h^{(2)}. \tag{6.26}$$

Since $(g^{(1)},g^{(2)},h^{(2)})^\top$ is a solution of the homogeneous system (6.5)–(6.8), it is clear that $(U^{(1)},U^{(2)})$ will be a solution of the homogeneous boundary-transmission problem $(TD)_\tau$. Then from the uniqueness theorem of problem $(TD)_\tau$, it follows that

$$U^{(1)}\equiv 0\quad\text{in }\ \Omega_1,$$
$$U^{(2)}\equiv 0\quad\text{in }\ \Omega_2.$$

Since the single layer potentials are continuous in space $\mathbb{R}^3$, we have

$$\{U^{(1)}\}^+=\{U^{(1)}\}^-\quad\text{on }\ S_1$$

and

$$\{U^{(2)}\}^+=\{U^{(2)}\}^-\quad\text{on }\ S_1\cup S_2,$$

hence

$$\{U^{(1)}\}^-=0\quad\text{on }\ S_1,$$
$$\{U^{(2)}\}^-=0\quad\text{on }\ S_1\cup S_2.$$

Therefore we find that the vector functions $U^{(1)}$ and $U^{(2)}$ satisfy the following Dirichlet problems:

$$\begin{cases}A^{(1)}(\partial_x,\tau)U^{(1)}=0 & \text{in }\ \mathbb{R}^3\setminus\overline{\Omega_1},\\ \{U^{(1)}\}^-=0 & \text{in }\ S_1,\end{cases}$$

and

$$\begin{cases}A^{(2)}(\partial_x,\tau)U^{(2)}=0 & \text{in }\ \mathbb{R}^3\setminus\overline{\Omega_2},\\ \{U^{(2)}\}^-=0 & \text{in }\ S_1\cup S_2.\end{cases}$$

From the uniqueness of the solutions of the Dirichlet problem it follows that these problems have only trivial solution, i.e.,

$$U^{(1)}\equiv 0\quad\text{in }\ \mathbb{R}^3\setminus\overline{\Omega_1},$$
$$U^{(2)}\equiv 0\quad\text{in }\ \mathbb{R}^3\setminus\overline{\Omega_2},$$

thus

$$U^{(1)}\equiv 0\quad\text{in }\ \mathbb{R}^3,$$
$$U^{(2)}\equiv 0\quad\text{in }\ \mathbb{R}^3.$$

Now, applying the jump formulas of the potentials (6.25) and (6.26), we get

$$\{T^{(1)}U^{(1)}\}^- - \{T^{(1)}U^{(1)}\}^+ = g^{(1)}=0\quad\text{on }\ S_1,$$
$$\{T^{(2)}U^{(2)}\}^- - \{T^{(2)}U^{(2)}\}^+ = g^{(2)}=0\quad\text{on }\ S_1,$$
$$\{T^{(2)}U^{(2)}\}^- - \{T^{(2)}U^{(2)}\}^+ = h^{(2)}=0\quad\text{on }\ S_2.$$

Therefore we obtain

$$\operatorname{Ker}\mathscr{N}=\{0\},$$

and since the index of the operator $\mathscr{N}$ equals zero, we have

$$\operatorname{Ker} \mathscr{N}^* = \{0\}.$$

This implies that $\mathscr{N}$ is surjection. Thus we find that the operator

$$\mathscr{N} : \mathscr{X} \to \mathscr{Y}$$

is invertible, and Theorem 6.1 is proved. □

The invertibility of the operator $\mathscr{N}$ implies the unique solvability of systems (6.1)–(6.4), (6.5)–(6.8) and hence we obtain the unique solvability of the Dirichlet boundary-transmission problem $(TD)_\tau$.

Theorem 6.3. *Let* $S_1, S_2 \in C^\infty$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 > 0$, $\omega \in \mathbb{R}$, *and* $f_j^{(1)} \in H^{\frac12}(S_1)$, $f_j^{(2)} \in H^{-\frac12}(S_1)$, $j = \overline{1,4}$, $Q_j^{(2)} \in H^{\frac12}(S_1)$, $j = \overline{5,9}$, $p^{(2)} \in [H^{\frac12}(S_2)]^9$. *Then the Dirichlet boundary-transmission problem* $(TD)_\tau$ *has a unique solution*

$$(U^{(1)}, U^{(2)}) \in [H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9,$$

which is represented as follows:

$$U^{(1)} = V_{S_1}^{(1)} (\mathscr{H}_{S_1}^{(1)})^{-1} g^{(1)} \quad in\ \Omega_1, \tag{6.27}$$

$$U^{(2)} = V_{S_1}^{(2)} (\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)} + V_{S_2}^{(2)} (\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)} \quad in\ \Omega_2, \tag{6.28}$$

where $g^{(1)}$, $g^{(2)}$, $h^{(2)}$ *are solutions of the uniquely solvable system* (6.1)–(6.4).

The following regularity theorem holds.

Theorem 6.4. *Let* $S_1, S_2 \in C^{m,a}$, $0 < \beta < \alpha \le 1$, $m \ge 2$, $m \in \mathbb{N}$, *and*

$$f_j^{(1)} \in C^{k,\beta}(S_1), \ \ f_j^{(2)} \in C^{k-1,\beta}(S_1), \ \ j = \overline{1,4},$$

$$Q_j^{(2)} \in C^{k,\beta}(S_1), \ \ j = \overline{5,9}, \quad p^{(2)} \in [C^{k,\beta}(S_2)]^9, \ \ k = \overline{1, m-1}.$$

Then the Dirichlet boundary-transmission problem $(TD)_\tau$ *has a unique solution*

$$(U^{(1)}, U^{(2)}) \in [C^{k,\beta}(\overline{\Omega_1})]^4 \times [C^{k,\beta}(\overline{\Omega_2})]^9,$$

which is represented as single layer potentials (6.27), (6.28), *where*

$$g^{(1)} \in [C^{k,\beta}(S_1)]^4, \ \ g^{(2)} \in [C^{k,\beta}(S_1)]^9, \ \ h^{(2)} \in [C^{k,\beta}(S_2)]^9$$

are solutions of the uniquely solvable system (6.1)–(6.4) .

Proof. Since the operators

$$\mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)} : [H^{\frac12}(S_1)]^4 \to [H^{-\frac12}(S_1)]^4,$$

$$\mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)} : [C^{k,\beta}(S_1)]^4 \to [C^{k-1,\beta}(S_1)]^4$$

are first order strongly elliptic and have a common regularizer, we have (see [1, §10])

$$\dim \operatorname{Ker}(\mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)}) = \dim \operatorname{CoKer}(\mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)}).$$

This means that the operator

$$\mathscr{A}_{S_1}^{(1)} + \overline{\mathscr{A}}_{S_1}^{(2)} : [C^{k,\beta}(S_1)]^4 \to [C^{k-1,\beta}(S_1)]^4$$

is Fredholm with zero index. Therefore the operator

$$\mathscr{N} : [C^{k,\beta}(S_1)]^{13} \times [C^{k,\beta}(S_2)]^9 \to [C^{k,\beta}(S_1)]^9 \times [C^{k-1,\beta}(S_1)]^4 \times [C^{k,\beta}(S_1)]^9$$

will also be Fredholm with zero index. The invertibility of the operator $\mathscr{N}$ and the unique solvability of systems (6.1)–(6.4) in the space $[C^{k,\beta}(S_1)]^{13} \times [C^{k,\beta}(S_2)]^9$ can be shown similarly to the case of Sobolev spaces.

Now, if we use the boundedness of single layer potentials, invertibility and Fredholm properties of the corresponding boundary operators (see Theorem 6.5 and Theorem 6.6 below), then from the representation formulas of solutions (6.27) and (6.28), we obtain the regularity of solutions of the boundary-transmission problem $(TD)_\tau$ in the class $[C^{k,\beta}(\overline{\Omega_1})]^4 \times [C^{k,\beta}(\overline{\Omega_2})]^9$. □

In the proof of Theorem 6.4 we have used the following theorems (see [5], [15], [1]).

Theorem 6.5. *Let $S_1, S_2 \in C^{m,\alpha}$, $0 < \beta < \alpha \leq 1$, $k = \overline{1, m-1}$, $m \geq 2$, $m \in \mathbb{N}$. Then the single layer potentials*

$$V_{S_1}^{(1)} : [C^{k,\beta}(S_1)]^4 \to [C^{k+1,\beta}(\overline{\Omega_1})]^4,$$
$$V_{S_1}^{(2)} : [C^{k,\beta}(S_1)]^9 \to [C^{k+1,\beta}(\overline{\Omega_2})]^9,$$
$$V_{S_2}^{(2)} : [C^{k,\beta}(S_2)]^9 \to [C^{k+1,\beta}(\overline{\Omega_2})]^9$$

are bounded.

Theorem 6.6. *Let $S_1, S_2 \in C^{m,\alpha}$, $0 < \beta < \alpha \leq 1$, $k = \overline{1, m-1}$, $m \geq 2$, $m \in \mathbb{N}$. Then the boundary integral operators*

$$\mathscr{H}_{S_1}^{(1)} : [C^{k,\beta}(S_1)]^4 \to [C^{k+1,\beta}(S_1)]^4,$$
$$\mathscr{H}_{S_1}^{(2)} : [C^{k,\beta}(S_1)]^9 \to [C^{k+1,\beta}(S_1)]^9,$$
$$\mathscr{H}_{S_2}^{(2)} : [C^{k,\beta}(S_2)]^9 \to [C^{k+1,\beta}(S_2)]^9$$

are invertible, while the operators

$$-\frac{1}{2} I_4 + \mathscr{K}_{S_1}^{(1)} : [C^{k,\beta}(S_1)]^4 \to [C^{k,\beta}(S_1)]^4,$$
$$-\frac{1}{2} I_9 + \mathscr{K}_{S_1}^{(2)} : [C^{k,\beta}(S_1)]^9 \to [C^{k,\beta}(S_1)]^9,$$
$$-\frac{1}{2} I_9 + \mathscr{K}_{S_2}^{(2)} : [C^{k,\beta}(S_2)]^9 \to [C^{k,\beta}(S_2)]^9$$

are Fredholm with zero index.

It follows from Theorem 6.6 that the operators

$$\mathscr{A}_{S_1}^{(1)} : [C^{k,\beta}(S_1)]^4 \to [C^{k-1,\beta}(S_1)]^4,$$
$$\mathscr{A}_{S_1}^{(2)} : [C^{k,\beta}(S_2)]^9 \to [C^{k-1,\beta}(S_1)]^9,$$
$$\mathscr{A}_{S_2}^{(2)} : [C^{k,\beta}(S_2)]^9 \to [C^{k-1,\beta}(S_2)]^9$$

are Fredholm operators with zero index.

The following corollary holds.

Corollary 6.7. *Let $S_1, S_2 \in C^{\infty}$ and $f_j^{(1)} \in C^{\infty}(S_1)$, $f_j^{(2)} \in C^{\infty}(S_1)$, $j = \overline{1, 4}$, $Q_j^{(2)} \in C^{\infty}(S_1)$, $j = \overline{5, 9}$, $p^{(2)} \in [C^{\infty}(S_2)]^9$. Then the unique solution $(U^{(1)}, U^{(2)})$ of the Dirichlet problem $(TD)_\tau$ belongs to the class $[C^{\infty}(\overline{\Omega_1})]^4 \times [C^{\infty}(\overline{\Omega_2})]^9$, i.e.*

$$(U^{(1)}, U^{(2)}) \in [C^{\infty}(\overline{\Omega_1})]^4 \times [C^{\infty}(\overline{\Omega_2})]^9.$$

7. Existence of Solutions to the Neumann Boundary-transmission Problem $(TN)_\tau$ of Pseudo-oscillations

We seek for a solution of the Neumann boundary-transmission problem $(TN)_\tau$ in the form of the following single layer potentials:

$$U^{(1)} = V_{S_1}^{(1)} (\mathscr{H}_{S_1}^{(1)})^{-1} g^{(1)} \quad \text{in } \Omega_1,$$
$$U^{(2)} = V_{S_1}^{(2)} (\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)} + V_{S_2}^{(2)} (\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)} \quad \text{in } \Omega_2,$$

where the unknown densities $g^{(1)}$, $g^{(2)}$ and $h^{(2)}$ belong to the Sobolev spaces, $g^{(1)} = (g_1^{(1)}, \dots, g_4^{(1)})^\top \in [H^{\frac{1}{2}}(S_1)]^4$, $g^{(2)} = (g_1^{(2)}, \dots, g_9^{(2)})^\top \in [H^{\frac{1}{2}}(S_1)]^9$, $h^{(2)} = (h_1^{(2)}, \dots, h_9^{(2)})^\top \in [H^{\frac{1}{2}}(S_2)]^9$.

Taking into account the boundary and boundary-transmission conditions of problem $(TN)_\tau$, we obtain the following system of equations with respect to the vector functions $g^{(1)}$, $g^{(2)}$ and $h^{(2)}$:

$$g_j^{(2)} + r_{S_1} [V_{S_2}^{(2)} (\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)}]_j = Q_j^{(2)} \quad \text{on } S_1 \;\; j = \overline{5, 9}, \tag{7.1}$$

$$g_j^{(1)} - g_j^{(2)} - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j = f_j^{(1)} \quad \text{on } S_1, \ \ j=\overline{1,4}, \tag{7.2}$$

$$\Big[\Big(-\frac{1}{2}I_4 + \mathscr{K}_{S_1}^{(1)}\Big)(\mathscr{H}_{S_1}^{(1)})^{-1}g^{(1)}\Big]_j + \Big[\Big(-\frac{1}{2}I_9 + \mathscr{K}_{S_1}^{(2)}\Big)(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}\Big]_j + r_{S_1}[T^{(2)}V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j = f_j^{(2)} \quad \text{on } S_1, \ \ j=\overline{1,4}, \tag{7.3}$$

$$r_{S_2}[T^{(2)}V_{S_1}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}] + \Big(-\frac{1}{2}I_9 + \mathscr{K}_{S_2}^{(2)}\Big)(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)} = q^{(2)} \quad \text{on } S_2, \tag{7.4}$$

where r_{s_j} $(j=1,2)$ is the restriction operator on the surface S_j $(j=1,2)$.

Let us change positions of equations (7.1) and (7.2) of system (7.1)–(7.4) and multiply equation (7.1) by -1, i.e., we rewrite system (7.1)–(7.4) in the form of the following equivalent system of equations:

$$g_j^{(1)} - g_j^{(2)} - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j = f_j^{(1)} \quad \text{on } S_1, \ \ j=\overline{1,4}, \tag{7.5}$$

$$-g_j^{(2)} - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j = -Q_j^{(2)} \quad \text{on } S_1, \ \ j=\overline{5,9}, \tag{7.6}$$

$$\Big[\Big(-\frac{1}{2}I_4 + \mathscr{K}_{S_1}^{(1)}\Big)(\mathscr{H}_{S_1}^{(1)})^{-1}g^{(1)}\Big]_j + \Big[\Big(-\frac{1}{2}I_9 + \mathscr{K}_{S_1}^{(2)}\Big)(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}\Big]_j + r_{S_1}[T^{(2)}V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j = f_j^{(2)} \quad \text{on } S_1, \ \ j=\overline{1,4}, \tag{7.7}$$

$$r_{S_2}[T^{(2)}V_{S_1}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}] + \Big(-\frac{1}{2}I_9 + \mathscr{K}_{S_2}^{(2)}\Big)(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)} = q^{(2)} \quad \text{on } S_2. \tag{7.8}$$

The operator corresponding to system (7.5)–(7.8), has the following matrix form:

$$\mathscr{M} := \begin{bmatrix} I_{9\times 4} & -I_9 & -r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}]_{9\times 9} \\ \mathscr{A}_{S_1}^{(1)} & [\mathscr{A}_{S_1}^{(2)}]_{4\times 9} & r_{S_1}[T^{(2)}V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}]_{4\times 9} \\ [0]_{9\times 4} & r_{S_2}[T^{(2)}V_{S_1}(\mathscr{H}_{S_1}^{(2)})^{-1}]_{9\times 9} & \mathscr{A}_{S_2}^{(2)} \end{bmatrix}_{22\times 22},$$

where

$$\mathscr{A}_{S_1}^{(1)} := \Big(-\frac{1}{2}I_4 + \mathscr{K}_{S_1}^{(1)}\Big)(\mathscr{H}_{S_1}^{(1)})^{-1}, \quad \mathscr{A}_{S_1}^{(2)} := \Big(-\frac{1}{2}I_9 + \mathscr{K}_{S_1}^{(2)}\Big)(\mathscr{H}_{S_1}^{(2)})^{-1},$$
$$\mathscr{A}_{S_2}^{(2)} := \Big(-\frac{1}{2}I_9 + \mathscr{K}_{S_2}^{(2)}\Big)(\mathscr{H}_{S_2}^{(2)})^{-1}$$

are the Poincaré–Steklov type operators.

The operator $\mathscr{M}$ is bounded in the spaces

$$\mathscr{M} : \mathscr{X} \to \mathscr{Z},$$

where

$$\mathscr{X} := [H^{\frac12}(S_1)]^{13} \times [H^{\frac12}(S_2)]^9,$$
$$\mathscr{Z} := [H^{\frac12}(S_1)]^9 \times [H^{-\frac12}(S_1)]^4 \times [H^{-\frac12}(S_2)]^9.$$

The following theorem holds.

Theorem 7.1. *The operator*

$$\mathscr{M} : \mathscr{X} \to \mathscr{Z}$$

is invertible.

Proof. First, we show that the operator

$$\mathscr{M} : \mathscr{X} \to \mathscr{Z}$$

is Fredholm with index zero. Indeed, obviously, the operators

$$r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}]_{9\times 9} : [H^{\frac12}(S_2)]^9 \to [H^{\frac12}(S_1)]^9,$$
$$r_{S_1}[T^{(2)}V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}]_{4\times 9} : [H^{\frac12}(S_2)]^9 \to [H^{-\frac12}(S_1)]^4,$$
$$r_{S_2}[T^{(2)}V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}]_{9\times 9} : [H^{\frac12}(S_1)]^9 \to [H^{-\frac12}(S_2)]^9$$

are compact, because of $S_1 \cap S_2 = \varnothing$.

Now, let us consider the operator

$$\mathscr{M}_1 := \begin{bmatrix} I_{9\times 4} & -I_9 & [0]_{9\times 9} \\ \mathscr{A}^{(1)}_{S_1} & [\mathscr{A}^{(2)}_{S_1}]_{4\times 9} & [0]_{4\times 9} \\ [0]_{9\times 4} & [0]_{9\times 9} & \mathscr{A}^{(2)}_{S_2} \end{bmatrix}_{22\times 22},$$

where the operator

$$\mathscr{M} - \mathscr{M}_1 : \mathscr{X} \to \mathscr{Z}$$

is compact.

Write the corresponding system of the operator $\mathscr{M}_1$ as follows:

$$\widetilde{g}^{(1)}_j - \widetilde{g}^{(2)}_j = \widetilde{f}^{(1)}_j \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{7.9}$$

$$-\widetilde{g}^{(2)}_j = \widetilde{F}_j \quad \text{on } S_1, \ \ j = \overline{5,9}, \tag{7.10}$$

$$[\mathscr{A}^{(1)}_{S_1} \widetilde{g}^{(1)}]_j + [\mathscr{A}^{(2)}_{S_1} \widetilde{g}^{(2)}]_j = \widetilde{f}^{(2)}_j \quad \text{on } S_1, \ \ j = \overline{1,4}, \tag{7.11}$$

$$\mathscr{A}^{(2)}_{S_2} \widetilde{h}^{(2)} = \widetilde{q}^{(2)} \quad \text{on } S_2. \tag{7.12}$$

System (7.9)–(7.12) is equivalent to the following system:

$$\widetilde{g}^{(1)}_j - \widetilde{g}^{(2)}_j = \widetilde{f}^{(1)}_j \quad \text{on } S_1, j = \overline{1,4}, \tag{7.13}$$

$$-\widetilde{g}^{(2)}_j = \widetilde{F}_j \quad \text{on } S_1, \ \ j = \overline{5,9}, \tag{7.14}$$

$$(\mathscr{A}^{(1)}_{S_1} + \overline{\mathscr{A}}^{(2)}_{S_1}) \widetilde{g}^{(1)}_j = \Psi \quad \text{on } S_1, \tag{7.15}$$

$$\mathscr{A}^{(2)}_{S_2} \widetilde{h}^{(2)} = \widetilde{q}^{(2)} \quad \text{on } S_2, \tag{7.16}$$

where

$$\overline{\mathscr{A}}^{(2)}_{S_1} := [\mathscr{A}^{(2)}_{S_1,pq}]_{4\times 4}, \ \ p, q = \overline{1,4},$$

and Ψ is defined by (6.17).

The operator corresponding to system (7.13)–(7.16) has the following form:

$$\mathscr{M}_2 := \begin{bmatrix} I_{9\times 4} & -I_9 & [0]_{9\times 9} \\ \mathscr{A}^{(1)}_{S_1} + \overline{\mathscr{A}}^{(2)}_{S_1} & [0]_{4\times 9} & [0]_{4\times 9} \\ [0]_{9\times 4} & [0]_{9\times 9} & \mathscr{A}^{(2)}_{S_2} \end{bmatrix}_{22\times 22}.$$

Obviously, the operator

$$\mathscr{M}_2 : \mathscr{X} \to \mathscr{Z}$$

is bounded.

Consider the composition

$$\mathscr{M}_3 = \mathscr{M}_2 \circ \mathscr{Q},$$

where the operator

$$\mathscr{Q} : \mathscr{X} \to \mathscr{X}$$

is invertible (see Section 6).

The operator $\mathscr{M}_3$ has the form

$$\mathscr{M}_3 := \begin{bmatrix} -I_9 & I_{9\times 4} & [0]_{9\times 9} \\ [0]_{4\times 9} & \mathscr{A}^{(1)}_{S_1} + \overline{\mathscr{A}}^{(2)}_{S_1} & [0]_{4\times 9} \\ [0]_{9\times 9} & [0]_{9\times 4} & \mathscr{A}^{(2)}_{S_2} \end{bmatrix}_{22\times 22}.$$

To show that the operator

$$\mathscr{M} : \mathscr{X} \to \mathscr{Z}$$

is Fredholm with zero index, it suffices to show that the operator

$$\mathscr{M}_3 : \mathscr{X} \to \mathscr{Z}$$

is Fredholm with zero index.

Indeed, since the operator $\mathscr{M}_3$ is triangular diagonal, it suffices to show that the operators standing on the diagonal are Fredholm with zero index.

As we know, the operator (see Lemma 6.2)

$$\mathscr{A}^{(1)}_{S_1} + \overline{\mathscr{A}}^{(2)}_{S_1} : [H^{\frac{1}{2}}(S_1)]^4 \to [H^{-\frac{1}{2}}(S_1)]^4$$

is Fredholm with zero index.

It is also known that the operator

$$\mathscr{A}^{(2)}_{S_2} : [H^{\frac{1}{2}}(S_2)]^9 \to [H^{-\frac{1}{2}}(S_2)]^9$$

is Fredholm with zero index (see [5]). Therefore we obtain that the operator

$$\mathscr{M}_3 : \mathscr{X} \to \mathscr{Z}$$

is Fredholm with zero index. Then the operators

$$\mathscr{M}_2, \mathscr{M}_1 : \mathscr{X} \to \mathscr{Z}$$

will also be Fredholm with zero index, from which we derive that the operator

$$\mathscr{M} : \mathscr{X} \to \mathscr{Z}$$

is Fredholm with zero index.

It can be easily shown that the invertibility of the operator $\mathscr{M}$ follows from the uniqueness of solutions of the boundary-transmission problem $(TN)_\tau$. Indeed, let $(g^{(1)}, g^{(2)}, h^{(2)})^\top$ be a solution of the homogeneous equation

$$\mathscr{M}(g^{(1)}, g^{(2)}, h^{(2)})^\top = 0. \tag{7.17}$$

We construct the following potentials:

$$U^{(1)} = V^{(1)}_{S_1}(\mathscr{H}^{(1)}_{S_1})^{-1} g^{(1)}, \tag{7.18}$$

$$U^{(2)} = V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1} g^{(2)} + V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1} h^{(2)}. \tag{7.19}$$

Since $(g^{(1)}, g^{(2)}, h^{(2)})^\top$ is a solution of the homogeneous system (7.5)–(7.8), it is clear that $(U^{(1)}, U^{(2)})$ will be a solution of the homogeneous boundary-transmission problem $(TN)_\tau$. Then from the uniqueness theorem of problem $(TN)_\tau$ it follows that

$$U^{(1)} \equiv 0 \quad \text{in } \ \Omega_1, \tag{7.20}$$

$$U^{(2)} \equiv 0 \quad \text{in } \ \Omega_2. \tag{7.21}$$

Since the single layer potentials are continuous in space $\mathbb{R}^3$, we have

$$\{U^{(1)}\}^+ = \{U^{(1)}\}^- \quad \text{on } \ S_1,$$
$$\{U^{(2)}\}^+ = \{U^{(2)}\}^- \quad \text{on } \ S_1 \cup S_2,$$

whence it follows that

$$\{U^{(1)}\}^- = 0 \quad \text{on } \ S_1,$$
$$\{U^{(2)}\}^- = 0 \quad \text{in } \ S_1 \cup S_2.$$

Therefore we obtain that the vector functions $U^{(1)}$ and $U^{(2)}$ satisfy the following Dirichlet problems:

$$\begin{cases} (A^{(1)}(\partial_x, \tau)U^{(1)} = 0 & \text{in } \ \mathbb{R}^3 \setminus \overline{\Omega_1}, \\ \{U^{(1)}\}^- = 0 & \text{on } \ S_1, \end{cases}$$

and

$$\begin{cases} \mathscr{A}^{(2)}(\partial_x, \tau)U^{(2)} = 0 & \text{in } \ \mathbb{R}^3 \setminus \overline{\Omega_2}, \\ \{U^{(2)}\}^- = 0 & \text{on } \ S_1 \cup S_2. \end{cases}$$

From the uniqueness of solutions of the Dirichlet problem it follows that these Dirichlet problems have only trivial solution, i.e.,

$$U^{(1)} \equiv 0 \quad \text{in } \ \mathbb{R}^3 \setminus \overline{\Omega_1},$$

$$U^{(2)} \equiv 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_2}.$$

Then from (7.20) and (7.21), we get

$$U^{(1)} \equiv 0 \quad \text{in } \mathbb{R}^3,$$
$$U^{(2)} \equiv 0 \quad \text{in } \mathbb{R}^3.$$

Now, applying the jump formulas of potentials (7.18) and (7.19), we obtain

$$\{T^{(1)}U^{(1)}\}^- - \{T^{(1)}U^{(1)}\}^+ = g^{(1)} = 0 \quad \text{on } S_1,$$
$$\{T^{(2)}U^{(2)}\}^- - \{T^{(2)}U^{(2)}\}^+ = g^{(2)} = 0 \quad \text{on } S_1,$$
$$\{T^{(2)}U^{(2)}\}^- - \{T^{(2)}U^{(2)}\}^+ = h^{(2)} = 0 \quad \text{on } S_2.$$

Therefore we derive

$$\operatorname{Ker} \mathscr{M} = \{0\},$$

and since the index of the operator $\mathscr{M}$ equals zero, we have

$$\operatorname{Ker} \mathscr{M}^* = \{0\}.$$

From this it follows that $\mathscr{M}$ is a surjection. Thus we obtain that the operator

$$\mathscr{M} : \mathscr{X} \to \mathscr{Z}$$

is invertible, and Theorem 7.1 is proved. □

The invertibility of the operator $\mathscr{M}$ implies the unique solvability of systems (7.1)–(7.4), (7.5)–(7.8) and hence we obtain the unique solvability of the Neumann type boundary-transmission problem $(TN)_\tau$.

Thus we obtain the existence and uniqueness theorem of the Neumann type boundary-transmission problem $(TN)_\tau$.

Theorem 7.2. *Let $S_1, S_2 \in C^\infty$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 > 0$, $\omega \in \mathbb{R}$, and*

$$f_j^{(1)} \in H^{\frac{1}{2}}(S_1), \quad f_j^{(2)} \in H^{-\frac{1}{2}}(S_1), \;\; j = \overline{1,4},$$
$$Q_j^{(2)} \in H^{\frac{1}{2}}(S_1), \;\; j = \overline{5,9}, \quad q^{(2)} \in [H^{-\frac{1}{2}}(S_2)]^9.$$

Then the Neumann boundary-transmission problem $(TN)_\tau$ has a unique solution

$$(U^{(1)}, U^{(2)}) \in [H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9,$$

which is represented as follows:

$$U^{(1)} = V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1} g^{(1)} \qquad \text{in } \Omega_1, \tag{7.22}$$
$$U^{(2)} = V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)} + V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)} \quad \text{in } \Omega_2, \tag{7.23}$$

where $g^{(1)}$, $g^{(2)}$, $h^{(2)}$ are the unique solutions of system (7.1)–(7.4).

The following regularity theorem is proved in the same way as in the case of the Dirichlet boundary-transmission problem $(TD)_\tau$.

Theorem 7.3. *Let $S_1, S_2 \in C^{m,\alpha}$, $0 < \beta < \alpha \le 1$, $k = \overline{1, m-1}$, $m \ge 2$, $m \in \mathbb{N}$, and*

$$f_j^{(1)} \in C^{k,\beta}(S_1), \quad f_j^{(2)} \in C^{k-1,\beta}(S_1), \;\; j = \overline{1,4},$$
$$Q_j^{(2)} \in C^{k,\beta}(S_1), \;\; j = \overline{5,9}, \quad q^{(2)} \in [C^{k-1,\beta}(S_2))]^9, \;\; k = \overline{1, m-1}.$$

Then problem $(TN)_\tau$ has a unique solution

$$(U^{(1)}, U^{(2)}) \in [C^{k,\beta}(\overline{\Omega_1})]^4 \times [C^{k,\beta}(\overline{\Omega_2})]^9,$$

which is represented as single layer potentials (7.22), (7.23), *where*

$$g^{(1)} \in [C^{k,\beta}(S_1)]^4, \;\; g^{(2)} \in [C^{k,\beta}(S_1)]^9, \;\; h^{(2)} \in [C^{k,\beta}(S_2)]^9$$

are solutions of the uniquely solvable system (7.1)–(7.4).

Corollary 7.4. *Let the following conditions:*

$$S_1, S_2 \in C^\infty, \quad f_j^{(1)} \in C^\infty(S_1), \quad f_j^{(2)} \in C^\infty(S_1), \ \ j=\overline{1,4},$$
$$Q_j^{(2)} \in C^\infty(S_1), \ \ j=\overline{5,9}, \quad q^{(2)} \in [C^\infty(S_2)]^9,$$

be fulfilled, then the unique solution to problem $(TN)_\tau$ is infinitly differentiable, i.e.

$$(U^{(1)}, U^{(2)}) \in [C^\infty(\overline{\Omega_1})]^4 \times [C^\infty(\overline{\Omega_2})]^9.$$

8. Existence of Solution to the Mixed Boundary-transmission Problem $(TM)_\tau$ of Pseudo-oscillations

We are looking for a solution to the mixed boundary-transmission problem $(TM)_\tau$ in the form of the following single layer potentials:

$$U^{(1)} = V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1} g^{(1)} \qquad \text{in } \ \Omega_1,$$
$$U^{(2)} = V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1} g^{(2)} + V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1} h^{(2)} \quad \text{in } \ \Omega_2,$$

where the unknown densities $g^{(1)}$, $g^{(2)}$ and $h^{(2)}$ belong to the following Sobolev spaces:

$$g^{(1)} = (g_1^{(1)}, \dots, g_4^{(1)})^\top \in [H^{\frac12}(S_1)]^4, \quad g^{(2)} = (g_1^{(2)}, \dots, g_9^{(2)})^\top \in [H^{\frac12}(S_1)]^9,$$
$$h^{(2)} = (h_1^{(2)}, \dots, h_9^{(2)})^\top \in [H^{\frac12}(S_2)]^9.$$

Taking into account the boundary and boundary-transmission conditions of the mixed problem $(TM)_\tau$, we obtain the following system of equations with respect to the vector functions $g^{(1)}$, $g^{(2)}$ and $h^{(2)}$:

$$g_j^{(2)} + r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j = Q_j^{(2)} \qquad \text{on } \ S_1 \ \ j=\overline{5,9}, \tag{8.1}$$
$$g_j^{(1)} - g_j^{(2)} - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j = f_j^{(1)} \qquad \text{on } \ S_1, \ \ j=\overline{1,4}, \tag{8.2}$$
$$\begin{aligned}&\Big[\Big(-\frac12 I_4 + \mathscr{K}_{S_1}^{(1)}\Big)(\mathscr{H}_{S_1}^{(1)})^{-1}g^{(1)}\Big]_j + \Big[\Big(-\frac12 I_9 + \mathscr{K}_{S_1}^{(2)}\Big)(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}\Big]_j \\ &\qquad + r_{S_1}[T^{(2)}V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)}]_j = f_j^{(2)} \qquad \text{on } \ S_1, \ \ j=\overline{1,4},\end{aligned} \tag{8.3}$$
$$r_{S_2^{(D)}}[V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}] + r_{S_2^{(D)}}h^{(2)} = p_2^{(D)} \qquad \text{on } \ S_2^{(D)}, \tag{8.4}$$
$$r_{S_2^{(N)}}[T^{(2)}V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}] + r_{S_2^{(N)}}\Big(-\frac12 I_9 + \mathscr{K}_{S_2}^{(2)}\Big)(\mathscr{H}_{S_2}^{(2)})^{-1}h^{(2)} = q_2^{(N)} \ \text{on } \ S_2^{(N)}. \tag{8.5}$$

Equation (8.4) can be rewritten as follows:

$$r_{S_2}V_{S_1}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)} + h^{(2)} = \Phi_0^{(2)} + h_0^{(2)} \quad \text{on } \ S_2, \tag{8.6}$$

where $\Phi_0^{(2)} \in [H^{\frac12}(S_2)]^9$ is a fixed extension of the Dirichlet condition, the vector function $p_2^{(D)} \in [H^{\frac12}(S_2^{(D)})]^9$ over the entire surface S_2, and

$$h_0^{(2)} \in [\widetilde{H}^{\frac12}(S_2^{(N)})]^9, \quad \operatorname{supp} h_0^{(2)} \subset \overline{S_2^{(N)}}.$$

Let us determine $h^{(2)}$ from equations (8.6) in the following way:

$$h^{(2)} = \Phi_0^{(2)} + h_0^{(2)} - r_{S_2}[V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)}]$$

and insert it into equations (8.1), (8.2), (8.3), and (8.5) of system (8.1)–(8.5). At the same time, we change the places of equations (8.1) and (8.2), and multiply equation (8.1) by -1. In this case, we get the following equivalent system of equations with respect to the vector functions $g^{(1)}$, $g^{(2)}$ and $h_0^{(2)}$:

$$\begin{aligned}&g_j^{(1)} - g_j^{(2)} + r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}(r_{S_2}V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)})]_j \\ &\qquad - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h_0^{(2)}]_j = \widetilde{f}_j^{(1)} \qquad \text{on } \ S_1 \ \ j=\overline{1,4},\end{aligned} \tag{8.7}$$
$$\begin{aligned}&-g_j^{(2)} + r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}(r_{S_2}V_{S_1}^{(2)}(\mathscr{H}_{S_1}^{(2)})^{-1}g^{(2)})]_j \\ &\qquad - r_{S_1}[V_{S_2}^{(2)}(\mathscr{H}_{S_2}^{(2)})^{-1}h_0^{(2)}]_j = \widetilde{Q}_j^{(2)} \qquad \text{on } \ S_1, \ \ j=\overline{5,9},\end{aligned} \tag{8.8}$$

$$\Big[\Big(-\frac{1}{2}I_4+\mathscr{K}^{(1)}_{S_1}\Big)(\mathscr{H}^{(1)}_{S_1})^{-1}g^{(1)}\Big]_j \;+\; \Big[\Big(-\frac{1}{2}I_9+\mathscr{K}^{(2)}_{S_1}\Big)(\mathscr{H}^{(2)}_{S_1})^{-1}g^{(2)}\Big]_j$$
$$-\,r_{S_1}[T^{(2)}V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}(r_{S_2}V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}g^{(2)})]_j$$
$$+\,r_{S_1}[T^{(2)}V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}h^{(2)}_0]_j=\widetilde{f}^{\,(2)}_j \qquad \text{on } S_1,\ \ j=\overline{1,4}, \tag{8.9}$$
$$r_{S_2^{(N)}}[T^{(2)}V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}g^{(2)}]-r_{S_2^{(N)}}\Big[\Big(-\frac{1}{2}I_9+\mathscr{K}^{(2)}_{S_1}\Big)(\mathscr{H}^{(2)}_{S_1})^{-1}(r_{S_2}V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}g^{(2)})\Big]$$
$$+\,r_{S_2^{(N)}}\Big(-\frac{1}{2}I_9+\mathscr{K}^{(2)}_{S_2}\Big)(\mathscr{H}^{(2)}_{S_2})^{-1}h^{(2)}_0=\widetilde{q}^{(N)}_2 \qquad \text{on } S_2^{(N)}, \tag{8.10}$$

where

$$\widetilde{f}^{\,(1)}_j:=f^{(1)}_j+r_{S_1}[V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}\Phi^{(2)}_0],\ \ j=\overline{1,4},$$
$$\widetilde{Q}^{\,(2)}_j:=-Q^{(2)}_j+r_{S_1}[V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}\Phi^{(2)}_0]_j,\ \ j=\overline{5,9},$$
$$\widetilde{f}^{\,(2)}_j:=f^{(2)}_j-r_{S_1}[T^{(2)}V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}\Phi_0]_j,\ \ j=\overline{1,4},$$
$$\widetilde{q}^{(N)}_2:=q^{(N)}_2-r_{S_2^{(N)}}\Big(-\frac{1}{2}I_9+\mathscr{K}^{(2)}_{S_2}\Big)(\mathscr{H}^{(2)}_{S_2})^{-1}\Phi^{(2)}_0.$$

The operator corresponding to system (8.7)–(8.10), has the form

$$\mathscr{P}:=\begin{bmatrix} I_{9\times 4} & -I_9+\mathscr{B} & -r_{S_1}[V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}]_{9\times 9}\\ \mathscr{A}^{(1)}_{S_1} & [\mathscr{A}^{(2)}_{S_1}]_{4\times 9}+\mathscr{C} & r_{S_1}[T^{(2)}V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}]_{4\times 9}\\ [0]_{9\times 4} & \mathscr{D} & r_{S_2^{(N)}}\mathscr{A}^{(2)}_{S_2}\end{bmatrix}_{22\times 22},$$

where

$$\mathscr{A}^{(1)}_{S_1}:=\Big(-\frac{1}{2}I_4+\mathscr{K}^{(1)}_{S_1}\Big)(\mathscr{H}^{(1)}_{S_1})^{-1},$$
$$\mathscr{A}^{(2)}_{S_1}:=\Big(-\frac{1}{2}I_9+\mathscr{K}^{(2)}_{S_1}\Big)(\mathscr{H}^{(2)}_{S_1})^{-1},$$
$$\mathscr{A}^{(2)}_{S_2}:=\Big(-\frac{1}{2}I_9+\mathscr{K}^{(2)}_{S_2}\Big)(\mathscr{H}^{(2)}_{S_2})^{-1},$$
$$\mathscr{B}:=r_{S_1}[V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}(r_{S_2}V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1})]_{9\times 9},$$
$$\mathscr{C}:=-r_{S_1}[T^{(2)}V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}(r_{S_2}V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1})]_{4\times 9},$$
$$\mathscr{D}:=r_{S_2^{(N)}}[T^{(2)}V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}]_{9\times 9}-r_{S_2^{(N)}}\Big[\mathscr{A}^{(2)}_{S_1}\big(r_{S_2}V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}\big)\Big]_{9\times 9}.$$

The operator

$$\mathscr{P}:\mathscr{X}_1\to\mathscr{Y}_1$$

is bounded, where

$$\mathscr{X}_1:=[H^{\frac{1}{2}}(S_1)]^{13}\times[\widetilde{H}^{\frac{1}{2}}(S_2^{(N)})]^9,$$
$$\mathscr{Y}_1:=[H^{\frac{1}{2}}(S_1)]^{9}\times[H^{-\frac{1}{2}}(S_1)]^4\times[H^{-\frac{1}{2}}(S_2^{(N)})]^9.$$

The following theorem holds.

Theorem 8.1. *The operator*

$$\mathscr{P}:\mathscr{X}_1\to\mathscr{Y}_1$$

is invertible.

Proof. First, we show that the operator

$$\mathscr{P}:\mathscr{X}_1\to\mathscr{Y}_1$$

is Fredholm with zero index.

Indeed, obviously, the operators

$$r_{S_1}[V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}]_{9\times 9} : [\widetilde{H}^{\frac12}(S_2^{(N)})]^9 \to [H^{\frac12}(S_1)]^9,$$
$$r_{S_1}[T^{(2)}V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}]_{4\times 9} : [\widetilde{H}^{\frac12}(S_2^{(N)})]^9 \to [H^{-\frac12}(S_1)]^4,$$
$$\mathscr{B} : [H^{\frac12}(S_1)]^9 \to [H^{\frac12}(S_1)]^9,$$
$$\mathscr{C} : [H^{\frac12}(S_1)]^9 \to [H^{-\frac12}(S_1)]^4,$$
$$\mathscr{D} : [H^{\frac12}(S_1)]^9 \to [H^{-\frac12}(S_2^{(N)})]^9$$

are compact, since $S_1 \cap S_2 = \varnothing$.

Now, we consider the operator

$$\mathscr{P}_1 := \begin{bmatrix} I_{9\times 4} & -I_9 & [0]_{9\times 9} \\ \mathscr{A}^{(1)}_{S_1} & [\mathscr{A}^{(2)}_{S_1}]_{4\times 9} & [0]_{4\times 9} \\ [0]_{9\times 4} & [0]_{9\times 9} & r_{S_2^{(N)}}\mathscr{A}^{(2)}_{S_2} \end{bmatrix},$$

where the operator

$$\mathscr{P} - \mathscr{P}_1 : \mathscr{X}_1 \to \mathscr{Y}_1$$

is compact. If we show that the operator

$$\mathscr{P}_1 : \mathscr{X}_1 \to \mathscr{Y}_1$$

is Fredholm with zero index, then the operator

$$\mathscr{P} : \mathscr{X}_1 \to \mathscr{Y}_1$$

will be Fredholm with zero index.

Write the system corresponding to the operator $\mathscr{P}_1$ as follows:

$$\widetilde{g}^{\,(1)}_j - \widetilde{g}^{\,(2)}_j = \widetilde{f}^{\,(1)}_j \quad \text{on } S_1, \ \ j=\overline{1,4}, \tag{8.11}$$
$$-\widetilde{g}^{\,(2)}_j = \widetilde{F}_j \quad \text{on } S_1, \ \ j=\overline{5,9}, \tag{8.12}$$
$$[\mathscr{A}^{(1)}_{S_1}\widetilde{g}^{\,(1)}]_j + [\mathscr{A}^{(2)}_{S_1}\widetilde{g}^{\,(2)}]_j = \widetilde{f}^{\,(2)}_j \quad \text{on } S_1, \ \ j=\overline{1,4}, \tag{8.13}$$
$$r_{S_2^{(N)}}\mathscr{A}^{(2)}_{S_2}\widetilde{h}_0 = \widetilde{q}^{(N)}_2 \quad \text{on } S_2^{(N)}. \tag{8.14}$$

System (8.11)–(8.14) is equivalent to the following system:

$$\widetilde{g}^{\,(1)}_j - \widetilde{g}^{\,(2)}_j = \widetilde{f}^{\,(1)}_j \quad \text{on } S_1, \ \ j=\overline{1,4}, \tag{8.15}$$
$$-\widetilde{g}^{\,(2)}_j = \widetilde{F}_j \quad \text{on } S_1, \ \ j=\overline{5,9}, \tag{8.16}$$
$$(\mathscr{A}^{(1)}_{S_1} + \overline{\mathscr{A}}^{(2)}_{S_1})\widetilde{g}^{\,(1)} = \Psi \quad \text{on } S_1, \tag{8.17}$$
$$r_{S_2^{(N)}}\mathscr{A}^{(2)}_{S_2}\widetilde{h}^{(2)}_0 = \widetilde{q}^{(N)}_2 \quad \text{on } S_2^{(N)}, \tag{8.18}$$

where

$$\overline{\mathscr{A}}^{(2)}_{S_1} := \big[\mathscr{A}^{(2)}_{S_1,ji}\big]_{4\times 4}, \ \ j,i=\overline{1,4}, \quad \Psi = (\psi_1,\psi_2,\psi_3,\psi_4)^T,$$
$$\psi_j := \widetilde{f}^{\,(2)}_j + \sum_{i=1}^{4}\mathscr{A}^{(2)}_{S_1,ji}\widetilde{f}^{(1)}_i + \sum_{i=5}^{9}\mathscr{A}^{(2)}_{S_1,ji}\widetilde{F}_i, \ \ j=\overline{1,4}.$$

The operator corresponding to system (8.15)–(8.18), has the form

$$\mathscr{P}_2 := \begin{bmatrix} I_{9\times 4} & -I_9 & [0]_{9\times 9} \\ \mathscr{A}^{(1)}_{S_1} + \overline{\mathscr{A}}^{(2)}_{S_1} & [0]_{4\times 9} & [0]_{4\times 9} \\ [0]_{9\times 4} & [0]_{9\times 9} & r_{S_2^{(N)}}\mathscr{A}^{(2)}_{S_2} \end{bmatrix}_{22\times 22}.$$

Evidently, the operator

$$\mathscr{P}_2 : \mathscr{X}_1 \to \mathscr{Y}_1$$

is bounded.

Consider the composition

$$\mathscr{P}_3 := \mathscr{P}_2 \circ \mathscr{Q},$$

where the operator

$$\mathscr{Q} : \mathscr{X}_1 \to \mathscr{X}_1$$

is invertible (see Section 6).

The operator $\mathscr{P}_3$ has the form

$$\mathscr{P}_3 := \begin{bmatrix} -I_9 & I_{9\times 4} & [0]_{9\times 9} \\ [0]_{4\times 9} & \mathscr{A}^{(1)}_{S_1} + \overline{\mathscr{A}}^{(2)}_{S_1} & [0]_{4\times 9} \\ [0]_{9\times 9} & [0]_{9\times 4} & r_{S_2^{(N)}} \mathscr{A}^{(2)}_{S_2} \end{bmatrix}_{22\times 22}.$$

To show that the operator

$$\mathscr{P} : \mathscr{X}_1 \to \mathscr{Y}_1$$

is Fredholm with zero index, it suffices to show that the operator

$$\mathscr{P}_3 : \mathscr{X}_1 \to \mathscr{Y}_1$$

is Fredholm with zero index.

Since the operator $\mathscr{P}_3$ is triangular diagonal, it suffices to show that the operators standing on the diagonal are Fredholm with zero index.

As we know, from Lemma 6.2 it follows that the operator

$$\mathscr{A}^{(1)}_{S_1} + \overline{\mathscr{A}}^{(2)}_{S_1} : [H^{\frac{1}{2}}(s_1)]^4 \to [H^{-\frac{1}{2}}(s_1)]^4$$

is Fredholm with zero index, while the Poincaré–Steklov type operator

$$r_{S_2^{(N)}} \mathscr{A}^{(2)}_{S_2} : [\widetilde{H}^{\frac{1}{2}}(S_2^{(N)})]^9 \to [H^{-\frac{1}{2}}(S_2^{(N)})]^9$$

is invertible (see [5, Theorem 7.7]). Hence the operator

$$\mathscr{P}_3 : \mathscr{X}_1 \to \mathscr{Y}_1$$

is Fredholm with zero index. Then the operators

$$\mathscr{P}_1, \mathscr{P}_2 : \mathscr{X}_1 \to \mathscr{Y}_1$$

will also be Fredholm with zero index. Therefore the operator

$$\mathscr{P} : \mathscr{X}_1 \to \mathscr{Y}_1$$

is Fredholm with zero index.

Now we show that the operator

$$\mathscr{P} : \mathscr{X}_1 \to \mathscr{Y}_1$$

is invertible.

The invertibility of the operator $\mathscr{P}$ is derived from the uniqueness of the solution of the boundary-transmission problem $(TM)_\tau$.

Indeed, let $(g^{(1)}, g^{(2)}, h_0^{(2)}) \in \mathscr{X}_1$ be a solution of the homogeneous equation

$$\mathscr{P}(g^{(1)}, g^{(2)}, h_0^{(2)})^\top = 0. \tag{8.19}$$

We construct the following potentials:

$$U^{(1)} = V^{(1)}_{S_1} (\mathscr{H}^{(1)}_{S_1})^{-1} g^{(1)}, \tag{8.20}$$

$$U^{(2)} = V^{(2)}_{S_1} (\mathscr{H}^{(2)}_{S_1})^{-1} g^{(2)} + V^{(2)}_{S_2} (\mathscr{H}^{(2)}_{S_2})^{-1} h_0^{(2)}. \tag{8.21}$$

Since $(g^{(1)}, g^{(2)}, h_0^{(2)})^\top$ is a solution of the homogeneous equation (8.19), i.e., of the homogeneous system (8.7)–(8.10), it is clear that $(U^{(1)}, U^{(2)})$ will be a solution of the homogeneous boundary-transmission problem $(TM)_\tau$. Then from the uniqueness theorem of problem $(TM)_\tau$ there follows that

$$U^{(1)} \equiv 0 \quad \text{in } \ \Omega_1, \tag{8.22}$$

$$U^{(2)} \equiv 0 \quad \text{in } \Omega_2. \tag{8.23}$$

Since the single layer potentials are continuous in space $\mathbb{R}^3$, we have

$$\{U^{(1)}\}^+ = \{U^{(1)}\}^- = 0 \quad \text{on } S_1,$$
$$\{U^{(2)}\}^+ = \{U^{(2)}\}^- = 0 \quad \text{on } S_1 \cup S_2.$$

Hence the vector functions $U^{(1)}$ and $U^{(2)}$ satisfy the following Dirichlet problems:

$$\begin{cases} A^{(1)}(\partial_x, \tau)U^{(1)} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_1}, \\ \{U^{(1)}\}^- = 0 & \text{on } S_1, \end{cases}$$

and

$$\begin{cases} A^{(2)}(\partial_x, \tau)U^{(2)} = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega_2}, \\ \{U^{(2)}\}^- = 0 & \text{on } S_1 \cup S_2. \end{cases}$$

From the uniqueness of solutions of the Dirichlet problem it follows that these problems have only trivial solution, i.e.,

$$U^{(1)} \equiv 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_1},$$
$$U^{(2)} \equiv 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega_2}.$$

Hence from (8.22) and (8.23), we get

$$U^{(1)} \equiv 0 \quad \text{in } \mathbb{R}^3,$$
$$U^{(2)} \equiv 0 \quad \text{in } \mathbb{R}^3.$$

Then applying the jump formulas of potentials (8.20) and (8.21), we get

$$\{T^{(1)}U^{(1)}\}^- - \{T^{(1)}U^{(1)}\}^+ = g^{(1)} = 0 \quad \text{on } S_1,$$
$$\{T^{(2)}U^{(2)}\}^- - \{T^{(2)}U^{(2)}\}^+ = g^{(2)} = 0 \quad \text{on } S_1,$$
$$\{T^{(2)}U^{(2)}\}^- - \{T^{(2)}U^{(2)}\}^+ = h_0^{(2)} = 0 \quad \text{on } S_2.$$

Therefore we obtain

$$\operatorname{Ker} \mathscr{P} = \{0\}$$

and, since $\operatorname{ind} \mathscr{P} = 0$, we have

$$\operatorname{Ker} \mathscr{P}^* = \{0\}.$$

Thus the operator

$$\mathscr{P} : \mathscr{X}_1 \to \mathscr{Y}_1$$

is invertible, and Theorem 8.1 is proved. □

The invertibility of the operator $\mathscr{P}$ implies the unique solvability of systems (8.7)–(8.10) and (8.1)–(8.5). Consequently, we obtain the unique solvability of the mixed type boundary-transmission problem $(TM)_\tau$.

Thus we obtain the existence and uniqueness theorem of the mixed type boundary-transmission problem $(TM)_\tau$.

Theorem 8.2. *Let $S_1, S_2 \in C^\infty$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 > 0$, $\omega \in \mathbb{R}$, and*

$$f_j^{(1)} \in H^{\frac{1}{2}}(S_1), \quad f_j^{(2)} \in H^{-\frac{1}{2}}(S_2), \quad j = \overline{1,4},$$
$$Q_j^{(2)} \in H^{\frac{1}{2}}(S_1), \quad j = \overline{5,9}, \quad p_2^{(D)} \in [H^{\frac{1}{2}}(S_2^{(D)})]^9, \quad q_2^{(N)} \in [H^{-\frac{1}{2}}(S_2^{(N)})]^9.$$

Then the mixed boundary-transmission problem $(TM)_\tau$ has a unique solution

$$(U^{(1)}, U^{(2)}) \in [H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9,$$

which is presented in the following form:

$$U^{(1)} = V_{S_1}^{(1)}(\mathscr{H}_{S_1}^{(1)})^{-1} g^{(1)} \qquad \textit{in } \Omega_1,$$

$$U^{(2)} = V^{(2)}_{S_1}(\mathscr{H}^{(2)}_{S_1})^{-1}g^{(2)} + V^{(2)}_{S_2}(\mathscr{H}^{(2)}_{S_2})^{-1}h^{(2)} \quad in\ \Omega_2,$$

where $g^{(1)}$, $g^{(2)}$, $h^{(2)}$ *are the unique solutions of system* (8.1)–(8.5).

Let us introduce the notation

$$d := \frac{c\,b_0^{(2)} + p\,\lambda_1^{(2)} + q\,\nu_2^{(2)}}{2\gamma^{(2)}},$$

where

$$c := \frac{1}{2}(b_0^{(2)}b_{11} + \lambda_1^{(2)}b_{21} + \nu_2^{(2)}b_{31}), \quad p := \frac{1}{2}(b_0^{(2)}b_{12} + \lambda_1^{(2)}b_{22} + \nu_2^{(2)}b_{32}),$$

$$q := \frac{1}{2}(b_0^{(2)}b_{13} + \lambda_1^{(2)}b_{23} + \nu_2^{(2)}b_{33}),$$

$$[b_{jk}]_{3\times 3} := \begin{bmatrix} a_0^{(2)} & -\lambda_2^{(2)} & \nu_1^{(2)} \\ \lambda_2^{(2)} & \chi^{(2)} & \nu_3^{(2)} \\ \nu_1^{(2)} & -\nu_3^{(2)} & k^{(2)} \end{bmatrix}^{-1}.$$

The following regularity theorem holds.

Theorem 8.3. *Suppose* $S_1, S_2 \in C^\infty$ *and*

$$f_j^{(1)} \in C^\infty(S_1), \quad f_j^{(2)} \in C^\infty(S_1), \quad j = \overline{1,4},$$

$$Q_j^{(2)} \in C^\infty(S_1), \quad j = \overline{5,9}, \quad p_2^{(D)} \in [C^\infty(\overline{S_2^{(D)}})]^9, \quad q_2^{(N)} \in [C^\infty(\overline{S_2^{(N)}})]^9.$$

Then

1) *If* $d < 0$, *then the unique solution* $(U^{(1)}, U^{(2)})$ *to the mixed boundary-transmission problem* $(TM)_\tau$ *belongs to* $[C^\infty(\overline{\Omega_1})]^4 \times [C^{\gamma_1}(\overline{\Omega_2})]^9$, *i.e.*

$$(U^{(1)}, U^{(2)}) \in [C^\infty(\overline{\Omega_1})]^4 \times [C^{\gamma_1}(\overline{\Omega_2})]^9,$$

where $\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\operatorname{arctg} 2\sqrt{-d}$, γ_1 *depends on the material constants, does not depend on the geometry of the exceptional line* $\ell_m = \partial S_2^{(D)} = \partial S_2^{(N)}$ *and may take any values from the interval* $(0, \frac{1}{2})$.

2) *If* $d \geq 0$, *then the unique solution to the corresponding boundary-transmission problem* $(TM)_\tau$

$$(U^{(1)}, U^{(2)}) \in [C^\infty(\overline{\Omega_1})]^4 \times [C^{\frac{1}{2}}(\overline{\Omega_2})]^9.$$

Proof of this theorem follows from the work [5] (see Section 9), where the asymptotic properties and the smoothness of solutions of mixed problem are studied near the change of boundary conditions, i.e., near the line ℓ_m (cf. [2,3]). Note that the smoothness of the vector function $U^{(2)}$ is finite in a neighborhood of ℓ_m, but taking into account the data conditions of the Theorem 8.3, outside this neighborhood it is infinitely differentiable, i.e., belongs to the class C^∞.

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